The Derivative Nonlinear Schrödinger Equation with periodic boundary conditions

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Introduction

The derivative nonlinear Schrödinger equation (DNLS) was first given by Rogister [Rog71] as a model for the propagation of Alfvén waves in magnetized plasma with constant magnetic field, but appeared in other contexts as well and is now regarded as one of the canonical nonlinear equations in physics. It also plays a special role since it is among a few equations that is also completely integrable. In this direction, Kaup and Newell [KN78] applied the inverse scattering method to solve the equation with vanishing boundary conditions and they also point out the existence of solitons for this equation.

The linear part of this evolution equation is a Schrödinger dispersion which is counteracted by the steepening tendency of the cubic nonlinearity with one spatial derivative. It is therefore a semi-linear evolution equation which shares features with an algebraic NLS (both having second order dispersion, and through a nonlinear transformation, DNLS resembles a perturbed focusing quintic NLS), but is also close to the KdV equation (which is also completely integrable and has derivative in the nonlinearity).

This equation was studied extensively in the last two decades both in the Euclidean setting (spatial variable being allowed on the entire real line) and in the periodic setting (when the spatial variable is constricted to a bounded interval). Understanding the behavior of solutions corresponding to these two settings is of interest on its own; however, periodic solutions are of interest partly because numerical simulations typically use periodic boundary conditions.

The scope of this report is to review and clarify the application of the I-method in the periodic setting as was performed in [Win10], where it is claimed that the equation is globally well-posed in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$. Specifically, we would like to investigate here whether the frequency restriction on the derivative-cubic nonlinearity estimate is essential or not and, if possible, to quantify the mass threshold that is involved in the smallness condition required on the initial data.

The state-of-the-art approach to the well-posedness theory for this problem is considered to be the work of the "I-team" in [CKS+01, CKS+02] obtaining global well-posedness first for $\frac{2}{3} < s < 1$ and then by using a refinement of the same method down to $s > \frac{1}{2}$, both under the same smallness of mass condition (see assumption (A)). The end-point case $s = \frac{1}{2}$ was settled in [MWX11] also under (A). The *I-method*, or *the almost conserved energy method* is itself a refinement of Bourgain's high-low method in which instead of a smooth cutoff (as the I-operator does), the initial data is sharply truncated in the frequency space. Bourgain's method was applied for the DNLS on the real line by Takaoka [Tak01], and it turned out to be less successful (the regularity managed was $s > \frac{32}{33}$) than the CKSTT's method.

It is worth emphasizing that the application of the I-method in the non-periodic setting heavily relies on the local smoothing and maximal function estimates, as well as a refined bi-linear estimate which are not available in the periodic setting. Also, on the periodic box, Strichartz estimates for the free (linear) evolution are not available (mainly due to the lack of dispersive properties when one restricts the spatial variable to a bounded domain), the only exception being the L^4-Strichartz estimate which was proved in a purely Fourier analytic method by Bourgain (1993). Hence, at the technical level the two settings need to be treated separately. The technical tool that is used in the periodic setting is a bi-linear L^2-Strichartz estimate (Lemma 2.3.6) which provides a constant that decays as the spatial domain inflates. The result that establishes the global well-posedness in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$ (Theorem 2.12.2) also provides a mass threshold which was previously not quantified. The report concludes with a remark on potential improvements/continuations of the analysis of this equation.

Chapter 1

DNLS on $\mathbb{R}$

1.1 Basic features of the equation

Consider the derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u = i\mu\, \partial_x(|u|^2 u) \tag{DNLS}$$

where $t \in \mathbb{R}$ and either $x \in \mathbb{R}$ or $x \in \mathbb{T}$ (the periodic setting); the unknown $(t,x) \mapsto u(t,x)$ is $\mathbb{C}$-valued.

Without loss of generality, we may assume that $\mu = 1$. Indeed, if $\mu < 0$, we may use the time-reversal transformation $u(t,x) \mapsto \overline{u(-t,x)}$ and for any $\mu > 0$ we can use the rescaling $u(t,x) \mapsto \mu^{-1/2}u(t,x)$.

The nonlinearity can be easily expanded: $\partial_x(|u|^2u) = 2|u|^2\partial_x u + u^2\partial_x\overline{u}$. The first term is considerably worse than the second one since for the later there are no known estimates.

The associated Cauchy problem

$$\begin{cases} i\partial_t u + \partial_x^2 u = i(2|u|^2\partial_x u + u^2\partial_x\overline{u}) \\ u_{|t=0} = u_0 \end{cases} \tag{1.1}$$

with initial data $u_0 \in H^s$, for some $s \in \mathbb{R}$, has been studied extensively by various authors in the past two decades, both in the Euclidean and periodic settings.

Scaling. It is clear that the linear Schrödinger equation is invariant under the transformation $u(t,x) \mapsto \lambda^c\, u(\lambda^2 t, \lambda x)$, for any $\lambda > 0$ and any $c \in \mathbb{R}$. The presence of the nonlinearity forces a concrete value of c, namely the equation (DNLS) is invariant under the scaling transformation

$$u(t,x) \mapsto \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x) =: u_\lambda(t,x) \tag{1.2}$$

The homogeneous Sobolev norm that does not "see" this transformation is that of $L^2(\mathbb{R})$, i.e. $\|u_\lambda(t)\|_{L^2} = \|u(t)\|_{L^2}$ at any time t. Thus, *the problem* (1.1) *is* L^2*-critical.*

Complete integrability. For this equation new conserved quantities may be written down by identifying coefficients in the series expansion of $\ln a(\lambda)$, where $a(\lambda)$ is a scattering coefficient (see [KN78]). Alternatively, by Noether's theorem, if a quantity $Q(u)$ Poisson bracket commutes with the Hamiltonian $H(u)$ of the equation, then Q is conserved by the H-flow (see [Tao07, Thm. 1.29]).

