Algebras with restricted growth

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Golod-Shafarevich Groups and Algebras and Rank Gradient

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Growth of groups

• Let S be a finite generating set for a group G. Let $f_S(n)$ be the number of elements of G that can be written as words of length at most n in $S \cup S^{-1}$. Then f_S is the growth function of G with respect to S.

Famous Theorem (Gromov) f_S is polynomially bounded if and only if G is virtually nilpotent.

• Prior work by Shvarts, Milnor, Wolfe, Tits, Bass

• Recently published "How groups grow", by A Mann, LMS Lecture Notes Series vol 395

• Let k be a field and $A = kG$. Let V be the k-subspace of A generated by $\{1\} \cup S \cup S^{-1}$. Then $A = k[V]$ and $f_S(n) = \dim(V^n)$.

Growth of algebras

- Let A be a k-algebra that is finitely generated as an algebra.

- A finite dimensional vector subspace V such that $1 \in V$ and $A = k[V]$ is called a frame of A.

- The growth function of A (relative to the frame V) is the function
$$f(n) := \dim(V^n)$$

- We are interested in the (asymptotic) rate of growth of $f(n)$.

- Note, the rate of growth is independent of the choice of frame.

Trichotomy

(i) If $\dim(V^n)$ is bounded above by a polynomial in n then $A = k[V]$ has polynomially bounded growth.

(ii) If $\dim(V^n) > c^n$ for some $c > 1$ then $A = k[V]$ has exponential growth

(iii) Otherwise, A has intermediate growth.

- The union of (i) and (iii) is subexponential growth.

Examples: polynomially bounded growth

• Commutative polynomial algebras $A = k[x_1, \ldots, x_d]$, with $V = k + kx_1 + \cdots + kx_n$
then
$$\dim(V^n) = \binom{n+d}{d}$$
which is a polynomial in n of degree d.

• The group algebra of a finitely generated virtually nilpotent group.

• The Weyl algebra $A = k[x, y]$ with $xy - yx = 1$.

Ring theoretic Corollary to Gromov If kG has polynomially bounded growth then kG is a noetherian ring. (But not vice-versa.)

Examples: exponential growth

- The free algebra $k\langle x, y\rangle$.
- The group algebra of a free group on more than one generator.
- The group algebra of a polycyclic, non virtually nilpotent group.
- Golod-Shafarevich algebras

Examples: intermediate growth

• The group algebras of groups of intermediate growth such as those constructed by Grigorchuk. (It is still open whether Gupta-Sidki groups have intermediate growth.)

• The universal enveloping algebra U of the Virasoro Lie algebra: $L = {}_k < v_1, v_2, \cdots >$ subject to $[v_n, v_m] := (n - m)v_{n+m}$.

• Note that $U = k[v_1, v_2]$.

In fact, Ufnarovskii shows that U is finitely presented (but this is not so obvious,).

Subexponential growth

Theorem (Jategaonkar, and others)
Domains with subexponential growth have division rings of fractions

Proof Otherwise the Ore condition fails and there are nonzero elements such that $aR \cap bR = 0$.

Then $k\langle a, b\rangle$ is a free subalgebra and this gives exponential growth.

Subexponential growth

Theorem (Stephenson and Zhang) Graded noetherian algebras have subexponential growth.

Example If G is a polycylic group that is not virtually nilpotent then kG is a noetherian algebra, but G and kG have exponential growth.

Informal definition

If $\dim(V^n)$ grows like n^r then $\mathrm{GKdim}(A) = r$.

Official definition

$$\mathrm{GKdim}(A) \;=\; \limsup_{n\to\infty} \frac{\log(\dim(V^n))}{\log(n)}$$

for any frame V of A.

• For example, there are algebras that grow like $n^2 \log(n)$, and these algebras have GKdim equal to 2, but grow faster than n^2.

Historical remark

• The nth Weyl Algebra $A_n := k[x_1, \ldots, x_n; y_1, \ldots, y_n]$ with x_i pairwise commuting, y_j pairwise commuting and $x_i y_j - y_j x_i = \delta_{ij}$.

• The algebra A_n is a noetherian domain and so has a division ring of fractions D_n. These division rings often occur in Lie theory.

• **Gelfand-Kirillov conjecture** The universal enveloping algebra of a finite dimensional algebraic Lie algebra has a division ring of fractions that is isomorphic to a suitable D_n. (Now known not to be true in general)

• The first significant use of the ideas of Gelfand-Kirillov dimension was in 1966 when Gelfand and Kirillov used related dimensions to show that $D_n \cong D_m$ precisely when $n = m$.

Possible values for GKdim

$$\boxed{\{0\} \cup \{1\} \;\cup\; [2, \infty]}$$

- $\mathrm{GKdim}(A) = 0$ if and only if A is finite dimensional

- **Small, Stafford and Warfield**: If $\mathrm{GKdim}(A) = 1$ then A is nearly commutative (technically: A is a polynomial identity algebra (PI))

- **Borho and Kraft**: any value in $[2, \infty)$ is possible

- **Bergman's Gap Theorem**: values in the open interval $(1, 2)$ are not possible

Graded algebras

• A is a (connected) graded algebra if A has a vector space decomposition

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \ldots$$

with $A_i A_j \subseteq A_{i+j}$.

