A NOTE ON THE STOCHASTIC NONLINEAR WAVE EQUATIONS WITH A MULTIPLICATIVE SPACE-TIME WHITE NOISE ON THE CIRCLE

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Abstract. In this note, we review local well-posedness of the one-dimensional stochastic nonlinear wave equations with a multiplicative space-time white noise.

1. Introduction

1.1. **Stochastic nonlinear wave equations with a multiplicative noise.** We consider the stochastic nonlinear wave equations (SNLW) on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a multiplicative space-time white noise:

$$\begin{cases} \partial_t^2 u = \partial_x^2 u \pm u^k + u^m \frac{\partial^2 B}{\partial t \partial x} \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1) \in \mathcal{H}^s := H^s \times H^{s-1}, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}_+, \tag{1.1}$$

where $k \geq 2$ and $m \geq 1$ are integers and $\frac{\partial^2 B}{\partial t \partial x}$ denotes a (Gaussian) space-time white noise on $\mathbb{T} \times \mathbb{R}_+$. In the following, we restrict our discussion to the real-valued setting.

By letting $v = \partial_t u$, we can write (1.1) in the following Ito formulation:

$$\begin{cases} d \begin{pmatrix} u \\ v \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \pm u^k \end{pmatrix} \right\} dt + \begin{pmatrix} 0 \\ u^m \end{pmatrix} dW \\ (u, v)|_{t=0} = (\phi_0, \phi_1). \end{cases} \tag{1.2}$$

Here, $W(t) = \frac{\partial B}{\partial x}$ denotes a cylindrical Wiener process on $L^2(\mathbb{T})$. More precisely, by letting $e_n(x) = e^{2\pi i n x}$, we have[1]

$$\begin{aligned} W(t) &= \beta_0(t) e_0 + \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \beta_n(t) e_n \\ &= \beta_0(t) e_0 + \sum_{n \in \mathbb{N}} \Big[\operatorname{Re}(\beta_n(t)) \cdot \sqrt{2} \cos(2\pi n x) - \operatorname{Im}(\beta_n(t)) \cdot \sqrt{2} \sin(2\pi n x) \Big], \end{aligned} \tag{1.3}$$

where $\{\beta_n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a family of mutually independent complex-valued Brownian motions[2] on a fixed probability space $(\Omega, \mathcal{F}, P)$ and $\beta_{-n} := \overline{\beta_n}$ for $n \in \mathbb{Z}_{\geq 0}$. Note that $\operatorname{Var}(\beta_n(t)) = 2t$

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[1] Note that $\{1, \sqrt{2}\cos(2\pi n x), \sqrt{2}\sin(2\pi n x) : n \in \mathbb{N}\}$ forms an orthonormal basis of $L^2(\mathbb{T})$ in the real valued setting.

[2] Here, we take β_0 to be real-valued.

for $n \in \mathbb{Z} \setminus \{0\}$, while $\text{Var}(\beta_0(t)) = t$. From the random Fourier series representation, it is easy to see that W almost surely lies in[3]

$$H^b_{t,\text{loc}} H_x^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\frac{1}{2}}_{t,\text{loc}} H_x^{-\frac{1}{2}}(\mathbb{T})$$

for any $b < \frac{1}{2}$ and $\varepsilon > 0$.

Let $S(t)$ be the propagator for the linear wave equation given by

$$S(t)(\phi_0, \phi_1) := \cos(t\sqrt{-\Delta})\phi_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1.$$

Then, the mild formulation of the Cauchy problem (1.1) (and (1.2)) is given by

$$u(t) = S(t)(\phi_0, \phi_1) \pm \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} u^k(t')dt' + \Psi(u^m)(t), \tag{1.4}$$

where the stochastic convolution $\Psi(v)$ is defined by

$$\Psi(v)(t) = \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} \big[v(t')dW(t')\big]. \tag{1.5}$$

Here, it is understood that $\frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} e_0 = t - t'$. We also use the notation $\Psi(v)(t_0, t)$ defined by

$$\Psi(v)(t_0, t) = \int_{t_0}^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} \big[v(t')dW(t')\big]. \tag{1.6}$$

In the following, we study local well-posedness of the mild formulation (1.4). As such, the defocusing/focusing nature of the equation does not play any role and thus we restrict our attention to the focusing case (i.e. with the + sign in (1.1) and (1.4)) in the following.