Conserved quantities. For the present study, the consequence of the complete integrability that we are using is the presence of an infinite family of conservation laws for (DNLS) (cf. [KN78]). Among them we have the three important functionals:

$$M(u) := \int_{\mathbb{R}} |u|^2 dx \qquad \text{(Mass)}$$

$$P(u) := \int_{\mathbb{R}} \left(\frac{1}{2}|u|^4 + \mathrm{Im}(u\,\partial_x\overline{u})\right) dx \qquad \text{(Momentum)}$$

$$E(u) := \int_{\mathbb{R}} \left(|\partial_x u|^2 + \frac{1}{2}|u|^6 + \frac{3}{2}\,|u|^2\,\mathrm{Im}(u\,\partial_x\overline{u})\right) dx \qquad \text{(Energy)}$$

Gauge transformations. The nonlinear map $\mathcal{G}_\nu : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\mathcal{G}_\nu f(x) := e^{-i\nu\int_{-\infty}^x |f(y)|^2 dy} f(x) \tag{1.3}$$

(where $\nu \in \mathbb{R}$), is usually employed to transform the (DNLS) into other Schrödinger-type equations (and was used in [Oza96, CKS+01, CKS+02, MWX11]). It is immediate that $|\mathcal{G}_\nu f(x)| = |f(x)|$ (consequently $\|\mathcal{G}_\nu f\|_{L^p} = \|f\|_{L^p}$ for any p) and we can invert the transformation via

$$\mathcal{G}_\nu^{-1} g(x) = e^{i\nu\int_{-\infty}^x |g(y)|^2 dy} g(x)$$

Moreover, $\mathcal{G}_\nu$ maps $H^s(\mathbb{R})$ into itself continuously (and so does $\mathcal{G}_\nu^{-1}$) for any $0 \leqslant s \leqslant 1$ (cf. [CKS+01, Lem. 3.2]).

First, by multiplying (DNLS) with $-i\,\overline{u}$, and conjugating respectively, we get

$$(\partial_t u)\overline{u} - i(\partial_x^2 u)\overline{u} = \partial_x(|u|^2 u)\overline{u} \quad , \quad (\partial_t \overline{u})u + i(\partial_x^2 \overline{u})u = \partial_x(|u|^2\overline{u})u \ .$$

It follows that $\partial_t(u\overline{u}) - i\,(\partial_x(\partial_x u\,\overline{u}) - \partial_x(\partial_x\overline{u}\,u)) = \frac{3}{2}\partial_x(|u|^4)$ or equivalently

$$\partial_t(|u|^2) = 2\partial_x \mathrm{Im}(u\partial_x\overline{u}) + \frac{3}{2}\partial_x(|u|^4) \ . \tag{1.4}$$

Next, set $w(t) := \mathcal{G}_\nu u(t)$ and by straightforward computations (easily justified for $u \in \mathcal{S}_{t,x}$) we successively obtain:

$$\begin{aligned}
\partial_t w &= e^{-i\nu\int_{-\infty}^x |u|^2 dy}\left[\partial_t u - i\nu\, u\int_{-\infty}^x \partial_t|u|^2 dy\right] \\
u\partial_x\overline{u} &= w\partial_x\overline{w} - i\nu|w|^4 \quad ; \quad \mathrm{Im}(u\partial_x\overline{u}) = \mathrm{Im}(w\partial_x\overline{w}) - \nu|w|^4 \\
i\partial_t u &= e^{i\nu\int_{-\infty}^x |w|^2 dy}\left[i\partial_t w - \nu\left(\frac{3}{2} - \nu\right)|w|^4 w - 2\nu\mathrm{Im}(w\partial_x\overline{w})\right] \\
\partial_x^2 u &= e^{i\nu\int_{-\infty}^x |w|^2 dy}\left[\partial_x^2 w + i\nu\partial_x(|w|^2 w) + i\nu|w|^2\partial_x w - \nu^2|w|^4 w\right] \\
i\partial_x(|u|^2 u) &= e^{i\nu\int_{-\infty}^x |w|^2 dy}\left[i\partial_x(|w|^2 w) - \nu|w|^4 w\right]
\end{aligned}$$

Thus u solves (DNLS) if and only if w solves

$$i\partial_t w + \partial_x^2 w \;=\; i(1-\nu)\,\partial_x(|w|^2 w) \;-\; i\nu\, w^2\partial_x\overline{w} \;+\; \nu\left(\frac{1}{2} - \nu\right)|w|^4 w \ . \tag{DNLS$_\nu$}$$

We will also find it useful to have how the conserved quantities translate under the gauge transform.

$$P(u) \;=\; \left(\frac{1}{2} - \nu\right)\|w\|_{L^4}^4 + \int_{\mathbb{R}} \mathrm{Im}(w\partial_x\overline{w})dx \tag{1.5}$$

$$E(u) \;=\; \|\partial_x w\|_{L^2}^2 + (\nu^2 - \frac{3}{2}\nu + \frac{1}{2})\|w\|_{L^6}^6 + \left(\frac{3}{2} - 2\nu\right)\int_{\mathbb{R}} |w|^2\mathrm{Im}(w\partial_x\overline{w})dx \tag{1.6}$$

We introduce the following notations:

$$P_{gDNLS_\nu} = P \circ \mathcal{G}_{-\nu} \qquad , \qquad E_{gDNLS_\nu} := E \circ \mathcal{G}_{-\nu}$$

and if used E_{DNLS}, P_{DNLS} refer to the same functionals as E, P respectively.

