1. Local Well-Posedness in $H^1(\mathbb{R}^d)$

1.1. **NLS and scaling.** First, we consider the local well-posedness of the nonlinear Schrödinger equation (NLS) on $\mathbb{R}^d$:

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1}u \\ u\big|_{t=0} = u_0. \end{cases} \tag{1.1}$$

In the following, $S(t)$ denote the linear semigroup corresponding to (1.1). i.e. $S(t) = e^{it\Delta}$.

Recall that NLS (1.1) has the natural scaling: $u(t,x) \to \lambda^{-\frac{2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. This leaves the homogeneous Sobolev norm in $\dot{H}^{s_c}$ invariant, where the critical Sobolev index is given by $s_c = \frac{d}{2} - \frac{2}{p-1}$. In the following, we will mainly focus on the H^1-subcritical/critical case in $\mathbb{R}^3$, i.e. $d = 3$.

(a) H^1-subcritical: $p < \frac{d+2}{d-2}$ (only for $d \geq 3$.)
(b) L^2-subcritical: $p < 1 + \frac{4}{d}$.

1.2. **Strichartz estimate.** A pair (q,r) of exponents are called (Schrödinger) *admissible* if $2 \leq q, r, \leq \infty$,

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2},$$

and $(q, r, d) \neq (2, \infty, 2)$.

Lemma 1.1 (Strichartz estimate)**.** *Let (q,r) and $(\widetilde{q}, \widetilde{r})$ be admissible pairs. Then, we have*
(a) *homogeneous Strichartz estimate*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} \tag{1.2}$$

(b) *dual homogeneous Strichartz estimate*

$$\Big\| \int_{\mathbb{R}} S(-t')F(t')dt' \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)} \tag{1.3}$$

(c) *nonhomogeneous (retarded) Strichartz estimate*

$$\Big\| \int_{t_0}^t S(t-t')F(t')dt' \Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{1.4}$$

Given any time interval I, define the Strichartz space $S^0(I \times \mathbb{R}^d)$ via the norm

$$\|u\|_{S^0(I\times\mathbb{R}^d)} := \sup_{(q,r),\text{ admissible}} \|u\|_{L^q_t L^r_x(I\times\mathbb{R}^d)}.$$

Since $(q,r) = (\infty, 2)$ is admissible, we have $\|u\|_{C(I;L^2_x)} \leq \|u\|_{S^0(I\times\mathbb{R}^d)}$. We also define the dual $N^0(I\times\mathbb{R}^d) := (S^0(I\times\mathbb{R}^d))^*$, and thus we have

$$\|F\|_{N^0(I\times\mathbb{R}^d)} \leq \|F\|_{L^{q'}_t L^{r'}_x(I\times\mathbb{R}^d)}. \tag{1.5}$$

In discussing the Cauchy problem in H^1, it is convenient to introduce the following norms:

$$\|u\|_{S^1} = \|u\|_{S^0} + \|\nabla u\|_{S^0}, \quad \text{and} \quad \|u\|_{N^1} = \|u\|_{N^0} + \|\nabla u\|_{N^0}.$$

Then, from Lemma 1.1, we have

$$\|u\|_{S^1(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{H^1} + \|\mathcal{N}(u)\|_{N^1(I\times\mathbb{R}^d)}, \tag{1.6}$$

where $\mathcal{N}(u) = \lambda|u|^{p-1}u$.

1.3. H^1**-subcritical local theory.**

Proposition 1.2. *For $2 \leq p < 5$, NLS (1.1) is locally well-posed in $H^1(\mathbb{R}^3)$ in the subcritical sense. i.e. the time of local existence T is given by $T = T(\|u_0\|)$.*

Remark 1.3. In the defocusing case, this immediately provides the global well-posedness of (1.1). In the focusing case, the global well-posedness follows (a) unconditionally in the L^2-subcritical case, (b) for u_0 with the L^2 norm small than the L^2 norm of the ground state in the L^2-critical case, and (c) for u_0 with small H^1 norm in the L^2-supercritical and H^1-subcritical case.

