STRAUSS EXPONENT AND SMALL DATA GLOBAL EXISTENCE: STRICHARTZ ESTIMATE FOR NON-ADMISSBLE PAIR

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1. Introduction

Consider the nonlinear Schrödinger equation (NLS) with a power-type nonlinearity:

$$\begin{cases} iu_t + \Delta u = \mathcal{N}_p(u), \\ u\big|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \tag{1.1}$$

where the nonlinearity $\mathcal{N}_p(u) = \mathcal{N}_p(u, \overline{u})$ is homogeneous of order p.

Local well-posedness of (1.1) easily follows from the Strichartz estimates, which we state here for convenience. A pair (q, r) of exponents are called *Schrödinger admissible* if $2 \leq q, r, \leq \infty$,

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \tag{1.2}$$

and $(q, r, d) \neq (2, \infty, 2)$. Then, we have the following *Strichartz estimates*:

Lemma 1.1 (Strichartz estimate)**.** *Let (q, r) and $(\widetilde{q}, \widetilde{r})$ be admissible pairs. Then, we have* (a) *homogeneous Strichartz estimate:*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} \tag{1.3}$$

(b) *dual homogeneous Strichartz estimate:*

$$\Big\| \int_{\mathbb{R}} S(-t')F(t')dt' \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)} \tag{1.4}$$

(c) *nonhomogeneous (retarded) Strichartz estimate:*

$$\Big\| \int_{t_0}^t S(t-t')F(t')dt' \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{1.5}$$

The contraction argument with the Strichartz estimates yields small data global existence [1] in the scaling critical regularity $s_c = \frac{d}{2} - \frac{2}{p-1}$; if $\|u_0\|_{H^{s_c}}$ is sufficiently small, then the corresponding solution u exists globally in time.

In establishing global well-posedness, there are basically two kinds of results available; (i) global existence by conservation laws and (ii) small data global existence. When $\mathcal{N}_p(u) = \pm|u|^{p-1}u$, (1.1) enjoys the conservation of the L^2-norm and the Hamiltonian. In particular, in the defocusing case, i.e. $\mathcal{N}_p(u) = |u|^{p-1}u$, we obtain a uniform control of the H^1-norm of a solution. This allows us to iterate the local argument and construct global-in-tim solutions in the energy-subcritical regime: $1 < p < 1 + \frac{4}{d-2}$. In the focusing case, global existence holds for $p < 1 + \frac{4}{d}$. In general, some finite-time blowup solutions are known, but global

1. For some global existence results mentioned in this note, scattering results are also known. However, we will not dicuss them. Note also that below the short range exponent, i.e. $p \leq 1 + \frac{2}{d}$, it is known that scattering does not always hold.

existence still holds provided that the H^1-norm (or L^2-norm if $p = 1 + \frac{4}{d}$) are sufficiently small. In the energy-critical case, the situation is more subtle. Nonetheless, global existence holds in the defocusing case (and in the focusing case under some conditions.) Finally, no global result is known in the energy-supercritical case. The uniform bound on the H^1-norm of a solution does not help us iterate the local argument in H^s, $s > 1$.

In the following, we discuss small data global existence above the Strauss exponent p_S given by

$$p_S = \frac{d + 2 + \sqrt{d^2 + 12d + 4}}{2d}. \tag{1.6}$$

Here, p_S appears as the unique positive root of

$$dp^2 - (d + 2)p - 2 = 0. \tag{1.7}$$

Note that p_S lies between the short range exponent $p_s = 1 + \frac{2}{d}$ and the scaling L^2-critical exponent $p_{L^2} = 1 + \frac{4}{d}$.