Choices of $\nu = \frac{1}{2}$, $\nu = \frac{3}{4}$ and respectively $\nu = 1$ have been used in the literature, and the corresponding equations are:

$$(\text{DNLS}_{\frac{1}{2}}) \qquad i\partial_t w + \partial_x^2 w = i\,|w|^2 \partial_x w$$

$$(\text{DNLS}_{\frac{3}{4}}) \qquad i\partial_t w + \partial_x^2 w = \frac{i}{2}|w|^2\partial_x w - \frac{i}{2} w^2 \partial_x \overline{w} - \frac{3}{16}|w|^4 w$$

$$(\text{DNLS}_1) \qquad i\partial_t w + \partial_x^2 w = -i\, w^2 \partial_x \overline{w} - \frac{1}{2}|w|^4 w$$

Thus the well-posedness analysis of (1.1) is equivalent with that of the Cauchy problem formed with any of ($\text{DNLS}_{\frac{1}{2}}$), ($\text{DNLS}_{\frac{3}{4}}$) or ($\text{DNLS}_1$) and initial data $w_0 := \mathcal{G}u_0$; notice that (A) is equipvalent with $\|w_0\|_{L^2} < \sqrt{2\pi}$. While ($\text{DNLS}_{\frac{1}{2}}$) looks simpler than ($\text{DNLS}_1$), the term $|w|^2\partial_x\overline{w}$ is nicer in the analysis than $|w|^2\partial_x w$; the choice $\nu = \frac{3}{4}$ (also used in [Wu13, Sect. 4] in the half-line setting), is particularly useful to simplify the energy functional to $E_{\text{gDNLS}_{\frac{3}{4}}}(w) = \|\partial_x w\|_{L^2}^2 - \frac{1}{16}\|w\|_{L^6}^6$.

Assumption. Many of the results in the literature work under the following smallness condition on the initial data

$$(\text{A}) \qquad \|u_0\|_{L^2}^2 < 2\pi$$

which has the following immediate consequence:

Lemma 1.1.1. *Suppose $u_0 \in H^1(\mathbb{R})$ satisfies (A). Then*

$$\|u_0\|_{L^6}^6 + \|\partial_x u_0\|_{L^2}^2 \lesssim_{\|u_0\|_{L^2},\nu} E_{gDNLS_\nu}(u_0).$$

In particular, $E_{DNLS}(u_0) > 0$.

Proof. By taking $w_0 = \mathcal{G}_{\frac{3}{4}-\nu}u_0$, we have $\|w_0\|_{L^2} = \|u_0\|_{L^2}$ and

$$E_{gDNLS_\nu}(u_0) = E_{gDNLS_{\frac{3}{4}}}(w_0) = \|\partial_x w_0\|_{L^2}^2 - \frac{1}{16}\|w_0\|_{L^6}^6$$

The sharp Gagliardo-Nirenberg inequality $\|f\|_{L^6}^6 \leqslant \frac{4}{\pi^2}\|f\|_{L^2}^4\|\partial_x f\|_{L^2}^2$, allows us to further get

$$E_{gDNLS_\nu}(u_0) \geqslant \|\partial_x w_0\|_{L^2}^2 - \frac{1}{4\pi^2}\|w_0\|_{L^2}^4\|\partial_x w_0\|_{L^2}^2 = \|\partial_x w_0\|_{L^2}^2\left(1 - \left(\frac{\|u_0\|_{L^2}}{\sqrt{2\pi}}\right)^4\right) ,$$

$$E_{gDNLS_\nu}(u_0) \geqslant \frac{\pi^2}{4}\frac{\|w_0\|_{L^6}^6}{\|w_0\|_{L^2}^4} - \frac{1}{16}\|w_0\|_{L^6}^6 = \|w_0\|_{L^6}^6\frac{1}{16}\left(\left(\frac{\sqrt{2\pi}}{\|u_0\|_{L^2}}\right)^4 - 1\right)$$

and also that

$$\|\partial_x u_0\|_{L_x^2} = \|(\nu - \frac{3}{4})|w_0|^2 w_0 + \partial_x w_0\|_{L_x^2} \lesssim_\nu \|w_0\|_{L_x^6}^3 + \|\partial_x w_0\|_{L_x^2} \lesssim_{\nu,\|w_0\|_{L_x^2}} \|\partial_x w_0\|_{L_x^2}.$$

Taking into account (A) and $\|u_0\|_{L^6} = \|w_0\|_{L^6}$, the conclusion follows. □

1.2 Review of well-posedness results in the non-periodic setting

The local well-posedness theory in the energy space $H^1(\mathbb{R})$ is credited to Hayashi-Ozawa. In [Tak99], Takaoka showed that (1.1) is locally well-posed in H^s for $s \geqslant \frac{1}{2}$. Previously, there were the works of Tsutsumi and Fukuda [TF80, TF81]. This result is sharp since for $s < \frac{1}{2}$, in [Tak01] it is shown that the solution map $u_0 \mapsto u(t)$ fails to be C^3, while [BL01, Theorem 1.2] proves that the solution map is not uniformly continuous. Consequently, one cannot construct solutions $u(t) \in H^s(\mathbb{R})$ by the standard fixed point argument below $s = \frac{1}{2}$.

Theorem 1.2.1 (Takaoka, 1999)**.** *Let $s \geqslant \frac{1}{2}$. The Cauchy problem associated to the gauged DNLS equation* (DNLS$_1$) *is (unconditionally) locally well-posed in $H^s(\mathbb{R})$. More precisely, for any $R > 0$, there exists $T = T(R^{-\theta}) > 0$ such that for all $u_0 \in H^s(\mathbb{R})$ with $\|u_0\|_{H^s(\mathbb{R})} \leqslant R$ there exists a unique solution $u \in C([-T, T], H^s(\mathbb{R}))$ of* (DNLS$_1$)*. Moreover the solution map $\Phi : u_0 \mapsto u$ is Lipschitz continuous.*

Global existence of solutions in the Schwartz class was obtained by Lee in [Lee89] and also in [HO92]. Global well-posedness in the energy space was obtained by Hayashi in [Hay93] under the assumption that $\|u_0\|_{L^2}$ is sufficiently small.

