Counting invariants and wall-crossing

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1. Moduli spaces and counting invariants

Let X be a smooth complex projective variety.

Consider moduli spaces of coherent sheaves on X.

We shall insist on moduli spaces that are varieties rather than algebraic stacks so we can use them to define counting invariants.

The standard method is to choose a polarization $\ell = c_1(L)$ and restrict attention to semistable sheaves.

Geometric invariant theory then constructs a projective variety $\mathcal{M}_\ell(\alpha)$ which is a coarse moduli space for semistable sheaves of fixed Chern character α.

There is usually no universal family, basically because objects have non-trivial automorphisms.

The most naive counting invariant associated to a moduli space $\mathcal{M}$ is its Euler characteristic $e(\mathcal{M})$ in the analytic topology.

If X is a Calabi-Yau threefold and $\mathcal{M}$ is a moduli space of stable sheaves there is a more sophisticated approach using virtual cycles. The resulting integers are called Donaldson-Thomas invariants, and are invariant under deformations of X.

Behrend discovered a more local definition. He associates a weight $\nu(E) \in \mathbb{Z}$ to any sheaf E. Given a family of sheaves over a base $\mathcal{M}$ the resulting function $\nu\colon \mathcal{M} \to \mathbb{Z}$ is constructible, and one can consider the weighted Euler characteristic

$$e(\mathcal{M}, \nu) := \sum_{n \in \mathbb{Z}} n \cdot e(\nu^{-1}(n)).$$

When $\mathcal{M} = \mathcal{M}_\ell(\alpha)$ consists of stable objects, this number $e(\mathcal{M}, \nu)$ coincides with the Donaldson-Thomas invariant.

Recently, Joyce and Song have shown that the moduli stack of coherent sheaves is locally of the form

$$(dW = 0)/\operatorname{GL}(n)$$

where T is a complex manifold, and $W\colon T \to \mathbb{C}$ is holomorphic.

It follows that

$$\nu(E) = (-1)^{\dim(T/G)}\big(1 - e(\mathrm{MF}_W(E))\big).$$

where $\mathrm{MF}_W(E)$ is the Milnor fibre of the function W at the point corresponding to E.

Surprisingly, all the results in this talk apply both to the naive Euler characteristic invariants and to the DT invariants.

2. Framed invariants

An alternative approach to constructing moduli varieties is to consider framed sheaves.

Fix a sheaf P and consider sheaves E equipped with a surjective map $f\colon P \twoheadrightarrow E$.

There is a projective variety $\mathrm{Quot}^P(\alpha)$ which is a fine moduli space for such maps.

There are no non-trivial automorphisms in this case.

We can use Behrend's approach to associate counting invariants to these framed moduli spaces.

Example 1. Let X be a Calabi-Yau threefold.

Fix $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$.

There is a variety $\mathrm{Hilb}(\beta, n)$ parameterizing

(a) surjections $\mathcal{O}_X \twoheadrightarrow F$ with $\mathrm{ch}(F) = (0, 0, \beta, n)$,

(b) stable sheaves E with $\mathrm{ch}(E) = (1, 0, -\beta, -n)$ and trivial determinant,

(c) subschemes $C \subset X$ of dimension $\leqslant 1$ with $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$.

The corresponding DT invariants $I(\beta, n)$ are the curve-counting invariants studied by [MNOP].

As we vary the polarization ℓ the moduli spaces $\mathcal{M}^\ell(\alpha)$ change, and so do the associated counting invariants.

In many cases we get a wall-and-chamber structure. Recent work of Joyce and Kontsevich-Soibelman studies wall-crossing behaviour of the counting invariants.

What is the analogue of varying ℓ in the framed case?

Consider the derived category $D(X) := D^b \operatorname{Coh}(X)$.

Recall that $\operatorname{Coh}(X) \subset D(X)$.

Consider quotients of P in different abelian subcategories $\mathcal{A} \subset D(X)$ containing P.

Example 2. Let X be a Calabi-Yau threefold.

Suppose we want to understand curve-counting invariants for birationally equivalent Calabi-Yau varieties Y.

For any such Y there is an equivalence

$$D(Y) \xrightarrow{\Phi} D(X), \quad \Phi(\mathcal{O}_Y) = \mathcal{O}_X.$$

Setting $\mathcal{A}_Y = \Phi(\operatorname{Coh}(Y))$ we have $\mathcal{O}_X \in \mathcal{A}_Y \subset D(X)$.