In this note, we discuss the following small data global existence result above the Strauss exponent:

Theorem 1.2. *Let $p_S < p < 1 + \frac{4}{d-2}$ and $a > 0$ be as in* (2.9) *below. Then, there exists $\varepsilon > 0$ such that if u_0 satisfies $\|e^{it\Delta}u_0\|_{L^a_t([0,\infty);L^{p+1}_x)} \le \varepsilon$, then there exists a global solution u to* (1.1) *with initial condition u_0. Moreover, $u \in L^a_t([0,\infty); L^{p+1}_x(\mathbb{R}^d))$.*

Remark 1.3. If $u_0 \in H^1$, then it follows from Sobolev embedding $\|u_0\|_{L^{p+1}} \lesssim \|u_0\|_{H^1}$ that there exists $\varepsilon' > 0$ such that $\|u_0\|_{H^1} \le \varepsilon'$ implies global existence. Moreover, $u \in L^{\widetilde{q}}_t([0,\infty); W^{1,\widetilde{r}}_x(\mathbb{R}^d))$ for any Schrödinger admissible pair $(\widetilde{q}, \widetilde{r})$.

If $u_0 \in L^2 \cap L^{\frac{p+1}{p}}$ with $p_S < p \le 1 + \frac{4}{d}$, then the conclusion of Theorem 1.2 also holds, i.e. that there exists $\varepsilon' > 0$ such that $\|u_0\|_{L^2 \cap L^{\frac{p+1}{p}}} \le \varepsilon'$ implies global existence.

There are time-decay and scattering results as well, but we will not discuss them here.

Theorem 1.2 holds regardless of the precise form of the nonlinearity $\mathcal{N}_p(u)$. Below the Strauss exponent, however, the situation is quite different. On the one hand, Germain-Masmoudi-Shatah [2] proved small data global existence (and scattering) for 2-d quadratic NLS, i.e. $p = 1 + \frac{2}{d}$ if the nonlinearity is given by $\mathcal{N}_2(u) = \alpha u^2 + \beta \overline{u}^2$. On the other hand, Ikeda-Wakasugi [3] showed that for the nonlinearity $\mathcal{N}_p(u) = \lambda |u|^p$ with $1 < p \le 1 + \frac{2}{d}$, there is a finite-time blowup and hence there is no global solution, provided

$$\operatorname{Re}\lambda \cdot \operatorname{Im} \int_{\mathbb{R}^d} u_0(x)\, dx < 0, \quad \text{or} \quad \operatorname{Im}\lambda \cdot \operatorname{Re} \int_{\mathbb{R}^d} u_0(x)\, dx > 0. \tag{1.8}$$

In particular, there is no global well-posedness even for small data. Note that this result applies to 2-d quadratic NLS with $\mathcal{N}_2(u) = \lambda |u|^2$.

Lastly, we point out that the proof of Theorem 1.2 is based on the dispersive estimate (see (2.1) below). Such a dispersive estimate does not hold in the periodic setting. Indeed, the recent result by Oh [4] shows a finite-time blowup for NLS on $\mathbb{T}^d$ with the nonlinearity $\mathcal{N}_p(u) = \lambda |u|^p$, $p > 1$, as long as the condition (1.8) is satisfied. Hence, a general small data global existence result as in Theorem 1.2 does *not* hold on $\mathbb{T}^d$.

2. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on the argument in Section 6.3 of Cazenave [1]. Recall the following *dispersive estimate*:

$$\|S(t)u_0\|_{L^\infty_x} \lesssim |t|^{-\frac{d}{2}}\|u_0\|_{L^1_x}, \quad t \neq 0. \tag{2.1}$$

By interpolating (2.1) with $\|S(t)u_0\|_{L^2_x} = \|u_0\|_{L^2_x}$, we obtain the $L^{r'}$-L^r estimate:

$$\|S(t)u_0\|_{L^r_x} \lesssim |t|^{-(\frac{d}{2}-\frac{d}{r})}\|u_0\|_{L^{r'}_x}, \quad t \neq 0, \tag{2.2}$$

for $1 \leq r' \leq r \leq \infty$, where p' is the Hölder conjugate of r.

Write (1.1) in the Duhamel formulation:

$$u(t) = \Gamma(t)u := S(t)u_0 + \int_0^t S(t-t')\mathcal{N}_p(u)(t')dt', \tag{2.3}$$

where $S(t) = e^{it\Delta}$ denotes the linear semigroup. The goal is to construct a solution to (2.3) by the contraction mapping principle.