Notation: Let $S^1_T = S^1([-T,T]\times\mathbb{R}^3)$, $N^1_T = N^1([-T,T]\times\mathbb{R}^3)$, and $L^q_T = L^q_t([-T,T])$.

Proof. Let $\Gamma u = S(t)u_0 - i\lambda\int_0^t S(t-t')|u|^{p-1}u(t')dt'$. By (1.6), we have

$$\|\Gamma u - \Gamma v\|_{S^1_T} \lesssim \||u|^{p-1}u - |v|^{p-1}v\|_{N^1_T}$$

Note that $(q,r) = (2,6)$ is admissible. Then, by (1.5),

$$\leq \||u|^{p-1}u - |v|^{p-1}v\|_{L^2_T W^{1,\frac{6}{5}}_x}. \tag{1.7}$$

In the following, we only consider the contribution from the gradient of $|u|^{p-1}u$. More specifically, we only consider the part $F(u)\nabla u$, where $F(z) = O(|z|^{p-1})$ and $F'(z) = O(|z|^{p-2})$. i.e. $F(u) - F(v) = O\big(|u|^{p-2}+|v|^{p-2})(u-v)\big)$. Thus, we have

$$\begin{aligned} F(u)\nabla u - F(v)\nabla v &= F(u)\nabla(u-v) + (F(u)-F(v))\nabla v \\ &= O\big(|u|^{p-1}\nabla(u-v)\big) + O\big(|u|^{p-2}+|v|^{p-2})(u-v)\nabla u\big) \end{aligned}$$

By Sobolev inequality, we have

$$\|f\|_{L^{\frac{5(p-1)}{2}}} \lesssim \|f\|_{W^{1,\frac{30}{13}}} \tag{1.8}$$

since $\frac{13}{30} - \frac{2}{5(p-1)} \leq \frac{1}{3}$ for $p \leq 5$. Then, by Hölder inequality ($\frac{1}{2} = \frac{5-p}{10} + \frac{1}{10}p$ with $p<5$ in time and $\frac{5}{6} = \frac{2}{5(p-1)}(p-1) + \frac{13}{30}$ in space) followed by (1.8), we have

$$\begin{aligned} (1.7) \lesssim T^{\frac{5-p}{10}} \big(&\|u\|^{p-1}_{L^{10}_T L_x^{\frac{5(p-1)}{2}}} \|\nabla(u-v)\|_{L^{10}_T L_x^{\frac{30}{13}}} \\ &+ \|u\|^{p-2}_{L^{10}_T L_x^{\frac{5(p-1)}{2}}} \|u-v\|_{L^{10}_T L_x^{\frac{5(p-1)}{2}}} \|\nabla u\|_{L^{10}_T L_x^{\frac{30}{13}}}\big) \\ &\lesssim T^{\frac{5-p}{10}}(\|u\|^{p-1}_{L^{10}_T W^{1,\frac{30}{13}}} + \|v\|^{p-1}_{L^{10}_T W^{1,\frac{30}{13}}})\|u-v\|_{L^{10}_T W^{1,\frac{30}{13}}} \\ &\lesssim T^{\frac{5-p}{10}}(\|u\|^{p-1}_{S^1_T} + \|v\|^{p-1}_{S^1_T})\|u-v\|_{S^1_T}, \end{aligned}$$

where the last inequality follows since $(q,r) = (10, \frac{13}{30})$ is admissible. This shows that Γ is a contraction on a ball $\{u : \|u\|_{S^1_T} \leq 2\|u_0\|_{H^1}\}$ for $T \sim \|u_0\|^{-\alpha}_{H^1}$ for some $\alpha > 0$. $\square$

1.4. H^1**-critical local theory.** For $p = 5$, the Hölder inequality in time the above proof no longer provide a small constant in front of $\|u-v\|_{S^1_T}$.