The state-of-the-art approach to the well-posedness theory for this problem is considered to be the work of the "I-team" in [CKS+01, CKS+02] obtaining global well-posedness first for $\frac{2}{3} < s < 1$ and then by using a refinement of the same method down to $s > \frac{1}{2}$, both under (A). The end-point case $s = \frac{1}{2}$ was settled in [MWX11] also under (A). The *I-method*, or *the almost conserved energy method* is itself a refinement of Bourgain's high-low method in which instead of smooth cutoff, the initial data is sharply truncated in the frequency space. Bourgain's method was applied for DNLS by Takaoka [Tak01], turning out to be less successful since the regularity managed was $s > \frac{32}{33}$.

Recently, Wu in [Wu13, Wu14] improves the mass threshold for the GWP in $H^1(\mathbb{R})$ for initial data with mass up to 4π. He also remarks that the situation is different in the half-line setting where blow-up solutions with negative energy (thus with $\|u_0\|_{L^2}^2 > 2\pi$) do occur.

Chapter 2

DNLS on $\mathbb{T}$

The local well-posedness in $H^s(\mathbb{T})$ for $s \geqslant \frac{1}{2}$ was established by Herr in [Her06b, Her06a]. Global well-posedness in $H^s(\mathbb{T})$ was proved/attempted by Win for $s > \frac{1}{2}$ assuming smallness on the L^2-norm of the initial data (the threshold not being quantified); see [Win10]. It seems that the expression of the almost conserved energy that is used there is incomplete: a term coming from the particular modification of the gauge transform is missing and this quantity is not conserved (see Remark 2.1.2 and Lemma 2.1.3 below).

We aim to establish when the local-in-time H^s-solutions exist globally in time. The necessary and sufficient condition for $u \in C([-T,T]; H^s(\mathbb{T}))$ as in [Her06a, Theorem 1.1] to be extended to a global solution in $C(\mathbb{R}; H^s(\mathbb{T}))$ is

$$\sup_{-T \leqslant t \leqslant T} \|u(t)\|_{\dot{H}^s_x} < \infty, \text{ for all } T > 0. \tag{2.1}$$

Since DNLS enjoys the time-reversibility symmetry, we can concentrate on proving that the $\dot{H}^s_x$-norm of the solution stays finite on any time interval $[0,T]$.

2.1 Gauge transformation

Because we need to make use of the natural scaling (1.2) of the equation, we work on $\mathbb{T}_\lambda = \mathbb{R}/2\pi\lambda\mathbb{Z} \simeq [0, 2\pi\lambda)$ (for simplicity we set $\mathbb{T} = \mathbb{T}_1$). Then, the Fourier modes belong to the lattice $\mathbb{Z}_\lambda := \frac{1}{\lambda}\mathbb{Z}$. The conventions for the Fourier transform, H^s and $X^{s,b}$ definitions are as in [CKS$^+$03, Sect. 7]; see also section 2.2 below.

In order to deal with the derivative in the nonlinearity, we use the following gauge transformation for the periodic problem, which was introduced in [Her06a, Her06b] when proving local well-posedness for DNLS in $H^s(\mathbb{T})$, $\frac{1}{2} \leqslant s < 1$:

$$\mathcal{G}_\nu : L^2(\mathbb{T}_\lambda) \to L^2(\mathbb{T}_\lambda) \quad , \quad \mathcal{G}_\nu(f)(x) := e^{-i\nu\mathcal{I}(f)(x)} f(x) \quad , \tag{2.2}$$

where

$$\mathcal{I}(f)(x) := \frac{1}{2\pi\lambda} \int_0^{2\pi\lambda} \int_\theta^x |f(y)|^2 - \frac{1}{2\pi\lambda}\|f\|^2_{L^2(\mathbb{T}_\lambda)}\, dy\, d\theta$$

is the antiderivative of the mean-zero function $|f|^2 - \mu(f)$, where

$$\mu(f) := \frac{1}{2\pi\lambda}\|f\|^2_{L^2(\mathbb{T}_\lambda)} \tag{2.3}$$

(notice that $\mu(f) = \mu(\mathcal{G}_\nu(f))$ so we might write in short μ instead of $\mu(f)$). Indeed, since

$$\int_x^{x+2\pi\lambda} |f(y)|^2 - \frac{1}{2\pi\lambda}\|f\|^2_{L^2(\mathbb{T}_\lambda)}\, dy = 0 \ ,$$

we get that $\mathcal{I}(f)$ is $2\pi\lambda$-periodic, provided that $f \in L^2(\mathbb{T}_\lambda)$. Hence $\mathcal{G}_\nu$ is well-defined. We notice that $\mathcal{G}_\nu$

is an isometry on $L^2(\mathbb{T}_\lambda)$ and its inverse is $\mathcal{G}_{-\nu}$.

It can be shown that $\mathcal{G}_\nu$ is bi-Lipschitz from H^s into itself, for any $s \geqslant 0$ [Her06b, Ch. 3], hence any well-posedness result on a gauged DNLS can be stated for the original DNLS equation.