First, we consider the nonlinear part:

$$\Phi(t)(u) := \int_0^t S(t-t')\mathcal{N}_p(u)(t')dt'. \tag{2.4}$$

With $r = p+1$ in (1.2), the Schrödinger admissible exponent q is given by see that

$$q = \frac{2}{\beta}, \quad \text{where } \beta := \frac{d}{2} - \frac{d}{p+1} > 0. \tag{2.5}$$

By (2.2) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \|\Phi(t)(u)\|_{L^a_T L^{p+1}_x} &\lesssim \left\| \int_0^t |t-t'|^{-(\frac{d}{2}-\frac{d}{p+1})}\|\mathcal{N}_p(u)(t')\|_{L_x^{\frac{p+1}{p}}} dt' \right\|_{L^a_T} \\ &\lesssim \|\mathcal{N}_p(u)\|_{L^b_T L_x^{\frac{p+1}{p}}}, \end{aligned} \tag{2.6}$$

where $L^a_T := L^a([0,T])$ and

$$1 + \frac{1}{a} = \beta + \frac{1}{b}. \tag{2.7}$$

Now, we *impose* the following condition

$$pb = a. \tag{2.8}$$

Namely, we set

$$a = \frac{p-1}{1-\beta} = \frac{2(p-1)(p+1)}{2p+2-dp+d} \tag{2.9}$$

The denominator in (2.9) is positive if and only if $p < 1 + \frac{4}{d-2}$; the Cauchy problem (1.1) is energy-subcritical. With this choice of a and b, (2.6) yields

$$\|\Phi(t)(u)\|_{L^a_T L^{p+1}_x} \lesssim \|u\|_{L^a_T L^{p+1}_x}. \tag{2.10}$$

Note that in applying Hardy-Littlewood-Sobolev inequality in (2.6), we need $1 < a, b, \frac{1}{\beta} < \infty$. On the one hand, by the condition $\beta < 1$, we obtain $p < 1 + \frac{4}{d-2}$. On the other hand, by (2.8), we have $p < a < \infty$. With (2.9), this implies that

$$dp^2 - (d+2)p - 2 > 0 \quad \Longrightarrow \quad p > p_S.$$

Hence, we must have $p_S < p < 1 + \frac{4}{d-2}$.

Once we establish the nonlinear estimate (2.10), the standard argument shows that $\Gamma(t)$ is a contraction on the ball

$$B_\varepsilon = \{u \in L^a_T L^{p+1}_x : \|u\|_{L^a_T L^{p+1}_x} \leq C\varepsilon\}$$

for any $T > 0$.

In the remaining part of this note, we discuss the results in Remark 1.3. First, we estimate the linear part in the $L^a_T L^{p+1}_x$-norm. First, consider the case $t \geq 1$. Here, we *impose* the condition:

$$2a > q. \tag{2.11}$$

In particular, we have $\beta a > 1$ from (2.5) and (2.11). Then, by (2.2) with (2.5), we have

$$\Big\| \|S(t)u_0\|_{L^{p+1}_x} \Big\|_{L^a_t([1,T])} \lesssim \Big\| |t|^{-\beta} \|u_0\|_{L_x^{\frac{p+1}{p}}} \Big\|_{L^a_t([1,T])} \lesssim \|u_0\|_{L_x^{\frac{p+1}{p}}}, \tag{2.12}$$

where the implicit constant is independent of T. Next, we consider the case $t < 1$. In this case, we *impose* the condition

$$q \geq a. \tag{2.13}$$

Then, by the Strichartz estimate (1.3), we have

$$\|S(t)u_0\|_{L^a_t([0,1))L^{p+1}_x} \leq \|S(t)u_0\|_{L^q_t([0,1))L^{p+1}_x} \lesssim \|u_0\|_{L^2_x}. \tag{2.14}$$

From (2.12) and (2.14), we conclude that

$$\|S(t)u_0\|_{L^a_t([0,1))L^{p+1}_x} \leq \|S(t)u_0\|_{L^q_t([0,1))L^{p+1}_x} \lesssim \|u_0\|_{L^2_x \cap L_x^{\frac{p+1}{p}}}. \tag{2.15}$$

Let us now consider the conditions (2.11) and (2.13). From (2.5) and (2.9), we see that the condition (2.11) is equivalent to $p > p_S$. Similarly, it follows from (2.5) and (2.9), we see that the condition (2.13) is equivalent to $p \leq 1 + \frac{4}{d}$.