1.2. **Main result.** Before we state our main result, we first need to discuss critical regularities for the deterministic nonlinear wave equations (NLW):

$$\partial_t^2 u = \Delta u \pm u^k. \tag{1.7}$$

On the one hand, NLW on $\mathbb{R}^d$ enjoys the scaling symmetry, which induces the so-called scaling critical Sobolev index: $s_1 = \frac{d}{2} - \frac{2}{k-1}$. On the other hand, NLW also enjoys the conformal symmetry, which yields its own critical regularity: $s_2 = \frac{d+1}{4} - \frac{1}{k-1}$. In the one-dimensional case, there is another critical regularity due to lack of dispersion. In particular, there are no Strichartz estimates. Hence, Sobolev's inequality plays an essential role in the analysis, thus yielding a critical regularity: $s_3 = \frac{1}{2} - \frac{1}{k}$. Indeed, (1.7) on $\mathbb{R}$ is known to be ill-posedness in $\mathcal{H}^s$ for $s < s_3$. See [3]. Note that $s_3 > s_2 > s_1$.

Since we consider the multiplicative noise, it also induces a critical regularity: $s_4 = \frac{1}{2} - \frac{1}{2m}$. We set s_{crit} by

$$s_{\text{crit}} := \max(s_3, s_4, 0).$$

[3]In fact, W lies almost surely in

$$W^{b,\infty}_{t,\text{loc}} W_x^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}) \setminus W^{\frac{1}{2},\infty}_{t,\text{loc}} W_x^{-\frac{1}{2},\infty}(\mathbb{T})$$

for any $b < \frac{1}{2}$ and $\varepsilon > 0$.

Theorem 1.1. *Let $k \geq 2$ and $m \geq 1$ be integers and $s_{\text{crit}} \leq s < \frac{1}{2}$. Then, given $(\phi_0, \phi_1) \in \mathcal{H}^s$, there exist a stopping time $t_* = t_*(\phi_0, \phi_1)$, almost surely positive, and a unique adapted mild solution $u \in L^2_{\text{ad}}(\Omega; C([0, t_*); H^s))$ to the mild formulation* (1.4) *of the SNLW* (1.1) *with paths almost surely in $C([0, t_*); H^s)$. Moreover, the following blowup alternative holds:*

$$\text{(i) } t_* = \infty \qquad \textit{or} \qquad \text{(ii) } \lim_{t \to t_*-} \|u(t)\|_{H^s} = \infty \tag{1.8}$$

almost surely.

2. Proof of Theorem 1.1

We employ the truncation argument in de Bouard-Debussche [4]. In Subsection 2.1, we first establish a moment estimate on the stochastic integral $\Psi(v)$ in (1.5). In Subsection 2.2, we then prove local well-posedness of the truncated SNLW (see (2.8) below) and use it to prove Theorem 1.1 in Subsection 2.3.

2.1. Stochastic estimate.

In this subsection, we estimate the stochastic integral $\Psi(v)$ in (1.5). In the following, we denote by $L^p_{\text{ad}}(\Omega; C_T H^s) = L^p_{\text{ad}}(\Omega; C([0, T]; H^s(\mathbb{T})))$ (and $L^p_{\text{ad}}(\Omega; L^\infty_T H^s)$, respectively) the subclass of adapted processes in $L^p(\Omega; C_T H^s)$ (and $L^p(\Omega; L^\infty_T H^s)$, respectively). We also use $HS(H_1; H_2)$ to denote the class of Hilbert-Schmidt operators from H_1 to H_2.