We recall the conserved quantities which are of interest in our analysis:

$$M(u) := \int_{\mathbb{T}_\lambda} |u|^2 dx \qquad \text{(Mass)}$$

$$H(u) := \int_{\mathbb{T}_\lambda} \left(\frac{1}{2}|u|^4 + \operatorname{Im}(u\,\partial_x\overline{u})\right) dx \qquad \text{(Hamiltonian)}$$

$$E(u) := \int_{\mathbb{T}_\lambda} \left(|\partial_x u|^2 + \frac{1}{2}|u|^6 + \frac{3}{2}\,|u|^2 \operatorname{Im}(u\,\partial_x\overline{u})\right) dx \qquad \text{(Energy)}$$

We would like to see how the equation and these quantities transform under $\mathcal{G}_\nu$. So assume u has enough regularity to justify all the calculus below and let $w(t,x) := \mathcal{G}_\nu(u(t))(x)$. We use the shorthand u_x instead of $\partial_x u$ and the obvious analogues

$$\begin{aligned} u_x &= e^{i\nu\mathcal{I}(w(t))}\left(i\nu(\mathcal{I}(w))_x w + w_x\right) \\ u_t &= e^{i\nu\mathcal{I}(w(t))}\left(i\nu(\mathcal{I}(w))_t w + w_t\right) \end{aligned}$$

Therefore, we have

$$(\mathcal{I}(w))_x = \left(\frac{1}{2\pi\lambda}\int_0^{2\pi\lambda} |w(t,x)|^2 - \frac{1}{2\pi\lambda}\|w(t)\|^2_{L^2(\mathbb{T}_\lambda)}\,d\theta\right) = |w|^2 - \mu$$

which leads to

$$u_x = e^{i\nu\mathcal{I}(w(t))}\left[w_x + i\nu(|w|^2-\mu)w\right]$$

and

$$\operatorname{Im}(u\overline{u}_x) = \operatorname{Im}\left(w\overline{w}_x - i\nu(|w|^2-\mu)|w|^2\right) = \operatorname{Im}(w\overline{w}_x) - \nu|w|^4 + \nu\mu|w|^2$$

Using

$$\begin{aligned} \partial_t|u|^2 &= 2\operatorname{Re}(u_t\,\overline{u}) = 2\operatorname{Re}\left(iu_{xx}\overline{u} + (|u|^2u)_x\overline{u}\right) \\ &= 2\operatorname{Re}\left(i(u_x\overline{u})_x - iu_x\overline{u}_x + (|u|^2u\overline{u})_x - |u|^2u\overline{u}_x\right) \\ &= 2(\operatorname{Im}(u\overline{u}_x))_x + 2(|u|^4)_x - \frac{1}{2}(|u|^4)_x \\ &= \partial_x\left(2\operatorname{Im}(u\overline{u}_x) + \frac{3}{2}|u|^4\right) \end{aligned}$$

we deduce

$$\begin{aligned} (\mathcal{I}(w))_t = \partial_t(\mathcal{I}(u)) &= 2\operatorname{Im}(u\overline{u}_x) + \frac{3}{2}|u|^4 - \frac{1}{2\pi\lambda}\int_0^{2\pi\lambda}\left(2\operatorname{Im}(u\overline{u}_x) + \frac{3}{2}|u|^4\right)(t,\theta)\,d\theta \\ &= 2\left(\operatorname{Im}(w\overline{w}_x) - \nu|w|^4 + \nu\mu|w|^2\right) + \frac{3}{2}|w|^4 - \phi(w), \end{aligned}$$

where

$$\phi(w) := \fint_{\mathbb{T}_\lambda} 2\left(\operatorname{Im}(w\overline{w}_x) - \nu|w|^4 + \nu\mu|w|^2\right)(t,\theta) + \frac{3}{2}|w|^4(t,\theta)d\theta.$$

This gives the first term of the equation under the gauge transform

$$iu_t = e^{i\nu\mathcal{I}(w)}\left(iw_t - \nu\left(2\operatorname{Im}(w\overline{w}_x) + (\frac{3}{2} - 2\nu)|w|^4 + 2\nu\mu|w|^2 - \phi(w)\right)w\right)$$

For the second term of the equation, we have

$$\begin{aligned}
u_{xx} &= e^{i\nu\mathcal{I}(w)}\big(i\nu\mathcal{I}(w)_x\left(i\nu(\mathcal{I}(w))_x w + w_x\right) + i\nu(\mathcal{I}(w)_x w)_x + w_{xx}\big)\\
&= e^{i\nu\mathcal{I}(w)}\big(-\nu^2(\mathcal{I}(w)_x)^2 w + 2i\nu\mathcal{I}(w)_x w_x + i\nu\mathcal{I}(w)_{xx}w + w_{xx}\big)\\
&= e^{i\nu\mathcal{I}(w)}\big(w_{xx} - \nu^2(|w|^4 w - 2\mu|w|^2 w + \mu^2 w) + i\nu(3|w|^2 w_x + w^2\overline{w}_x - 2\mu w_x)\big)
\end{aligned}$$

The nonlinearity transforms into

$$\begin{aligned}
i(|u|^2u)_x &= e^{i\nu\mathcal{I}(w)}\big(-\nu\mathcal{I}(w)_x|w|^2 w + i(|w|^2 w)_x\big)\\
&= e^{i\nu\mathcal{I}(w)}\big(-\nu|w|^4 w + \nu\mu_0|w|^2 w + 2i|w|^2 w_x + iw^2\overline{w}_x\big)
\end{aligned}$$

The equation satisfied by w is

$$\begin{aligned}
iw_t + w_{xx} &= \nu\left(i|w|^2 w_x - iw^2\overline{w}_x + (\frac{3}{2} - 2\nu)|w|^4 w + 2\nu\mu|w|^2 w - \phi(w)w\right)\\
&\quad + \nu^2(|w|^4 w - 2\mu|w|^2 w + \mu^2 w) - i\nu(3|w|^2 w_x + w^2\overline{w}_x - 2\mu w_x)\\
&\quad + \big(-\nu|w|^4 w + \nu\mu|w|^2 w + 2i|w|^2 w_x + iw^2\overline{w}_x\big)\\
&= 2i(1-\nu)|w|^2 w_x + i(1-2\nu)w^2\overline{w}_x + \nu\mu|w|^2 w + \nu(\frac{1}{2} - \nu)|w|^4 w\\
&\quad + 2i\nu\mu w_x - \nu(\phi(w) - \nu\mu^2)w
\end{aligned}$$