Next, we consider the case $u_0 \in H^1$. Then, for $t \leq 1$, by Sobolev embedding, we have

$$\|S(t)u_0\|_{L^a_t([0,1);L^{p+1}_x)} \lesssim \|S(t)u_0\|_{L^a_t([0,1);H^1_x)} \lesssim \|u_0\|_{H^1}. \tag{2.16}$$

Note that in applying Sobolev embedding, we used the fact that $p < 1 + \frac{4}{d-2}$. For $t \geq 1$, we can proceed as in (2.12) along with Sobolev inequality.

$$\|S(t)u_0\|_{L^a_t([1,\infty);L^{p+1}_x)} \lesssim \|u_0\|_{L_x^{\frac{p+1}{p}}} \lesssim \|u_0\|_{H^1}. \tag{2.17}$$

Hence, from (2.16) and (2.17), we conclude that $\|u_0\|_{H^1} \leq \varepsilon'$ implies Theorem 1.2. Let $r = p+1$ and q as in (2.5). Then, we have $pr' = p$ and $\frac{1}{q'} = \frac{1}{q} + \frac{p-1}{a}$, where a is as in (2.9). Hence, given any Schrödinger admissible pair $(\widetilde{q}, \widetilde{r})$, from (1.5), we have

$$\|\Phi(t)(u)\|_{L^{\widetilde{q}}_t L W^{1,\widetilde{r}}_x} \lesssim \|\mathcal{N}_p(u)\|_{L^{q'}_t W^{1,r'}_x} \leq \|u\|^{p-1}_{L^a_t L^r_x} \|u\|_{L^q_t W^{1,r}_x} \tag{2.18}$$

First, by (2.18) with $\|u\|_{L^a_t L^r_x} \lesssim \varepsilon$, we can control $\|u\|_{L^q_t W^{1,r}_x}$, i.e. $(\widetilde{q}, \widetilde{r}) = (q, r)$. This provides a control on $\|u\|_{L^{\widetilde{q}}_t W^{1,\widetilde{r}}_x}$ for any Schrödinger admissible pair $(\widetilde{q}, \widetilde{r})$.

Appendix A. Local well-posedness in high regularity setting: $s > \frac{d}{2}$

In this appendix, we prove local well-posedness of (1.1) in the high regularity setting $s > \frac{d}{2}$ via Sobolev embedding: $\|u\|_{L^\infty(M)} \lesssim \|u\|_{H^s(M)}$, where $M = \mathbb{R}^d$ or $\mathbb{T}^d$. In particular, we do not use Strichartz estimates. In the following, we consider $\mathcal{N}_p(u) = |u|^p$, $p \geq 2$.

First, we consider the case when p is an even integer. Let $C_T H^s = C([-T,T]; H^s(M))$. Then, from (2.3), we have

$$\|\Gamma u\|_{C_T H^s} \leq \|u_0\|_{H^s} + T\|u\|^p_{C_T H^s}, \tag{A.1}$$

$$\|\Gamma u - \Gamma v\|_{C_T H^s} \leq +T\big(\|u\|^{p-1}_{C_T H^s} + \|v\|^{p-1}_{C_T H^s}\big)\|u - v\|_{C_T H^s}. \tag{A.2}$$

Here, we used the algebra property of H^s, $s > \frac{d}{2}$ along with the fact that we have an algebraic nonlinearity $|u|^p = u^{\frac{p}{2}}\overline{u}^{\frac{p}{2}}$ when p is an even integer. Then, by setting $R = 2\|u_0\|_{H^s}$ and $T = T(R)$ sufficiently small, namely $2TR^{p-1} < 1$, we see that Γ is a contraction on the ball B_R of radius R in $C([-T,T]; H^s)$. Note that one also has *unconditional uniqueness* in this case.[2] Here, unconditional uniqueness means uniqueness in $C([-T,T]; H^s)$ without intersecting any auxiliary function spaces. For examples, if we use Strichartz estimates to prove local well-posedness, then we have *conditional* uniqueness only in $C([-T,T]; H^s) \cap L^q_T L^r_x$ in such a case. A separate argument is required to prove unconditional uniqueness. See Kato [5]