Lemma 2.1. *Let $p \geq 2$, $s < \frac{1}{2}$, and $T > 0$. Given $v \in L^p_{\text{ad}}(\Omega; L^\infty([0, T]; H^s))$, let $\Psi(v)$ be as in* (1.5). *Then, we have*

$$\mathbb{E}\Big[\|\Psi(v)\|^p_{L^\infty_T H^s_x}\Big] \lesssim \max(T^{\frac{p}{2}}, T^{\frac{3p}{2}}) \mathbb{E}\big[\|v(t)\|^p_{L^\infty_T L^2_x}\big]. \tag{2.1}$$

Moreover, $\Psi(v)$ is pathwise continuous with values in H^s.

Proof. Let $K(t)$ denote the propagator of the half wave equation given by

$$\mathcal{F}_x(K(t)f)(n) = e^{it|n|}\widehat{f}(n).$$

Furthermore, define $K_+(t)$ and $K_-(t)$ by

$$K_+(t) = \frac{K(-t)}{2i\sqrt{-\Delta}}\mathbf{P}_{\neq 0} \qquad \text{and} \qquad K_-(t) = \frac{K(t)}{2i\sqrt{-\Delta}}\mathbf{P}_{\neq 0},$$

where $\mathbf{P}_{\neq 0} = \text{Id} - \mathbf{P}_0$ denotes the projection onto the non-zero (spatial) frequencies. Then, it follows from (1.5) that

$$\begin{aligned} \Psi(f)(t) &= \int_0^t (t - t')\mathbf{P}_0 \circ M_v(t')(dW(t')) \\ &\qquad + K(t) \int_0^t K_+ \circ M_v(t')(dW(t')) - K(-t) \int_0^t K_- \circ M_v(t')(dW(t')) \\ &=: \mathcal{I}_0(t) + \mathcal{I}_+(t) + \mathcal{I}_-(t), \end{aligned} \tag{2.2}$$

where M_v is the multiplication operator by v.

Note that the Hilbert-Schmidt norm of $K_+ \circ M_v(t)$ is given by

$$\begin{aligned}\|K_+\circ M_v(t)\|_{HS(L^2_x;H^s_x)} &= \bigg(\sum_{k\in\mathbb{Z}} \|K_+ \circ M_v(t)e_k\|^2_{H^s}\bigg)^{\frac{1}{2}}\\ &\sim \bigg(\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s-2}\bigg|\sum_{n=n_1+n_2}\widehat{v}(n_1,t)\delta_{n_2k}\bigg|^2\bigg)^{\frac{1}{2}}\\ &\sim \bigg(\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s-2}|\widehat{v}(n-k,t)|^2\bigg)^{\frac{1}{2}} \sim \|v(t)\|_{L^2_x} \end{aligned}\tag{2.3}$$

since $s < \frac{1}{2}$. Similarly, we have

$$\|K_-\circ M_v(t)\|_{HS(L^2_x;H^s_x)} \sim \|v(t)\|_{L^2_x}. \tag{2.4}$$

By Burkholder-Davis-Gundy inequality ([2, Theorem 4.36], see also [1]) with (2.3), we have

$$\begin{aligned}\mathbb{E}\Big[\|\mathcal{I}_+\|^p_{L^\infty_T H^s}\Big] = \mathbb{E}\Big[\|K(-t)\mathcal{I}^+\|^p_{L^\infty_T H^s}\Big] &\lesssim \mathbb{E}\bigg[\bigg(\int_0^T \|K_+\circ M_v(t)\|^2_{HS(L^2_x;H^s_x)}dt\bigg)^{\frac{p}{2}}\bigg]\\ &\lesssim T^{\frac{p}{2}}\mathbb{E}\big[\|v(t)\|^p_{L^\infty_T L^2_x}\big]. \end{aligned}\tag{2.5}$$

Similarly, it follows from Burkholder-Davis-Gundy inequality and (2.4) that

$$\mathbb{E}\Big[\|\mathcal{I}_-\|^p_{L^\infty_T H^s}\Big] \lesssim T^{\frac{p}{2}}\mathbb{E}\big[\|v(t)\|^p_{L^\infty_T L^2_x}\big]. \tag{2.6}$$

