NOTE ON A LOWER BOUND OF THE WEYL SUM IN BOURGAIN'S NLS PAPER (GAFA '93)

TADAHIRO OH

1. Introduction

In this note, we go over Bourgain's counterexample [2] to the periodic L^6-Strichartz estimate for the Schrödinger equation on $\mathbb{T}$. In [2], Bourgain proved the periodic L^6-Strichartz estimate with a slight loos of derivative:

$$\Big\| \sum_{|n|\leq N} a_n e^{2\pi i (nx + n^2 t)} \Big\|_{L^6(\mathbb{T}^2)} \leq C_N \big\| \{a_n\} \big\|_{\ell^2_{|n|\leq N}}, \tag{1.1}$$

where the constant C_N is bounded above by

$$C_N \lesssim e^{c \frac{\log N}{\log \log N}} \ll N^\varepsilon, \qquad \text{for any } \varepsilon > 0. \tag{1.2}$$

The proof is based on a simple divisor counting argument, and the loss basically comes from the number of divisors of an integer N.

In the same paper, he also showed that some loss of derivative in (1.1) was indeed necessary. More precisely, it was shown that

$$C_N \gtrsim (\log N)^{\frac{1}{6}} \tag{1.3}$$

for the initial condition $a_n = \chi_{[0,N]}(n)$. The main part of the argument is based on the following (lower) bound on the Weyl sum:

$$\Big| \sum_{n=0}^{N} e^{2\pi i (nx + n^2 t)} \Big| \sim \frac{N}{\sqrt{q}} \tag{1.4}$$

for fixed x and t in the major arc $\mathcal{M}_0(q,a,b)$.[1] See Proposition 3.1 below. Also, see Theorem 2.3. Here, the major arc $\mathcal{M}_0(q,a,b)$ is defined for q, a, and b, satisfying

$$1 \leq a < q \leq N^{\frac{1}{2}}, \quad (a,q) = 1, \quad 0 \leq b < q, \tag{1.5}$$

and is given by

$$\mathcal{M}_0(q,a,b) = \bigg\{ (x,t) \in [0,1]^2 : \Big| x - \frac{b}{q} \Big| \leq \frac{1}{100N}, \quad \Big| t - \frac{a}{q} \Big| \leq \frac{1}{100N^2} \bigg\}. \tag{1.6}$$

[1]In the application of the Hardy-Littlewood circle method, one often divides the sum into dyadic blocks and define major and minor arcs for each dyadic block. Here, we do not need such a dyadic decomposition.

2. Dirichlet's theorem, Gauss sum, and Weyl sum

Recall the following theorem.

Theorem 2.1 (Dirichlet)**.** *Let $\theta \in [0,1]$ and $N \geq 1$. Then, there exist integers a and q satisfying $1 \leq a \leq q \leq N$ and $(a,q) = 1$ such that*

$$\left\| \theta - \frac{a}{q} \right\| \leq \frac{1}{qN}, \tag{2.1}$$

where $\| \cdot \|$ denotes the distance to the close integer.

Proof. Consider the $N+1$ numbers $j\theta \pmod 1$ for $j = 0, 1, \dots, N$. By the pigeon hole principle, there eixts two distinct integers $m, n \in \{0, 1, \dots, N\}$ with $m > n$ such that

$$|m\theta - n\theta - a'| \leq \tfrac{1}{N} \tag{2.2}$$

for some non-negative integer a'. Let $q' = m - n \geq 1$. If $1 \leq a' \leq q'$ and $(a', q') = 1$, then (2.1) holds with $a = a'$ and $q = q'$ after dividing (2.2) by q'. It remains to consider the following three cases.