Considering quotients of $\mathcal{O}_X$ in $\mathcal{A}_Y \subset D(X)$ gives the invariants for Y.

Thus curve-counting invariants for all Calabi-Yau varieties birational to X occur as invariants counting quotients of $\mathcal{O}_X$.

3. Hearts and tilting

The analogue of wall-crossing in the framed situation is a basic operation in homological algebra called tilting.

Fix a triangulated category D such as $D(X)$.

First we give the definition of a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$.

Then we define a special class of abelian subcategories $\mathcal{A} \subset D$ called hearts.

Finally we define the tilting operation

$$(\mathcal{T}, \mathcal{F}) \subset \mathcal{A} \subset D \quad \longleftrightarrow \quad (\mathcal{T}', \mathcal{F}') \subset \mathcal{A}' \subset D.$$

The rest of the talk will contain several examples of this construction.

Let $\mathcal{A}$ be an abelian category.

A torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a pair of full subcategories such that

(a) $\operatorname{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$,

(b) for every object $E \in \mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The basic example is when $\mathcal{A} = \mathrm{Coh}(X)$ and $\mathcal{T}$ and $\mathcal{F}$ consist of torsion and torsion-free sheaves respectively.

Let D be a triangulated category.

A heart $\mathcal{A} \subset D$ is a full subcategory such that:

(a) $\mathrm{Hom}_D(A[j], B[k]) = 0$ for all $A, B \in \mathcal{A}$ and $j > k$.

(b) for every object $E \in D$ there is a finite sequence of triangles

$$\begin{array}{ccccccccc} 0 = E_{m-1} & \longrightarrow & E_m & \to \cdots \to & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & H_m & & & & H_n & \end{array}$$

with $H_j[j] \in \mathcal{A}$.

It follows that $\mathcal{A}$ is abelian. The basic example is $\mathcal{A} \subset D^b(\mathcal{A})$.

Suppose $\mathcal{A} \subset D$ is a heart, and $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a torsion pair.

There is a new heart $\mathcal{A}' \subset D$ consisting of objects E that fit into a triangle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & T[-1] & \end{array}$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

There is also a torsion pair in $\mathcal{A}'$, namely $(\mathcal{F}, \mathcal{T}[-1])$. Tilting again gives back the heart $\mathcal{A}$ with a shift.

4. Stable pairs

Let X be a Calabi-Yau threefold and tilt $\mathrm{Coh}(X) \subset D(X)$ with respect to the torsion pair

$$\mathcal{T} = \mathrm{Coh}_{\leqslant 0}(\mathcal{A}), \quad \mathcal{F} = \mathrm{Coh}_{\geqslant 1}(X).$$

Quotients of $\mathcal{O}_X$ in the tilted heart are maps of sheaves

$$\mathcal{O}_X \xrightarrow{f} E$$

with $E \in \mathrm{Coh}_{\geqslant 1}(X)$ and $\mathrm{Coker}(f) \in \mathrm{Coh}_{\leqslant 0}(X)$.

These are the stable pairs studied by Pandharipande and Thomas.

Using methods of Joyce or Kontsevich-Soibelman one can prove

$$\sum_n \mathrm{DT}(\beta, n) q^n = \sum_n \mathrm{DT}(0, n) q^n \cdot \sum_n \mathrm{PT}(\beta, n) q^n.$$

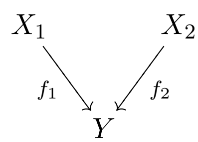
This can be thought of as a wall-crossing formula.

5. Threefold flops

Consider the threefold ordinary double point

$$Y = (xy - zw) \subset \mathbb{C}^4$$

and its small resolutions

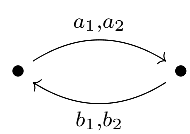


There are equivalences

$$D(X_1) \cong D(A) \cong D(X_2).$$

where A is a certain non-commutative algebra.

The algebra A is defined by the quiver



with the Klebanov-Witten potential

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

Explicitly, the relations are

$$b_1 a_i b_2 = b_2 a_i b_1, \quad a_1 b_i a_2 = a_2 b_i a_1.$$

Identifying the derived categories with a single category D we obtain three hearts

$$\mathrm{Coh}(X_1), \quad \mathrm{Mod}(A), \quad \mathrm{Coh}(X_2).$$

The hearts $\mathrm{Coh}(X_1)$ and $\mathrm{Mod}(A)$ are related by a tilt with respect to torsion theory

$$\mathcal{T} = \{E \in \mathrm{Coh}(X_1) : \mathbf{R}f_1, *(E) = 0\}.$$

Similarly for the hearts $\mathrm{Coh}(X_2)$ and $\mathrm{Mod}(A)$.