First, note that $\|u\|_{L^{10}_{t,x}} \lesssim \|u\|_{L^{10}_t \dot{W}^{1,\frac{30}{13}}}$ by Sobolev inequality. The latter norm is then bounded by $\|u\|_{\dot{S}^1}$, where $\|u\|_{\dot{S}^1} := \|\nabla u\|_{S^0}$, since $(10, \frac{30}{13})$ is admissible.

Note that given $\varepsilon > 0$, we have $\|S(t)u_0\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} < \varepsilon$ by taking T sufficiently small. Let $B = \{u : \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \leq 2\|S(t)u_0\|_{L^{10}_T \dot W^{1,\frac{30}{13}}}\}$. By repeating the previous computation, we have

$$\begin{aligned}\|\Gamma u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} &\leq \|S(t)u_0\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} + C\|u\|^4_{L^{10}_{T,x}} \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \\ &\leq (1+O(\varepsilon^4))\|S(t)u_0\|_{L^{10}_T \dot W^{1,\frac{30}{13}}}.\end{aligned}$$

Thus, Γ maps B into itself. Moreover, we have

$$\begin{aligned}\|\Gamma u - \Gamma v\|_{\dot S^1_T} &\lesssim \|u\|^4_{L^{10}_{T,x}} \|u-v\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} + \|u\|^3_{L^{10}_{T,x}} \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \|u-v\|_{L^{10}_{T,x}} \\ &\lesssim O(\varepsilon^4)\|u-v\|_{L^{10}_T \dot W^{1,\frac{30}{13}}}. \qquad (1.9)\end{aligned}$$

Hence, Γ is a contraction. Note that the unique solution we constructed is in $L^{10}_T \dot W^{1,\frac{30}{13}}$.

Now, we show that $u \in C([-T,T];\dot H^1)$. Clearly, we have $S(t)u_0 \in C([-T,T];\dot H^1)$. As for the Duhamel term, we have

$$\begin{aligned}\|u - S(t)u_0\|_{C([-T,T];\dot H^1)} &\leq \|u - S(t)u_0\|_{\dot S^1_T} \lesssim \|u\|^4_{L^{10}_{T,x}} \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \\ &\lesssim \|S(t)u_0\|^5_{L^{10}_T \dot W^{1,\frac{30}{13}}} < \infty.\end{aligned}$$

Hence, the Duhamel part is also in $C([-T,T];\dot H^1)$.

Remark 1.4. We indeed need to work a little more to show the continuity of the Duhamel part. Consider

$$\begin{aligned}&\int_0^{t+h} S(t+h-t')F(t')dt' - \int_0^t S(t-t')F(t')dt' \\ &\quad = \int_t^{t+h} S(t+h-t')F(t')dt' + (S(h)-1)\int_0^t S(t-t')F(t')dt'.\end{aligned}$$

Then, the first term goes to zero in $\dot H^1$ by the above estimate with shrinking interval $[t,t+h]$, whereas the second goes to zero by the continuity of $S(h)$.

Remark 1.5. We can consider the above argument with $\|u_0\|_{\dot H^1} = R_0 \leq R$ and $\|S(t)u_0\|_{L^{10}_{T,x}} = \varepsilon_0 < \varepsilon = \varepsilon(R)$. i.e. under weak assumption.

Let $B = \{u : \|u\|_{L^{10}_{T,x}} \leq 2\varepsilon_0,\ \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \leq CR_0\}$. Then, we have

$$\|\Gamma u\|_{L^{10}_{T,x}} \leq \|S(t)u_0\|_{L^{10}_{T,x}} + C\|u\|^4_{L^{10}_{T,x}} \|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \leq (1+O(R_0\varepsilon_0^3)\varepsilon_0 \leq 2\varepsilon_0.$$

By the homogeneous Strichartz estimate, we have

$$\|\Gamma u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \lesssim \|u_0\|_{\dot H^1} + \varepsilon_0^4 R_0 \lesssim R_0.$$