(a) If $a' = 0$, then it follows from (2.2) that $|\theta| \leq \frac{1}{q'N} \leq \frac{1}{N}$. Hence, (2.1) holds with $a = q = 1$.
(b) If $a' > q'$, then from (2.2), we have $\frac{1}{N} \geq a' - q'\theta \geq q'(1-\theta)$. Once again, (2.1) holds with $a = q = 1$.
(c) If $(a', q') \neq 1$ (but $1 \leq a' \leq q' \leq N$), then we can write $a' = ka$ and $q' = kq$ for some $k \geq 2$ such that $(a,q) = 1$. Then, from (2.2), we obtain $\frac{1}{qN} \geq \frac{1}{q'N} \geq |\theta - \frac{a'}{q'}| = \theta - |\frac{a}{q}|$. Hence, (2.1) holds in this case as well. □

Next, we recall the estimate of the Gauss sum. Given positive integers a and q with $(a,q) = 1$, the Gauss sum $S(a,q)$ is defined by

$$S(a,q) := \sum_{n=1}^{q} e^{2\pi i \frac{a}{q} n^2}. \tag{2.3}$$

More generally, for $a, q \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $(a,q) = 1$, we can define the Gauss sum $S(a,b,q)$ by

$$S(a,b,q) := \sum_{n=1}^{q} e^{2\pi i (\frac{a}{q} n^2 + \frac{b}{q} n)}. \tag{2.4}$$

Namely, we have $S(a,q) = S(a,0,q)$.

Theorem 2.2 (Gauss sum)**.** *Let $a, q \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $(a,q) = 1$. Then, the following holds for the Gauss sums:*

(a) *When b is even,*

$$|S(a,b,q)| = \begin{cases} \sqrt{q}, & \text{if } q \text{ is odd}, \\ 0, & \text{if } q \equiv 2 \pmod 4, \\ \sqrt{2q}, & \text{if } q \equiv 0 \pmod 4. \end{cases} \tag{2.5}$$

(a) *When b is odd,*

$$|S(a,b,q)| = \begin{cases} \sqrt{q}, & \text{if } q \text{ is odd,} \\ \sqrt{2q}, & \text{if } q \equiv 2 \pmod 4, \\ 0, & \text{if } q \equiv 0 \pmod 4. \end{cases} \tag{2.6}$$

Proof. First, note that the Gauss sum (2.4) is invariant if we shift the range of summation. Thus, we have

$$\begin{aligned} |S(a,b,q)|^2 = S(a,b,q)\overline{S(a,b,q)} &= \sum_{n=1}^{q}\sum_{m=1}^{q} e^{2\pi i\{\frac{a}{q}(m^2-n^2)+\frac{b}{q}(m-n)\}} \\ &= \sum_{n=1}^{q}\sum_{\ell=1}^{q} e^{2\pi i\{\frac{a}{q}((n+\ell)^2-n^2)+\frac{b}{q}((n+\ell)-n)\}} \\ &= \sum_{\ell=1}^{q}\bigg(\sum_{n=1}^{q} e^{2\pi i(2\ell\frac{a}{q})n}\bigg) e^{2\pi i(\frac{a}{q}\ell^2+\frac{b}{q}\ell)}. \end{aligned}$$

Here, the inner sum is 0 unless

$$2\ell a \equiv 0 \pmod q. \tag{2.7}$$

If (2.7) holds, the inner sum is equal to q.

• **Case 1:** Suppose that q is odd. Since $(a,q) = 1 = (2,q)$, it follows from (2.7) that $\ell = q$. Thus, we have

$$|S(a,b,q)|^2 = q.$$

• **Case 2:** Suppose that $q \equiv 2 \pmod 4$. Since $(a,q) = 1$, we have $2\ell \equiv 0 \pmod q$. Namely, $\ell = \frac{q}{2}$ or q. Thus, we have

$$|S(a,b,q)|^2 = q(e^{\pi i(\frac{qa}{2}+b} + e^{2\pi i(qa+b)}) = \begin{cases} 0, & \text{if } b \text{ is even,} \\ 2q, & \text{if } b \text{ is odd.} \end{cases}.$$

• **Case 3:** Lastly, suppose that $q \equiv 0 \pmod 4$. In this case, $\frac{qa}{2}$ is an even number. Thus, we have

$$|S(a,b,q)|^2 = q(e^{\pi i(\frac{qa}{2}+b} + e^{2\pi i(qa+b)}) = \begin{cases} 2q, & \text{if } b \text{ is even,} \\ 0, & \text{if } b \text{ is odd.} \end{cases}.$$

This proves (2.5) and (2.6). □

Lastly, we state the classical estimate on the Weyl sum.