Denoting

$$\psi(w) := \nu\phi(w) - \nu^2\mu^2 = \nu \fint_{\mathbb{T}_\lambda}\left(2\mathrm{Im}(w\overline{w}_x) + (\frac{3}{2} - 2\nu)|w|^4\right)(t,\theta)d\theta + \nu^2\mu^2\ ,$$

we can write

$$iw_t + w_{xx} - 2i\nu\mu w_x + \psi(w)w = 2i(1-\nu)|w|^2 w_x + i(1-2\nu)w^2\overline{w}_x + \nu\mu|w|^2 w + \nu(\frac{1}{2} - \nu)|w|^4 w$$

We cancel the linear term $2i\nu\mu w_x$ by using the transformation $w(t,x) \mapsto v(t, x + 2\nu\mu t)$, and so

$$(\mathcal{G}\mathrm{DNLS}_\nu) \qquad iv_t + v_{xx} + \psi(v)v = 2i(1-\nu)|v|^2 v_x + i(1-2\nu)v^2\overline{v}_x + \nu\mu|v|^2 v + \nu(\frac{1}{2} - \nu)|v|^4 v$$

For the conserved quantities listed above, we obtain $M(u) = M(w)$,

$$\begin{aligned}
H(u) &= \int_{\mathbb{T}_\lambda}\frac{1}{2}|w|^4 + \mathrm{Im}(w\overline{w}_x) - \nu|w|^4 + \nu\mu|w|^2\\
&= \left(\frac{1}{2} - \nu\right)\|w\|^4_{L^4(\mathbb{T})} + \int_{\mathbb{T}_\lambda}\mathrm{Im}(w\overline{w}_x)dx + \lambda\nu\mu^2
\end{aligned}$$

and

$$\begin{aligned}
E(u) &= \int_{\mathbb{T}_\lambda}|w_x + i\nu(|w|^2 - \mu)w|^2 + \frac{1}{2}|w|^6 + \frac{3}{2}|w|^2\left(\mathrm{Im}(w\overline{w}_x) - \nu|w|^4 + \nu\mu|w|^2\right)\\
&= \|w_x\|^2_{L^2(\mathbb{T}_\lambda)} + \left(\nu^2 - \frac{3}{2}\nu + \frac{1}{2}\right)\|w\|^6_{L^6(\mathbb{T}_\lambda)} + \left(\frac{3}{2} - 2\nu\right)\int_{\mathbb{T}_\lambda}|w|^2\mathrm{Im}(w\overline{w}_x)\\
&\quad + \mu\nu\left(\frac{3}{2} - 2\nu\right)\|w\|^4_{L^4} + 2\mu\nu\int_{\mathbb{T}_\lambda}\mathrm{Im}(w\overline{w}_x)dx + \lambda\mu^3\nu^2\\
&= \|w_x\|^2_{L^2(\mathbb{T}_\lambda)} + \left(\frac{3}{2} - 2\nu\right)\int_{\mathbb{T}_\lambda}|w|^2\mathrm{Im}(w\overline{w}_x) + \left(\nu^2 - \frac{3}{2}\nu + \frac{1}{2}\right)\|w\|^6_{L^6(\mathbb{T}_\lambda)} + \frac{\nu}{2}\mu\|w\|^4_{L^4(\mathbb{T}_\lambda)}\\
&\quad + 2\nu\mu H_\nu(w) - \lambda\nu^2\mu^3
\end{aligned}$$

By changing the variable of integration, we note that all the conserved quantities are invariant under the space translation transformation $w(t,x) \mapsto v(t, x+2\nu\mu t)$.

We introduce the functionals

$$
\begin{aligned}
(2.4)\qquad & H_\nu(v) := \mathrm{Im}\left(\int_{\mathbb{T}_\lambda} v\overline{v}_x dx\right) + \left(\frac{1}{2}-\nu\right)\|v\|^4_{L^4(\mathbb{T})} \\
& E_\nu(v) := \|v_x\|^2_{L^2(\mathbb{T}_\lambda)} + \left(\nu^2 - \frac{3}{2}\nu + \frac{1}{2}\right)\|v\|^6_{L^6(\mathbb{T}_\lambda)} + \left(\frac{3}{2}-2\nu\right)\int_{\mathbb{T}_\lambda}|v|^2\mathrm{Im}(v\overline{v}_x) + \frac{\nu}{2}\mu\|v\|^4_{L^4(\mathbb{T}_\lambda)}
\end{aligned}
$$

and we call them respectively the momentum and the energy of ($\mathcal{G}$DNLS$_\nu$). Since $\mu(v(t))$ and $H_\nu(v(t))$ are conserved, the above two quantities are also conserved provided v is a smooth enough solution of (2.8).

Lemma 2.1.1. *Let $f \in H^1(\mathbb{T})$ with $\|f\|^2_{L^2(\mathbb{T})} < \delta$ for some $\delta > 0$ small enough. Then, we have*

$$\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} \lesssim E_\nu(f)\ . \tag{2.5}$$

Moreover, the implicit constant above is independent of λ and we can take $\delta = 2\sqrt{2}$.