Note that B is a complete metric space with the metric $d(u,v) = \|u-v\|_{L^{10}_T \dot W^{1,\frac{30}{13}}}$. Suppose that u_n is Cauchy in B. Then, it converges to some u in $L^{10}_T \dot W^{1,\frac{30}{13}}$ and $\|u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} < CR_0$. We also have $\|u\|_{L^{10}_{T,x}} \leq 2\varepsilon_0$ since $\|u_n - u\|_{L^{10}_{T,x}} \lesssim \|u_n - u\|_{L^{10}_T \dot W^{1,\frac{30}{13}}} \to 0$. Then, it easily follows that Γ is a contraction on B. Note that we can replace $L^{10}_T \dot W^{1,\frac{30}{13}}$ by $\dot S^1_T$ and the same result still holds.

Remark 1.6. (a) Since $\|S(t)u_0\|_{L^{10}_{t,x}} \lesssim \|u_0\|_{\dot{H}^1}$, it follows that (1.1) is globally well-posed for sufficiently small initial data in $\dot{H}^1(\mathbb{R}^3)$.
(b) An a priori bound $\|u\|_{L^{10}_{t,x}} \leq C$ implies the globally well-posedness of (1.1).
(c) One can prove the unconditional uniqueness for this problem.

2. Scattering

2.1. **Scattering, existence of wave operator, asymptotic completeness.** We say that a global solution $u \in H^1$ *scatters in* H^1 as $t \to \infty$ if there exists $u_+ \in H^1$ such that $\|u(t) - S(t)u_+\|_{H^1} \to 0$. Or, equivalently $\|S(-t)u(t) - u_+\|_{H^1} \to 0$. i.e. $S(-t)u(t)$ converges in H^1 as $t \to \infty$.

By the Duhamel formula, we have

$$S(-t)u(t) = u_0 - i\lambda \int_0^t S(-t')|u|^{p-1}u(t')dt'.$$

Hence, the scattering problem is exactly considering the convergence of the improper integral

$$\int_0^\infty S(-t)|u|^{p-1}u(t)dt$$

in H^1. In this case, u_+ is given by

$$u_+ = u_0 - i\lambda \int_0^\infty S(-t')|u|^{p-1}u(t')dt'.$$

Conversely, one can ask; given $u_+ \in H^1$, can we find an initial data $u_0 \in H^1$ such that the global solution u of (1.1) scatters to u_+? If this holds, then we can define the *wave operator* $\Omega_+ : u_+ \in H^1 \mapsto u_0 \in H^1$. By the uniqueness of the well-posedness theory, Ω_+ is always injective. For this problem, it is convenient to write u in terms of u_+:

$$u(t) = S(t)u_+ + i\lambda \int_t^\infty S(t-t')|u|^{p-1}u(t')dt'. \tag{2.1}$$

Note that the issue of scattering is the invertibility of the wave operator Ω_+. If Ω_+ is invertible, we say that we have *asymptotic completeness.*

Moral: If $u(t)$ decays to zero, then $|u(t)|^{p-1}u(t)$ decays even faster. i.e. in the asymptotic theory, smaller exponents are more difficult to handle. On the contrary, larger exponents are more difficult in the local theory. e.g. we do not expect scattering in the long-range/critical range case $p \leq 1 + \frac{2}{d}$.

It is in general easier to prove the existence of the wave operator, which usually follows from dispersive estimates and from the local theory, especially under a smallness condition (both in focusing and defocusing cases.) The asymptotic completeness is harder, restricted to the defocusing case, and it requires a global space-time bound.

In the following, we consider the defocusing ($\lambda = 1$) cubic NLS on $\mathbb{R}^3$:

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u\big|_{t=0} = u_0 \in H^1. \end{cases} \tag{2.2}$$

2.2. **Existence of the wave operator Ω_+.** Using (2.1), we need to evolve from $t = \infty$ to $t = 0$. We do so in two steps: from $t = \infty$ to $t = T$, and from $t = T$ to $t = 0$.