Applying Burkholder-Davis-Gundy inequality once again, we have

$$\begin{aligned}\mathbb{E}\Big[\|\mathcal{I}_0(v)\|^p_{L^\infty_T H^s_x}\Big] &\lesssim \mathbb{E}\bigg[\bigg(\int_0^t (t-t')^2\|v(t')\|^2_{L^2_x}dt'\bigg)^{\frac{p}{2}}\bigg]\\ &\lesssim T^{\frac{3p}{2}}\mathbb{E}\big[\|v(t)\|^p_{L^\infty_T L^2_x}\big]. \end{aligned}\tag{2.7}$$

Therefore, (2.1) follows from (2.2), (2.5), (2.6), and (2.7).

Note that we have

$$K_+\circ M_v \in L^2_{\text{ad}}(\Omega; L^2([0,T]; HS(L^2_x; H^s_x)))$$

for $v \in L^p_{\text{ad}}(\Omega; L^\infty_T H^s)$. In particular, it follows from [1, Theorem 2.1] and the strong continuity of $S(t)$ that $\mathcal{I}_+(t)$ defined in (2.2) is pathwise continuous on $[0,T]$ (with values in H^s). A similar argument yields the pathwise continuity of $\mathcal{I}_-(t)$ and $\mathcal{I}_0(t)$. Therefore, $\Psi(v)$ is pathwise continuous with values in H^s. □

2.2. **Truncated SNLW.** Let η be a smooth cutoff function in $C^\infty(\mathbb{R}_+;[0,1])$ such that $\eta \equiv 1$ on $[0,1]$ and $\operatorname{supp}\eta \subset [0,2]$. Given $R>0$, set $\eta_R(x) = \eta(R^{-1}x)$.

Let $s\in\mathbb{R}$ be as in Theorem 1.1. Given $R>0$, we first consider the following truncated SNLW:

$$\begin{aligned}\Gamma_R(u)(t) = S(t)(\phi_0,\phi_1) &+ \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}}\eta_R(\|u\|_{C_{t'}H^s})u^k(t')dt'\\ &+ \Psi\big(\eta_R(\|u\|_{C_{t'}H^s})u^m\big)(t). \end{aligned}\tag{2.8}$$

In particular, we prove that the fixed point problem

$$\Gamma_R(u) = u \tag{2.9}$$

is *globally* well-posed for each $R > 0$.

Proposition 2.2. *Let $s_{\text{scrit}} \le s < \frac{1}{2}$. Then, given $(\phi_0, \phi_1) \in \mathcal{H}^s$ and $R > 0$, there exists a unique global adapted solution $u = u(R) \in L^2_{\text{ad}}(\Omega; C(\mathbb{R}_+; H^s))$ to the truncated SNLW (2.9) with paths almost surely in $C(\mathbb{R}_+; H^s)$.*

Proof. Fix $T > 0$. We first prove that $\Gamma_R(u)$ defined in (2.8) is pathwise continuous on $[0, T]$ (with values in H^s) for $u \in L^2_{\text{ad}}(\Omega; L^\infty_T H^s)$. Define Γ_1 by

$$\Gamma_1(v) = \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} v(t')dt'.$$

Then, by the mean value theorem, we have

$$\begin{aligned}
\|\Gamma_1(v)(t+h) - \Gamma_1(v)(t)\|_{H^s} \qquad& \\
\le \bigg\| \int_t^{t+h} \frac{\sin((t+h-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} v(t')dt' \bigg\|_{H^s}& \\
+ \bigg\| \int_0^t \frac{\sin((t+h-t')\sqrt{-\Delta}) - \sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} v(t')dt' \bigg\|_{H^s}& \\
\le |h| \|v\|_{L^\infty_T H^{s-1}} + T|h| \|v\|_{L^\infty_T H^s}&
\end{aligned} \tag{2.10}$$

for any $0 < |h| \ll 1$. This shows that $\Gamma_1(v) \in C_T H^s$ for $v \in L^\infty_T H^s$. Noting that $\eta_R(\|u\|_{C_{t'} H^s}) u^k \in L^2_{\text{ad}}(\Omega; L^\infty_T H^s)$ for $u \in L^2_{\text{ad}}(\Omega; L^\infty_T H^s)$, we conclude from (2.8), (2.10), and Lemma 2.1 that $\Gamma_R(u)$ is pathwise continuous with values in H^s.