• Growth calculations are often easier in graded algebras. Assume $A = k[A_1]$ then, with $V := k \oplus A_1$,

$$\dim(V^n) = \sum_{i=0}^{n} \dim A_i$$

• Noetherian connected graded algebras are the setting for noncommutative projective geometry.

Generating functions

- For an graded algebra A define the Poincaré series, $P_A(t)$, to be

$$P_A(t) := \sum_{n=0}^{\infty} \dim(A_n) t^n.$$

- A fundamental question is whether or not the Poincaré series is a rational function; that is, $P_A(t) = \frac{g(t)}{h(t)}$ for some polynomials g, h.

Theorem For an infinite dimensional algebra A with rational Poincaré series $P_A(t)$, the radius of convergence is ≤ 1, and

(i) $r < 1$ gives exponential growth,

(ii) $r = 1$ gives $\mathrm{GKdim}(A) = d \in \mathbb{N}$, where d is the order of the pole of $P_A(t)$ at 1.

Ufnarovskii's graph theorem

For a finite directed graph G, let $f(n)$ the number of paths of length n.

Theorem The growth of a finite directed graph G is

(i) exponential if and only if G contains overlapping cycles;

(ii) polynomial of degree d if and only if G does not contain overlapping cycles and d is the maximal possible number of distinct cycles a path on G may contain.

Proof If A is the adjacency matrix then the number of paths of length n is the sum of the entries in A^n.

Cayley-Hamilton implies linear recurrence and so the Poincaré series is rational.

Monomial algebras

Ufnarovskii Let $A = k\langle X\rangle / < R >$ be a finitely presented monomial algebra. Then either A has exponential growth, or $\mathrm{GKdim}(A) \in \mathbb{N}$.

Proof Let $r + 1$ be the maximal length of a word in R.

Let G be the graph whose vertices are nonzero words of length r and draw an arrow from v to w if there are $x, y \in X$ with $vx = yw \neq 0$.

The growth functions of A and G are essentially the same. The result then follows from the previous two slides.

Associated monomial algebras

• Given a finitely presented algebra A one can associate to it a finitely generated monomial algebra $\widetilde{A}$ with the same growth.

• Unfortunately, $\widetilde{A}$ often turns out not to be finitely presented.

• In the commutative case it is finitely presented. This is similar to the reasons why Gröbner bases methods work much better for commutative algebras than for noncommutative algebras.

Bergman's Gap Theorem

Let $A = k[V]$ be an algebra with growth less than quadratic.

Without loss of generality, by passing to the associated monomial algebra, we may assume that A is a finitely generated monomial algebra.

For some d there must be less than d words of length d in A.

Using only the relations up to length $d+1$, we may assume that A is finitely presented monomial algebra.

Thus, the growth of A is a natural number less than 2.

(**Ellingsen & Farkas, Kobayashi & Kobayashi**)

Groups of linear growth

Ideas very similar to the above can be used to show that a group G with linear growth must be virtually infinite cyclic.

(**Justin, Wilkie & Van den Dries, Imrich & Seifter**)

• The Heisenberg group $G = \langle x, y \rangle$ with $z := [x, y]$ central has growth of degree 4 and is the smallest example of a nilpotent group with an infinite cycle in the second term of the lower central series.

As a consequence, Gromov's theorem shows that any group G with growth less than 4 will be virtually abelian. As far as I know, there is no elementary proof of this fact.

Challenge to ring theorists Use the group algebra and ring theory to show, for example, that a group with quadratic growth must be a finite extension of $\mathbb{Z}^2$. (Or, at least that the group algebra is PI.)

Nil algebras

- An element r in an algebra R is nil if $r^n = 0$ for some n.

- A ring R (without 1) is nil if each element is nil.

Exercise (Hard?) Let R be a nil ring. Show that $M_2(R)$ is a nil ring.

- A ring R (without 1) is nilpotent if $R^n = 0$ for some n; that is, any product of n elements from R is zero.

- When is a finitely generated nil ring (or ideal) nilpotent?

Nil rings/ideals

- Many positive results.

Amitsur The Jacobson radical of any finitely generated algebra over an uncountable field is nil.

Levitski Every nil right ideal in a right noetherian ring is nilpotent.

- As a consequence, in this setting there is a largest nilpotent ideal N and a general strategy for noetherian rings appears: study R by studying R/N and the the chain of R modules

$$R \supseteq N \supseteq N^2 \supseteq \cdots \supseteq N^{s-1} \supseteq N^s = 0$$

using the fact that each factor is a finitely generated R/N-module.

Golod-Shafarevich

Golod-Shafarevich For every field k there is a finitely generated nil algebra R that is not nilpotent.

• Result used to construct counter-example to the General Burnside Problem

• The Golod-Shafarevich examples have exponential growth

Problem (Ufnarovskii, Small, others) Is there an example of a finitely generated nil but not nilpotent algebra with polynomially bounded growth?