Our first goal is to show the well-posedness on $[T, \infty)$ by exploiting the fact that all the Strichartz norms on $[T, \infty)$ (except for L^∞_t) go to zero as $T \to \infty$. For this purpose, define$\widetilde{S}^1$ by the norm

$$\|u\|_{\widetilde{S}^1} = \|u\|_{L^5_{t,x}} + \|u\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}}.$$

By Sobolev, we have $\|u\|_{L^5_{t,x}} \lesssim \|u\|_{L^5_t W^{1,\frac{30}{11}}}$, where $(5, \frac{30}{11})$ is admissible. Then, by taking T large, we have

$$\|S(t)u_+\|_{\widetilde{S}^1[T,\infty)} \leq \varepsilon$$

for $u_+ \in H^1$. Let $\widetilde{\Gamma} u(t)$ denote the right hand side of (2.1). Then, by Lemma 1.1 with admissible $(\frac{10}{3}, \frac{10}{3})$ followed by Hölder inequality, we have

$$\begin{aligned} \|\widetilde{\Gamma} u\|_{\widetilde{S}^1[T,\infty)} &\leq \varepsilon + C\big\||u|^2 u\big\|_{L_{t,x}^{\frac{10}{7}}([T,\infty)\times\mathbb{R}^3)} \leq \varepsilon + \|u\|^2_{L^5_{t,x}([T,\infty)\times\mathbb{R}^3)} \|u\|_{L_{t,x}^{\frac{10}{3}}([T,\infty)\times\mathbb{R}^3)} \\ &\leq \varepsilon + C\|u\|^3_{\widetilde{S}^1[T,\infty)}. \end{aligned}$$

Similarly, we have

$$\|\widetilde{\Gamma} u - \widetilde{\Gamma} v\|_{\widetilde{S}^1[T,\infty)} \leq C(\|u\|^2_{\widetilde{S}^1[T,\infty)} + \|v\|^2_{\widetilde{S}^1[T,\infty)})\|u - v\|_{\widetilde{S}^1[T,\infty)}.$$

Hence, $\widetilde{\Gamma}$ is a contraction on a ball $\{u : \|u\|_{\widetilde{S}^1[T,\infty)} \leq 2\varepsilon\}$. Lastly, we have, as before,

$$\begin{aligned} \|u\|_{C([T,\infty);H^1)} \leq \|u\|_{S^1[T,\infty)} &\leq C\|u_+\|_{H^1} + C\big\||u|^2 u\big\|_{L_{t,x}^{\frac{10}{7}}([T,\infty)\times\mathbb{R}^3)} \\ &\leq C\|u_+\|_{H^1} + C\|u\|^3_{\widetilde{S}^1[T,\infty)} \lesssim \|u_+\|_{H^1} < \infty. \end{aligned}$$

Now, by iterating the local theory along with the conservation laws, we can extend the solution to time $t = 0$.

2.3. **Asymptotic completeness.** By Lemma 1.1 (b)

$$\begin{aligned} \Big\| \int_0^\infty S(-t)|u|^{p-1}u(t)dt \Big\|_{H^1} &\lesssim \big\|\nabla(|u|^2 u)\big\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|u\|^2_{L^5_{t,x}} \|u\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}} \\ &\lesssim \|u\|^2_{L^5_t W^{1,\frac{30}{11}}} \|u\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}} \lesssim \|u\|^3_{S^1}. \end{aligned}$$

Hence, the strong space-time bound $\|u\|_{S^1} \lesssim 1$ implies scattering.

Next, we show that a weak space-time bound

$$\|u\|_{L^q_{t,x}} \lesssim 1, \tag{2.3}$$

for some $q \in [\frac{10}{3}, 10]$, implies the strong space-time bound $\|u\|_{S^1} \lesssim 1$. Note that the $L^q_{t,x}$-norm is weaker than the Strichartz S^1-norm. However, the Strichartz estimates in the subcritical setting allow us to control the strong norm by the weaker one.