Now, we show that Γ_R is a contraction in $L^2_{\text{ad}}(\Omega; C_T H^s)$ for some $T = T(R) > 0$. Let $u \in L^2_{\text{ad}}(\Omega; C_T H^s)$. By Sobolev inequality, Lemma 2.1, and the definition of η_R, we have

$$\begin{aligned}
\|\Gamma_R(u)\|_{L^2(\Omega; C_T H^s_x)} &\le \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + CT \|\eta_R(\|u\|_{C_t H^s}) u^k\|_{L^2(\Omega; C_T L^1_x)} \\
&\quad + CT^{\frac{1}{2}} \|\eta_R(\|u\|_{C_t H^s}) u^m\|_{L^2(\Omega; C_T L^2_x)} \\
&\le \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + CT \big\|\eta_R(\|u\|_{C_t H^s}) \|u(t)\|^k_{H^s_x}\big\|_{L^2_\omega C_T} \\
&\quad + CT^{\frac{1}{2}} \big\|\eta_R(\|u\|_{C_t H^s}) \|u(t)\|^m_{H^s_x}\big\|_{L^2_\omega C_T} \\
&\le \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + C_1(TR^k + T^{\frac{1}{2}} R^m).
\end{aligned} \tag{2.11}$$

for any $0 < T \le 1$.

Before we proceed to estimate a difference, we state an elementary deterministic lemma.

Lemma 2.3. *Let $k, m \in \mathbb{N}$, $T > 0$, and $s_{\text{scrit}} \le s < \frac{1}{2}$. Then, there exists $C > 0$ such that we have*

$$\big\|\eta_R(\|u_1\|_{C_t H^s}) u_1^k - \eta_R(\|u_2\|_{C_t H^s}) u_2^k\big\|_{C_T L^1_x} \le C R^{k-1} \|u_1 - u_2\|_{C_t H^s}, \tag{2.12}$$

$$\|\eta_R(\|u_1\|_{C_t H^s}) u_1^m - \eta_R(\|u_2\|_{C_t H^s}) u_2^m\big\|_{C_T L^2_x} \le C R^{m-1} \|u_1 - u_2\|_{C_t H^s}, \tag{2.13}$$

for any deterministic functions $u_1, u_2 \in C([0, T]; H^s)$ and $R > 0$.

Proof. We only present the proof of (2.12) since (2.13) follows in a similar manner. Given two deterministic functions $u_1, u_2 \in C([0,T]; H^s)$, define $t_{j,R}$, $j = 1, 2$, by

$$t_{j,R} = \sup\{t \in [0,T] : \|u_j\|_{C_t H^s} \leq 2R\}. \tag{2.14}$$

Without loss of generality, assume $t_{1,R} \leq t_{2,R}$.

We first estimate

$$\big\| \big(\eta_R(\|u_1\|_{C_t H^s}) - \eta_R(\|u_2\|_{C_t H^s})\big) u_2^k \big\|_{C_T L^1_x}. \tag{2.15}$$