Using Joyce's work on wall-crossing, Toda proved that for any flop of smooth projective threefolds, the expression

$$\frac{\sum_{(\beta,n)} \mathrm{DT}(\beta,n) x^\beta q^n}{\sum_{(\beta,n): f_*(\beta)=0} \mathrm{DT}(\beta,n) x^\beta q^n}$$

is invariant.

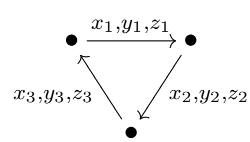
6. Local del Pezzo surfaces

Let X be the non-compact Calabi-Yau threefold $\mathcal{O}_{\mathbb{P}^2}(-3)$.

The McKay correspondence shows that there is an equivalence

$$D(X) \cong D(A)$$

where A is defined by the quiver



with potential

$$W = \sum_{i,j,k} \epsilon_{ijk} x_i y_j z_k.$$

Consider the heart $\mathcal{A} = \mathrm{Mod}(A) \subset D(X)$.

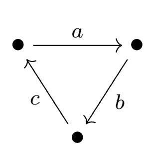
If S is a one-dimensional module define

$$\langle S \rangle = \{M \in \mathcal{A} : M = S^{\oplus n}\} \subset \mathcal{A}.$$

There are six torsion pairs in $\mathcal{A}$ obtained by taking either $\mathcal{T}$ or $\mathcal{F}$ to be $\langle S_i \rangle$ for some vertex i.

The resulting tilted hearts are all module categories. Repeating we get many algebras with $D(X) \cong D(A)$.

All are defined by quivers of the form



with $a^2 + b^2 + c^2 = abc$.

The combinatorics of the tilting process is described by the Cayley graph of the affine braid group

$$\langle \sigma_1, \sigma_2, \sigma_3 : \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \rangle$$

with respect to the generators $\sigma_i^{\pm 1}$.

Quotienting by the action of a subgroup of the autoequivalences of $D(X)$ gives the Markov tree, i.e. the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

Stern has shown how to perform similar constructions for any del Pezzo surface.

The resulting graphs were obtained earlier by Hanany and collaborators studying Seiberg duality for quiver gauge theories.

7. Cluster transformations

Kontsevich and Soibelman have recently explained how DT invariants change under this type of tilting operation.

Suppose A is a CY_3 algebra defined by a quiver with no loops or 2-cycles.

Label the vertices $1, \cdots, n$ and let n_{ij} be the number of arrows from vertex i to vertex j.

Set $v_{ij} = n_{ji} - n_{ij} = \chi(S_i, S_j)$.

Given a projective module P there are invariants $\mathrm{DT}^P(d)$ counting finite-dimensional quotient modules $P \twoheadrightarrow E$ with dimension vector $d = (d_1, \cdots, d_n)$.

Define a ring

$$R = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \otimes C[y_1^{\pm 1}, \cdots, y_n^{\pm 1}].$$

with an involution τ

$$\tau(x_i) = x_i^{-1}, \quad \tau(y_i) = y_i^{-1}.$$

There is a natural ideal $I \lhd R$ generated by $x_i - \prod y_j^{v_{ij}}$.

Encode the counting invariants in an automorphism

$$\mathrm{DT}(y_i) = y_i \sum_d \mathrm{DT}^{P_i}(d) x^d$$

of a suitable completion $R \subset \hat{R}$, preserving the ideal I.

Under a tilt

$$\begin{array}{ccc} \hat{R} & \xrightarrow{C_+} & \hat{R} \\ {\scriptstyle \mathrm{DT}^{(A)}}\downarrow & & \downarrow{\scriptstyle \mathrm{DT}^{(B)}} \\ \hat{R} & \xrightarrow{C_-} & \hat{R} \end{array}$$

where $\tau \circ C_- = C_+ \circ \tau$ and

$$C_+(x_j) = \begin{cases} {x_i}^{-1} & \text{if } j = i \\ x_j \cdot (1 + x_i)^{n_{ji}} (1 + {x_i}^{-1})^{-n_{ij}} & \text{if } j \neq i \end{cases}$$

and

$$C_+(y_k) = \begin{cases} {y_i}^{-1}(1 + {x_i}^{-1}) \prod_j {y_j}^{n_{ji}} & \text{if } k = i \\ y_k & \text{if } k \neq i \end{cases}$$

These are called cluster transformations.