Theorem 2.3 (Weyl sum)**.** *Let $x, t \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ such that $(a,q) = 1$. Moreover, assume that*

$$\left|t - \frac{a}{q}\right| \le \frac{1}{q^2}. \tag{2.8}$$

Then, the following bound holds:

$$\left|\sum_{n=0}^{N} e^{2\pi i(nx+n^2t)}\right| \lesssim \left(\frac{N}{q^{\frac{1}{2}}} + q^{\frac{1}{2}}\right)(\log q)^{\frac{1}{2}}. \tag{2.9}$$

Remark 2.4. (i) In general, let $p(n)$ be a polynomial of degree k such that the leading coefficient t satisfies (2.8). Then, we have

$$\bigg|\sum_{n=0}^{N} e^{2\pi i p(n)}\bigg| \lesssim C_{\varepsilon,k} N^{1+\varepsilon}\bigg(\frac{1}{N}+\frac{1}{q}+\frac{q}{N^k}\bigg)^{\frac{1}{2^k-1}}.$$

See Theorems 1 and 2 on p. 41 in [5].

(ii) Let $N \in \mathbb{N}$. Then, given $t \in \mathbb{R}$, it follows from Theorem 2.1 that there exists $a, q \in \mathbb{Z}$ such that $1 \le a \le q \le N$ and $(a,q) = 1$, satisfying (2.8).

3. Lower bound (1.4)

In this section, we prove the lower bound (1.4) under certain conditions on q and b. The basic idea is to use the bound on the Gauss sum (Theorem 2.2) after replacing a certain summation by integration (see (3.3)).

Proposition 3.1. *Let q, a, and b be as in* (1.5). *Then, for $(x,t) \in \mathcal{M}_0(q,a,b)$, we have the following lower bound on the Weyl sum:*

$$\bigg|\sum_{n=0}^{N} e^{2\pi i(nx+n^2t)}\bigg| \gtrsim \frac{N}{\sqrt{q}}, \tag{3.1}$$

provided that one of the following conditions holds:

(a) *q is odd,*
(b) *$q \equiv 0 \pmod 4$ and b is even, or*
(c) *$q \equiv 2 \pmod 4$ and b is odd.*

Remark 3.2. The following proof does not tell us what happens (i) $q \equiv 2 \pmod 4$ and b is even, or (ii) $q \equiv 0 \pmod 4$ and b is odd.

Proof. Let $\alpha = t - \frac{a}{q}$ and $\beta = x - \frac{b}{q}$. By writing $n = mq + \ell$ with $1 \le \ell \le q$, we have

$$\begin{aligned}\sum_{n=0}^{N} e^{2\pi i(nx+n^2t)} &= \sum_{n=1}^{[\frac{N}{q}]q} e^{2\pi i(nx+n^2t)} + O(q)\\ &= \sum_{\ell=1}^{q}\sum_{m=1}^{[\frac{N}{q}]} e^{2\pi i\{(mq+\ell)(\frac{b}{q}+\beta)+(mq+\ell)^2(\frac{a}{q}+\alpha)\}} + O(q)\\ &= \sum_{\ell=1}^{q} e^{2\pi i(\frac{a}{q}\ell^2+\frac{b}{q}\ell)}\sum_{m=1}^{[\frac{N}{q}]} e^{2\pi i\{(mq+\ell)^2\alpha+(mq+\ell)\beta\}} + O(q),\end{aligned} \tag{3.2}$$

since $mq+\ell \equiv \ell \pmod q$ and $(mq+\ell)^2 \equiv \ell^2 \pmod q$. Note that the error $O(q)$ in (3.2) is acceptable since $O(q) \lesssim N^{\frac{1}{2}} \ll N^{\frac{3}{4}} < \frac{N}{\sqrt{q}}$ under the assumption $q < N^{\frac{1}{2}}$.