Note that there is no contribution to (2.15) from the time interval $t_{2,R} \leq t \leq T$. Then, by the mean value theorem and Sobolev inequality with (2.14), we have

$$\begin{aligned}
\big\| \big(\eta_R(\|u_1\|_{C_t H^s}) &- \eta_R(\|u_2\|_{C_t H^s})\big) u_2^k \big\|_{C_T L^1_x} \\
&= \big\| \big(\eta_R(\|u_1\|_{C_t H^s}) - \eta_R(\|u_2\|_{C_t H^s})\big) u_2^k \big\|_{C([0,t_{2,R}]; L^1_x)} \\
&\lesssim \frac{\|\eta'\|_{L^\infty}}{R} \big\| (\|u_1\|_{C_t H^s} - \|u_2\|_{C_t H^s}) u_2^k \big\|_{C([0,t_{2,R}]; L^1_x)} \\
&\lesssim \frac{\|\eta'\|_{L^\infty}}{R} \big\| \|u_1 - u_2\|_{C_t H^s} u_2^k \big\|_{C([0,t_{2,R}]; L^1_x)} \\
&\lesssim \frac{\|\eta'\|_{L^\infty}}{R} \|u_1 - u_2\|_{C_T H^s} \|u_2\|^k_{C([0,t_{2,R}]; H^s)} \\
&\lesssim R^{k-1} \|u_1 - u_2\|_{C_T H^s_x}.
\end{aligned} \tag{2.16}$$

On the other hand, noting that $\|u_j\|_{C([0,t_{1,R}]; L^1_x)} \lesssim R$, $j = 1, 2$, it follows from Sobolev inequality that

$$\begin{aligned}
\big\| \eta_R(\|u_1\|_{C_t H^s}) \big(u_1^k - u_2^k\big) \big\|_{C_T L^1_x} &= \big\| \eta_R(\|u_1\|_{C_t H^s}) \big(u_1^k - u_2^k\big) \big\|_{C([0,t_{1,R}]; L^1_x)} \\
&\lesssim \big\| (|u_1|^{k-1} + |u_2|^{k-1})(u_1 - u_2) \big\|_{C([0,t_{1,R}]; L^1_x)} \\
&\lesssim \big(\|u_1\|^{k-1}_{C([0,t_{1,R}]; H^s)} + \|u_2\|^{k-1}_{C([0,t_{2,R}]; H^s)} \big) \|u_1 - u_2\|_{C_T H^s_x} \\
&\lesssim R^{k-1} \|u_1 - u_2\|_{C_T H^s_x}.
\end{aligned} \tag{2.17}$$

Hence, (2.12) follows from (2.16) and (2.17). □

Now, let $u_1, u_2 \in L^2_{\text{ad}}(\Omega; C_T H^s_x)$. Then, by Sobolev inequality, Lemma 2.1, and Lemma 2.3, we have

$$\|\Gamma_R(u_1) - \Gamma_R(u_2)\|_{L^2(\Omega; C_T H^s_x)} \leq C_2 (T R^{k-1} + T^{\frac{1}{2}} R^{m-1}) \|u_1 - u_2\|_{L^2(\Omega; C_T H^s_x)} \tag{2.18}$$

for any $0 < T \leq 1$. Choose $T = T(R) > 0$ sufficiently small such that

$$C_1 (T R^k + T^{\frac{1}{2}} R^m) \leq R, \tag{2.19}$$

$$C_2 (T R^{k-1} + T^{\frac{1}{2}} R^{m-1}) \leq \tfrac{1}{2}. \tag{2.20}$$

Then, it follows from (2.11) and (2.18) that Γ_R is a contraction in the entire $L^2_{\text{ad}}(\Omega; C([0,T]; H^s))$. In particular, the uniqueness of u holds in $C([0,T]; H^s)$ almost surely. Moreover, it follows from (2.11) and (2.19) that

$$\|u\|_{C_T H^s} \leq \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + R.$$

Next, we extend the solution u to (2.9) globally in time. For $t > T$, we have

$$\Gamma_R(u)(t) = S(t-T)(u(T), \partial_t u(T)) + \int_T^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} \eta_R(\|u\|_{C_{t'}H^s}) u^k(t') dt' \\ + \Psi\big(\eta_R(\|u\|_{C_{t'}H^s}) u^m\big)(T,t),$$

where $\Psi(f)(t_0, t)$ is defined in (1.6).