Proof. Consider $g := \mathcal{G}_{\frac{3}{4}-\nu} f$. Then $\|g\|_{L^2(\mathbb{T}_\lambda)} = \|f\|_{L^2(\mathbb{T}_\lambda)}$ and

$$E_\nu(f) = E_0(\mathcal{G}_{-\nu}f) = E_0(\mathcal{G}_{-\nu}\mathcal{G}_{\nu-\frac{3}{4}}g) = E_0(\mathcal{G}_{-\frac{3}{4}}g) = E_{\frac{3}{4}}(g).$$

By Cauchy-Schwartz and Gagliardo-Nirenberg inequalities (see Lemma 2.3.5), we have

$$\|\partial_x f\|^2_{L^2(\mathbb{T}_\lambda)} \lesssim \left(1 + (\nu - \frac{3}{4})\|f\|^2_{L^2}\right)^2 \|\partial_x f g|^2_{L^2(\mathbb{T}_\lambda)}. \tag{2.6}$$

It remains to show that

$$\|\partial_x g|^2_{L^2(\mathbb{T}_\lambda)} \lesssim E_{\frac{3}{4}}(g). \tag{2.7}$$

Gagliardo-Nirenberg implies

$$\|g\|^6_{L^6} \leqslant \left(\|\partial_x g\|_{L^2}\|g\|^2_{L^2} + \frac{1}{\sqrt{2\pi\lambda}}\|g\|_{L^2}\|g\|^2_{L^4}\right)^2 \leqslant 2\left(\|\partial_x g\|^2_{L^2}\|g\|^4_{L^2} + \frac{1}{2\pi\lambda}\|g\|^2_{L^2}\|g\|^4_{L^4}\right)$$

and therefore

$$
\begin{aligned}
E_{\frac{3}{4}}(g) \geqslant & \|\partial_x g\|^2_{L^2} - \frac{1}{8}\|\partial_x g\|^2_{L^2}\|g\|^4_{L^2} - \frac{1}{16\pi\lambda}\|g\|^2_{L^2}\|g\|^4_{L^4} + \frac{3}{16\pi\lambda}\|g\|^2_{L^2}\|g\|^4_{L^4} \\
= & \|\partial_x g\|^2_{L^2}\left(1 - \frac{1}{8}\|g\|^4_{L^2}\right) + \frac{1}{8\pi\lambda}\|g\|^2_{L^2}\|g\|^4_{L^4} \\
\geqslant & (1 - \frac{\|g\|^4_{L^2}}{8})\|\partial_x g\|^2_{L^2}
\end{aligned}
$$

Hence, as long as $\|g\|^2_{L^2} < 2\sqrt{2}$, we can obtain (2.7). □

In what follows, we work with $\nu = 1$ so that the "bad term" $|v|^2 v_x$ doesn't appear in the gauged equation ($\mathcal{G}$DNLS$_\nu$). We have

$$v_t - iv_{xx} = -v^2\overline{v}_x - i\mu|v|^2 v + \frac{i}{2}|v|^4 v + i\psi(v)v \tag{2.8}$$

with the corresponding energy functional (i.e. for $\nu = 1$, as defined in (2.4))

$$E(v) = \int_{\mathbb{T}_\lambda}\left(|v_x^2| - \frac{1}{2}|v|^2\mathrm{Im}(v\overline{v}_x) + \frac{1}{2}\mu(v)|v|^4\right)dx. \tag{2.9}$$

We can group the cubic and respectively quintic nonlinear terms in the right hand side of the equation and write

$$v_t - iv_{xx} = -\mathcal{T}(v) + \frac{i}{2}\mathcal{Q}(v) \ , \tag{2.10}$$

where

$$\begin{aligned} \mathcal{T}(v) &= \left(v\overline{v}_x - 2i \fint_{\mathbb{T}_\lambda} \mathrm{Im}(v\overline{v}_x)dx \right) v \quad , \\ \mathcal{Q}(v) &= \left(|v|^4 - \fint_{\mathbb{T}_\lambda} |v|^4 dx \right) v - 2\fint_{\mathbb{T}_\lambda} |v|^2 dx \left(|v|^2 - \fint_{\mathbb{T}_\lambda} |v|^2 dx \right) v. \end{aligned}$$

Remark 2.1.2. If v is a smooth solution of (2.8), $\|v\|_{L^4}$ is not necessarily conserved. Indeed,

$$\partial_t \|v\|_{L^4}^4 = 4\mathrm{Re} \int_{\mathbb{T}_\lambda} |v|^2 \overline{v} \partial_t v \, dx = 4\mathrm{Re} \int_{\mathbb{T}_\lambda} |v|^2 \overline{v} (iv_{xx} - \mathcal{T}(v) + \frac{i}{2}\mathcal{Q}(v)) \, dx = 4\mathrm{Re}\, i \int_{\mathbb{T}_\lambda} |v|^2 \overline{v} v_{xx} \, dx + h.o.t.$$

and

$$\mathrm{Re}\, i \int_{\mathbb{T}_\lambda} |v|^2 \overline{v} v_{xx} \, dx = \mathrm{Re}(-i) \int_{\mathbb{T}_\lambda} \partial_x (v\overline{v}^2) v_x \, dx = \mathrm{Im} \int_{\mathbb{T}_\lambda} (\overline{v}^2 v_x^2 + 2|v|^2 |v_x|^2) \, dx = \mathrm{Im} \int_{\mathbb{T}_\lambda} \overline{v}^2 v_x^2 \, dx$$

The higher order terms of $\partial_t \|v\|_{L^4}^4$ cannot cancel the fourth order term $4\mathrm{Im} \int_{\mathbb{T}_\lambda} \overline{v}^2 v_x^2 \, dx$.