The first sum (in ℓ) on the right-hand side of in (3.2) is basically the Gauss sum. However, we can not use Theorem 2.2 since the inner sum also depends on ℓ. Thus, we first need to replace the inner sum by an integral and get rid of the ℓ-dependence. (i.e. van der Corput approximation type argument.) Fix $m \in \mathbb{Z} \cap \big[0, [\frac{N}{q}]\big]$. Then, for $y \in [m, m+1]$, we have

$$\begin{aligned} \big| \{(mq+\ell)^2\alpha + (mq+\ell)\beta\} - \{(yq+\ell)^2\alpha + (yq+\ell)\beta\} \big| \\ = \big| ((m+y)q + 2\ell)(m-y)q\alpha + (m-y)q\beta \big| \leq \frac{1}{20N^{\frac{1}{2}}}. \end{aligned}$$

Hence, by Mean Value Theorem, we have

$$\begin{aligned} \sum_{m=1}^{[\frac{N}{q}]} e^{2\pi i \{(mq+\ell)^2\alpha + (mq+\ell)\beta\}} &= \int_0^{[\frac{N}{q}]+1} e^{2\pi i\{(yq+\ell)^2\alpha + (yq+\ell)\beta\}} dy + O\Big(\frac{N^{\frac{1}{2}}}{q}\Big) \\ &= \int_0^{\frac{N}{q}} e^{2\pi i\{(yq+\ell)^2\alpha + (yq+\ell)\beta\}} dy + O\Big(\frac{N^{\frac{1}{2}}}{q}\Big). \end{aligned} \tag{3.3}$$

The error $O(\frac{N^{\frac{1}{2}}}{q})$ becomes $O(N^{\frac{1}{2}})$ under the ℓ-summation in (3.2). Note that this is an acceptable error as before. By change of variables $z = yq + \ell$ (for fixed ℓ), the integral on the right-hand side of (3.3) becomes

$$\begin{aligned} \int_0^{\frac{N}{q}} e^{2\pi i\{(yq+\ell)^2\alpha + (yq+\ell)\beta\}} dy &= \frac{1}{q}\int_\ell^{N+\ell} e^{2\pi i(z^2\alpha + z\beta)} dz = \frac{1}{q}\int_0^{N} e^{2\pi i(z^2\alpha + z\beta)} dz + \underbrace{O\Big(\frac{\ell}{q}\Big)}_{=O(1)} \\ &= \frac{N}{q} + \frac{1}{q}\int_0^N (e^{2\pi i(z^2\alpha + z\beta)} - 1) dz + O(1) \\ &= \frac{N}{q} + O\Big(\frac{2\pi N}{50q}\Big) + O(1), \end{aligned} \tag{3.4}$$

where we used Mean Value Theorem in the last inequality. The error $O(1)$ in (3.4) becomes $O(q)$ under the ℓ-summation in (3.2), which is again acceptable.

Finally, the estimate (3.1) follows from Theorem 2.2 with (3.2), (3.3), and (3.4), provided one of the following conditions holds: (a) q is odd, (b) $q \equiv 0 \pmod 4$ and b is even, or (c) $q \equiv 2 \pmod 4$ and b is odd. □

4. Proof of (1.3)

In this section, we complete the construction of the counterexample to the periodic L^6-Strichartz estimate. Define f_N by

$$f_N(x,t) = \sum_{n=0}^{N} e^{2\pi i(nx + n^2 t)}.$$

Then, $\|f_N(\cdot, 0)\|_{L^2_x(\mathbb{T})} = N^{\frac{1}{2}}$.

Fix q, a, and b, satisfying (1.5). Then, from Proposition 3.1, we have

$$\int_{\mathcal{M}_0(q,a,b)} |f_N(x,t)|^6 dxdt \gtrsim \frac{N^3}{q^3}, \tag{4.1}$$

provided that q and b satisfies the conditions in Proposition 3.1.

Lemma 4.1 (Disjointness of the major arcs)**.** *Let $N \gg 1$. The major arcs defined in* (1.6) *are disjoint. More precisely, let q, a, b and q', a', b' satisfy* (1.5)*, respectively. Suppose that $\mathcal{M}_0(q,a,b) \cap \mathcal{M}_0(q',a',b') \neq \emptyset$. Then, $\mathcal{M}_0(q,a,b) = \mathcal{M}_0(q',a',b')$, i.e. $q = q'$, $a = a'$, and $b = b'$.*

Proof. Suppose that (x,t) belongs to two major arcs, i.e. $(x,t) \in \mathcal{M}_0(q,a,b) \cap \mathcal{M}_0(q',a',b')$, where q,a,b and q',a',b' satisfy (1.5), respectively.