Background to problem

Problem (Ufnarovskii, Small, others) Is there an example of a finitely generated nil but not nilpotent algebra with polynomially bounded growth?

• Early evidence suggested no is the answer.

• For example, Gromov's group theory result on growth implies that a finitely generated torsion group with polynomially bounded growth is finite.

Köthe Conjecture

Köthe Conjecture (1930) The ideal generated by the nil right ideals of R is nil.

• Holds in many settings, most notably for any algebra over an uncountable field (Amitsur)

• If you were to solve the earlier exercise (on $M_2(R)$), then you would be famous, as this is equivalent to the Köthe Conjecture.

• The sticking point was that no-one had any idea how to deal with the countable case.

Smoktunowicz's work

Smoktunowicz Over any countable field, there is a simple nil ring.

Lenagan & Smoktunowicz Over any countable field, there is a finitely generated nil algebra with polynomially bounded growth that is not nilpotent.

Lenagan, Smoktunowicz & Young Over any countable field, there is a finitely generated nil algebra with $\mathrm{GKdim} \leq 3$ that is not nilpotent.

Uncountable fields

• Recently **Bell & Young** have produced the first examples of finitely generated infinite dimensional nil algebras with subexponential growth over uncountable fields.

• In fact, for any monotonically increasing function β with intermediate growth, they produce an algebra with rate of growth bounded above by β.

• The exact rate of growth for these algebras is not known, but suspected to be intermediate growth rather than polynomially bounded growth.

• If this suspicion were correct, then the knowledge of intermediate growth for algebras would be ahead of the corresponding knowledge for groups, as, as far as I know, there are no known groups with intermediate growth less than $\exp(\sqrt{n})$.

Questions

- Is there a simple nil ring over an uncountable field?

- Is there a finitely generated nil but not nilpotent algebra with GKdim at most two?

- Is a finitely generated nil algebra with quadratically bounded growth finite dimensional?

- Is a finitely presented nil algebra nilpotent?

Quadratic growth

As algebras with linear growth are reasonably well understood by the work of Small, Stafford and Warfield; the next case to understand is quadratic growth.

An algebra $A = k[V]$ has quadratically bounded growth if there is a constant C such that $\dim(V^n) \leq Cn^2$ for all $n \geq 1$.

An algebra $A = k[V]$ has quadratic growth if it has quadratically bounded growth but not linearly bounded growth.

As nil algebras with quadratic growth are beyond us at present, we should restrict attention to settings such as prime or domain.

Quadratic growth: Graded algebras

The graded case is much simpler than the ungraded case.

Artin and Stafford classified graded domains with $\mathrm{GKdim} = 2$.

• One should think of these algebras as the homogeneous coordinate rings of (noncommutative) projective curves.

• Each such algebra is closely related to an elliptic curve and an automorphism of the elliptic curve, which twists the multiplication in the algebra.

• The order of the automorphism influences the behaviour of the algebra. If the automorphism has finite order then the algebra satisfies a polynomial identity, whereas if the automorphism has infinite order then the algebra is primitive.

Quadratic growth: Smoktunowicz Gap Theorem

• Artin and Stafford conjectured that there should be no graded domains with GKdim strictly between 2 and 3.

• They were able to show that there is no such algebra with $\mathrm{GKdim} \in (2, 11/5)$.

Smoktunowicz There are no graded domains with Gelfand-Kirillov dimension strictly between 2 and 3.

Quadratic growth: Small's question

Two extreme examples of quadratic growth

• $k[x, y]$, commutative polynomials in two variables.

• The Weyl Algebra $k[x, y]$ with $xy - yx = 1$ which is a simple algebra in characteristic zero, but a PI algebra in characteristic p (in fact, finite as a module over its centre).

Small's Question Suppose that $A = k[V]$ is noetherian prime with quadratic growth. Is A either primitive, or PI?

Artin & Stafford True for graded algebras

Bell Not true (for $\mathrm{GKdim} = 2$) if noetherian is dropped, true over uncountable fields.

Quadratic growth: Jason Bell's work

Bell Suppose that $A = k[V]$ is a simple Goldie algebra (ie, has a simple artinian ring of quotients) with quadratic growth. Then A is noetherian and has Krull dimension 1.

Corollary If A is a finitely generated simple domain with quadratic growth then A is noetherian with Krull dimension one and a result of Stafford then shows that every one-sided ideal can be generated by two elements.

Example The Weyl algebra in characteristic zero: $k[x, y]$ with $xy - yx = 1$.

Question Is a simple algebra with quadratic growth Goldie (and hence noetherian by Bell's results)?

Questions

• Prove something useful about finitely presented algebras with quadratic growth.

• Is there an example of a finitely presented algebra with $\mathrm{GKdim} = 2$ which does not have quadratic growth?

• Is there a finitely presented infinite dimensional nil algebra?

• Does a finitely presented algebra with exponential growth contain a free subalgebra? (Smoktunowicz recent arxiv paper: A note on Golod-Shafarevich algebras.)

• Is there a simple domain with GK between 2 and 3?