Let $I_1 = [T, 2T]$. Then, by repeating a similar computations in (2.11) with (2.19), we have

$$\begin{aligned} \|\Gamma_R(v)\|_{L^2(\Omega;C(I_1;H^s_x))} &\le \|(v(T), \partial_t v(T))\|_{L^2(\Omega;\mathcal{H}^s)} + C_1(TR^k + T^{\frac{1}{2}} R^m) \\ &\le \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + 2R \end{aligned} \tag{2.21}$$

for all $v \in L^2_{\text{ad}}(\Omega; C(I_1; H^s))$ such that $(v(T), \partial_t v(T)) = (u(T), \partial_t u(T))$ almost surely. Also, from (2.18) with (2.20), we have

$$\begin{aligned} \|\Gamma_R(u_1) - \Gamma_R(u_2)\|_{L^2(\Omega;C(I_1;H^s_x))} &\le C_2(TR^{k-1} + T^{\frac{1}{2}} R^{m-1}) \|u_1 - u_2\|_{L^2(\Omega;C(I_1;H^s_x))} \\ &\le \tfrac{1}{2} \|u_1 - u_2\|_{L^2(\Omega;C(I_1;H^s_x))} \end{aligned} \tag{2.22}$$

for all $u_1, u_2 \in L^2_{\text{ad}}(\Omega; C(I_1; H^s))$ such that $(u_j(T), \partial_t u_j(T)) = (u(T), \partial_t u(T))$ almost surely, $j = 1, 2$. Hence, it follows from (2.21) and (2.22) that Γ_R is a contraction in $L^2_{\text{ad}}(\Omega; C([T, 2T]; H^s))$.

Let $I_j = [jT, (j+1)T]$. Then, by iterating the argument inductively, we have

$$\begin{aligned} \|\Gamma_R(u)\|_{L^2(\Omega;C(I_j;H^s_x))} &\le \|u(jT)\|_{L^2(\Omega;\mathcal{H}^s)} + C_1(TR^k + T^{\frac{1}{2}} R^m) \\ &\le \|(\phi_0, \phi_1)\|_{\mathcal{H}^s} + (j+1)R \end{aligned}$$

and

$$\begin{aligned} \|\Gamma_R(u_1) - \Gamma_R(u_2)\|_{L^2(\Omega;C(I_j;H^s_x))} &\le C_2(TR^{k-1} + T^{\frac{1}{2}} R^{m-1}) \|u_1 - u_2\|_{L^2(\Omega;C(I_j;H^s_x))} \\ &\le \tfrac{1}{2} \|u_1 - u_2\|_{L^2(\Omega;C(I_j;H^s_x))}, \end{aligned}$$

allowing us to extend the solution u onto $I_j = [jT, (j+1)T]$ for any $j \in \mathbb{N}$. This completes the proof of Proposition 2.2. $\square$

2.3. **Proof of Theorem 1.1.** Given $R > 0$, let u_R be the global solution to the truncated problem (2.9) constructed in the previous subsection. Define a stopping time t_R by

$$t_R = \inf\{t \ge 0 : \|u_R\|_{C([0,t];H^s)} \ge R\}. \tag{2.23}$$

In view of (2.11), we see that $t_R > 0$, provided that $R > \|(\phi_0, \phi_1)\|_{\mathcal{H}^s}$.

It follows from the definition of the cutoff function η_R that u_R is a solution to the untruncated SNLW (1.4) on $[0, t_R]$. Since t_R is non-decreasing in R, we can define a stopping time

$$t_* = \lim_{R \to \infty} t_R. \tag{2.24}$$

Moreover, note that, given $\widetilde{R} \ge R$, we have $u_{\widetilde{R}} = u_R$ on $[0, t_R]$. This allows us to define u on $[0, t_*]$ by $u = u_R$ on $[0, t_R]$. In particular, u is a solution to (1.4) on $[0, t_*]$. It follows from the proof of Proposition 2.2 that the solution u is unique in $C([0, t_R]; H^s)$ for any $R > 0$ and hence u is unique in $C([0, t_*); H^s)$. Lastly, the blowup alternative (1.8) follows from (2.23) and (2.24).

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