If $\frac{a}{q} \neq \frac{a'}{q'}$, then we have

$$\frac{1}{50N^2} > \left|t - \frac{a}{q}\right| + \left|t - \frac{a'}{q'}\right| \geq \frac{|aq' - a'q|}{qq'} \geq \frac{1}{qq'} > \frac{1}{N}.$$

This is clearly a contradicition. Now, suppose that i.e.$q = q'$, $a = a'$, but $b \neq b'$.

$$\frac{1}{50N} > \left|x - \frac{b}{q}\right| + \left|x - \frac{b'}{q}\right| \geq \frac{|b - b'|}{q} \geq \frac{1}{q} > \frac{1}{N^{\frac{1}{2}}}.$$

This is again a contradiction. □

By Lemma 4.1 and (4.1), we have

$$\int_{\mathbb{T}^2} |f_N(x,t)|^6 dxdt \geq \sum_{q=1}^{N^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{b=1}^{q-1} \int_{\mathcal{M}_0(q,a,b)} |f_N(x,t)|^6 dxdt$$

$$\gtrsim N^3 \sum_{q=1}^{N^{\frac{1}{2}}} \frac{\varphi(q)}{q^2}.$$

where the summation in b is over (a) $b = 0, \dots q-1$, if q is odd, (b) even b, if $q \equiv 0$ (mod 4), and (c) odd b, if $q \equiv 2$ (mod 4). Here, $\varphi(q)$ is Euler's totient function, representing the number of positive integers $\leq q$ that are relatively prime to q. Finally, (1.3) follows once we prove the following lemma.[2]

Lemma 4.2. *Let $n \in \mathbb{N}$. Then, we have*

$$\sum_{q=1}^{N} \frac{\varphi(q)}{q^2} \gtrsim \log N. \tag{4.2}$$

Proof. Let $j \geq 0$. Then, we have

$$\sum_{2^j \leq q < 2^{j+1}} \frac{\varphi(q)}{q^2} \geq \frac{1}{2^{2j}} \sum_{2^j \leq q < 2^{j+1}} \varphi(q) \sim 1. \tag{4.3}$$

[2]In the previous version, in summing over only odd q, we simply used Theorem 328 in Hardy-Wright [3]:

$$\liminf_{n \to \infty} \frac{\varphi(n) \log\log n}{n} = e^{-\gamma},$$

where γ is Euler's constant given by $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.5772\dots$. Or rather, the following lower bound on φ:

$$\varphi(n) \geq \frac{n}{e^{\gamma} \log\log n + \frac{3}{\log\log n}}.$$

This was not efficient and introduced a $\log\log N$ loss.

Here, we used the fact that $\sum_{q=1}^{n} \varphi(q) = \frac{3n^2}{\pi^2} + O(n \log n)$. See Theorem 3.7 in [1]. Summing (4.3) over $j = 0, 1, \dots, \log N$ yields (4.2). □

Remark 4.3. (i) The same proof basically works to show that the L^4-Strichartz estimate on $\mathbb{T}^4$ fails with

$$C_N \gtrsim (\log N)^{\frac{1}{4}}. \tag{4.4}$$

Note that Takaoka-Tzvetkov [6] summed only over q prime, thus yielding only $C_N \gtrsim (\log \log N)^{\frac{1}{4}}$.

(ii) Recently, Kishimoto [4] gave a different proof of (1.3) for the periodic L^6-Strichartz estimate when $d = 1$ and (4.4) for the periodic L^4-Strichartz estimate when $d = 2$. When $d = 2$, he also showed that the periodic L^4-Strichartz estimate fails on almost all irrational tori. See [4].

(iii) In fact, one can derive a more precise asymptotic formula for $N \geq 2$:

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^2} = \frac{1}{\zeta(2)} \log N + \frac{\gamma}{\zeta(2)} - A + O\Big(\frac{\log N}{N}\Big), \tag{4.5}$$

where γ denotes Euler's constant and $A = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2}$. Here, $\zeta(\cdot)$ is the Riemann zeta function, while $\mu(\cdot)$ denotes the Möbius function. See Exercise 6 on p. 71 in [1].

The proof of (4.5) is based on

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d} \qquad \text{and} \qquad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s > 1.$$

See Theorem 2.3 in [1] and Theorem 287 in [3].

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Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road Edinburgh, EH9 3FD, United Kingdom

E-mail address: hiro.oh@ed.ac.uk