Given small $\varepsilon > 0$, divide the time interval into $I_1, \cdots, I_N$ such that

$$\|u\|_{L^q_{t,x}} \leq \varepsilon. \tag{2.4}$$

Let $I = (a, b)$ be one of the subintervals. Then, by (1.6), we have

$$\|u\|_{S^1(I\times\mathbb{R}^3)} \lesssim \|u(a)\|_{H^1} + \|\mathcal{N}(u)\|_{N^1(I\times\mathbb{R}^3)} \lesssim \|u(a)\|_{H^1} + \|u\|^2_{L^5_{t,x}(I\times\mathbb{R}^3)}\|u\|_{L_t^{\frac{10}{3}}W_x^{1,\frac{10}{3}}(I\times\mathbb{R}^3)}$$
$$\lesssim \|u(a)\|_{H^1} + \|u\|^2_{L^5_{t,x}(I\times\mathbb{R}^3)}\|u\|_{S^1(I\times\mathbb{R}^3)}.$$

Note the following.

$$\begin{cases} \|u\|_{L_{t,x}^{\frac{10}{3}}(I\times\mathbb{R}^3)} \leq \|u\|_{S^0(I\times\mathbb{R}^3)} \\ \|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|u\|_{L_t^{10}W_x^{1,\frac{30}{13}}(I\times\mathbb{R}^3)} \leq \|u\|_{S^1(I\times\mathbb{R}^3)} \end{cases}$$

By interpolating with (2.4), we have $\|u\|_{L^5_{t,x}(I\times\mathbb{R}^3)} \lesssim \varepsilon^\theta \|u\|_{S_I^1}^{1-\alpha}$. Hence, we have

$$\|u\|_{S^1(I\times\mathbb{R}^3)} \lesssim \|u(a)\|_{H^1} + \varepsilon^{2\theta}\|u\|^{3-2\theta}_{S^1(I\times\mathbb{R}^3)} \lesssim 1 + \varepsilon^{2\theta}\|u\|^{3-2\theta}_{S^1(I\times\mathbb{R}^3)}.$$

Now, let $F(t) = \|u\|_{S^1[a,a+t]}$. Then, one can prove that $F(t)$ is continuous (by interpolation, it is enough to show the continuity for the endpoints and $u \in C_t(H_x^1)$), and $X(t) \lesssim 1 + \varepsilon^{2\theta}X(t)^{3-2\theta}$. Then, for ε small, there exist $F_0 < F_1$ such that either $F(t) \leq F_0$ or $F(t) \geq F_1$. Hence, it follows from the continuity and $F(0) \lesssim 1$ that $F(t) \leq F_0$ for all $t \in I$. This holds for all the subintervals I_j, and therefore we obtained the strong global bound.

It remains to show the bound (2.3). We obtain such bounds by Morawetz inequality for radial functions, and by interaction Morawetz inequality for general functions.

• **Morawetz inequality:**

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{|u(t,x)|^4}{|x|}dxdt \lesssim \|u_0\|_{L^2}E^{\frac{1}{2}} \lesssim C(\|u_0\|_{H^1}). \tag{2.5}$$

Radial Sobolev inequality says: a radial function u satisfies $\big\||x||u|\big\|_{L_x^\infty(\mathbb{R}^d)} \lesssim C_{d,s}\|u\|_{H^1(\mathbb{R}^d)}$ for $\frac{d}{2} - 1 \leq s \leq \frac{d-1}{2}$. In particular, with the conservation of mass and energy, we have

$$\big\||x|u(t,x)\big\|_{L_t^\infty L_x^\infty(\mathbb{R}^d)} \lesssim 1. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\|u\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim C(\|u_0\|_{H^1}).$$

• **Interaction Morawetz inequality:** Interaction Morawetz inequality directly yields the global space-time bound for non-radial functions:

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim C(\|u_0\|_{H^1}).$$