Next, we consider the case $p > 2$. In this case, the nonlinearity $|u|^p$ is no longer algebraic and a care must be taken. With $F(z) = |z|^p = (z\overline{z})^{\frac{p}{2}}$, we have

$$\partial_z F(z) = \tfrac{p}{2}|z|^{p-2}\overline{z} \quad \text{and} \quad \partial_{\overline{z}} F(z) = \tfrac{p}{2}|z|^{p-2}z. \tag{A.3}$$

Then, we have

$$\begin{aligned} |u(x)|^p - |u(y)|^p &= F(u(x)) - F(u(y)) \\ &= \int_0^1 \partial_z F\big(u(y) + \theta(u(x) - u(y))\big)(u(x) - u(y)) \\ &\qquad + \partial_{\overline{z}} F\big(u(y) + \theta(u(x) - u(y))\big)(\overline{u}(x) - \overline{u}(y))d\theta. \end{aligned} \tag{A.4}$$

In particular, from (A.3) and (A.4), we have

$$\big||u(x)|^p - |u(y)|^p\big| \lesssim (|u(x)|^{p-1} + |u(y)|^{p-1})\,|u(x) - u(y)|. \tag{A.5}$$

Then, from (A.5)

$$\begin{aligned} \big\||u|^p\big\|_{\dot{H}^s} &= \bigg(\int_M \int_M \frac{\big||u(x)|^p - |u(y)|^p\big|^2}{|x-y|^{d+2s}} dxdy\bigg)^{\frac{1}{2}} \\ &\lesssim \big\||u|^{p-1}\big\|_{L^\infty}\bigg(\int_M \int_M \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy\bigg)^{\frac{1}{2}} \\ &= \|u\|^{p-1}_{L^\infty}\|u\|_{\dot{H}^s} \lesssim \|u\|^p_{H^s}. \end{aligned}$$

We also have $\big\||u|^p\big\|_{L^2} = \|u\|^p_{L^{2p}} \lesssim \|u\|^p_{H^s}$ from interpolation of L^{2p} with L^2 and L^∞ and Sobolev embedding. Hence, we obtain

$$\big\||u|^p\big\|_{H^s} \lesssim \|u\|^p_{H^s}. \tag{A.6}$$

Then, (A.1) follows from (A.6) in this case.

2. Since we just constructed a contraction on B_R, one still needs to eliminate the possibility of a solution u with $\|u\|_{C_T H^s} > R$ and $u|_{t=0} = u_0$ to prove unconditional uniqueness.

Next, we estimate the difference $\Gamma u - \Gamma v$. By proceeding as before, we have

$$\begin{aligned} |u|^p - |v|^p &= F(u) - F(v) \\ &= \int_0^1 \partial_z F\big(v + \theta(u-v)\big)(u-v) + \partial_{\overline{z}} F\big(v + \theta(u(-v)\big)(\overline{u} - \overline{v}) d\theta. \end{aligned} \tag{A.7}$$

From (A.3), (A.6) (with $p-1$ instead of p), we have

$$\begin{aligned} \big\| \partial_z F\big(v + \theta(u-v)\big)\big\|_{H^s} &\lesssim \big\| |v + \theta(u-v)|^{p-1} \big\|_{H^s} \\ &\lesssim \|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1} \end{aligned} \tag{A.8}$$

A similar estimate holds for $\partial_{\overline{z}} F\big(v + \theta(u(-v)\big)$. Then, from (A.7) and (A.8), we have

$$\big\| |u|^p - |v|^p \big\|_{H^s} \lesssim \big(\|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1} \big) \|u - v\|_{H^s}. \tag{A.9}$$

Hence, (A.2) follows in this case as well. The rest of the argument is the same as before and thus we omit details. Clearly, this argument fails for $p < 2$ precisely at (A.8) (since we need $p-1 > 1$). Nonetheless, (A.1) still holds even when $p < 2$ and one can show existence of solutions by parabolic regularization and the energy estimate (A.1).

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