However, by Sobolev embedding and interpolation of H^s spaces, we have

$$\|v\|_{L^4} \lesssim \|v\|_{H^{\frac{1}{4}}} \leqslant \|v\|_{L^2}^{\frac{3}{4}} \|v\|_{H^1}^{\frac{1}{4}}$$

and therefore

$$\frac{1}{2}\mu(v)\|v\|_{L^4} \lesssim \|v\|_{L^2}^5 \|v\|_{H^1} \lesssim \frac{1}{\varepsilon}\|v\|_{L^2}^{10} + \varepsilon \|v\|_{H^1}^2 \lesssim \frac{1}{\varepsilon}\|v\|_{L^2}^{10} + \varepsilon \|v\|_{L^2}^2 + \varepsilon \|\partial_x v\|_{L^2}^2. \tag{2.11}$$

We therefore consider the essential part of the energy functional (2.9), namely

$$\mathcal{E}(v) := \|\partial_x v\|_{L^2}^2 - \frac{1}{2}\mathrm{Im} \int_{\mathbb{T}_\lambda} |v| v \overline{v}_x \, dx \tag{2.12}$$

which in conduction with a mass term will still be able to control the square of the $\dot{H}^1$-norm of v. It is worthwhile mentioning that this is the same expression as the energy corresponding to (DNLS_1) on the real line.

Lemma 2.1.3. *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $f \in H^1(\mathbb{T}_\lambda)$ with $\|f\|_{L^2(\mathbb{T}_\lambda)}^2 < \delta$, we have*

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim \mathcal{E}(f) + \frac{1}{\varepsilon}\mu(f)^5 \tag{2.13}$$

with the implicit constant independent of λ. Moreover $\delta \nearrow 2\sqrt{2}$ as $\varepsilon \searrow 0$.

Proof. It follows from the proof of Lemma 2.1.1 and Remark 2.1.2. ∎

2.2 Fourier transform and periodic function spaces

The convention [1] we are using for the (spatial) Fourier transform of a $2\pi\lambda$-periodic function is

$$\widehat{f}(k) = \int_0^{2\pi\lambda} e^{-ikx} f(x)dx \quad , \quad k \in \mathbb{Z}_\lambda = \frac{1}{\lambda}\mathbb{Z}$$

which is inverted by

$$f(x) = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} e^{ikx} \widehat{f}(k) \quad , \quad x \in [0, 2\pi\lambda].$$

The convolution products on $\mathbb{T}_\lambda$ and $\mathbb{Z}_\lambda$ are computed by

$$f * g(x) = \int_0^{2\pi\lambda} f(x-y)g(y)\,dy \quad , \quad a \star b(k) = \frac{1}{2\pi\lambda} \sum_{h\in\mathbb{Z}_\lambda} a(k-h)b(h) \ ,$$

respectively. For clarity we should write $\star_\lambda$ for the convolution on $\mathbb{Z}_\lambda$ to emphasize the factor in front of the sum, but we ignore the subscript as we are using $\star$ instead of the regular $*$ symbol. As such,

$$\widehat{fg}(k) = \widehat{f} \star \widehat{g}(k)$$

By endowing $\mathbb{Z}_\lambda$ with $\frac{1}{2\pi\lambda}d\#$ (scaled counting measure), the $L^2(\mathbb{T}_\lambda)$ and $\ell^2(\mathbb{Z}_\lambda)$ inner products are

$$\langle f, g\rangle_{L^2(\mathbb{T}_\lambda)} = \int_0^{2\pi\lambda} f(x)\overline{g(x)}\,dx \quad , \quad \langle a, b\rangle_{\ell^2(\mathbb{Z}_\lambda)} = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} a(k)\overline{b(k)} \ .$$

Then, the Plancherel and Parseval identities are

$$\langle f, \check{a}\rangle_{L^2(\mathbb{T}_\lambda)} = \langle \widehat{f}, a\rangle_{\ell^2(\mathbb{Z}_\lambda)} \quad \Leftrightarrow \quad \int_0^{2\pi\lambda} f(x)\overline{\check{a}(x)}\,dx = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} \widehat{f}(k)a(k)$$

$$\|f\|_{L^2(\mathbb{T}_\lambda)} = \|\widehat{f}\|_{\ell^2(\mathbb{Z}_\lambda)} \quad \Leftrightarrow \quad \int_0^{2\pi\lambda} |f(x)|^2 dx = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} |\widehat{f}(k)|^2$$

The Sobolev space $H^s(\mathbb{T}_\lambda)$ is the completion of the $2\pi\lambda$-periodic C^∞ functions with respect to the norm given by

$$\|f\|^2_{H^s(\mathbb{T}_\lambda)} := \|\langle k\rangle^s \widehat{f}(k)\|^2_{\ell^2(\mathbb{Z}_\lambda)} = \frac{1}{2\pi\lambda} \sum_{k\in\mathbb{Z}_\lambda} \langle k\rangle^{2s} |\widehat{f}(k)|^2 \sim \frac{1}{\lambda} \sum_{k\in\mathbb{Z}_\lambda} \langle k\rangle^{2s} |\widehat{f}(k)|^2$$

We sometimes choose to ignore (powers of) 2π from every factor appearing in front of sums over (subsets of) $\mathbb{Z}_\lambda$ as they don't play a significant role in the estimates. However, we need to keep track of powers of λ as we will perform a rescaling of the associated Cauchy problem.

With a slight abuse of notation, the space-time Fourier transform of $u : \mathbb{R} \times \mathbb{T}_\lambda \to \mathbb{C}$ belonging to the class of Schwartz functions in t and $2\pi\lambda$-periodic C^∞ functions in x (class denoted $\mathcal{S}_{\text{per}}$) is

$$\widehat{u}(\tau, k) = \int_{\mathbb{R}\times\mathbb{T}_\lambda} e^{-i(\tau t + kx)} u(t, x)\,dt\,dx \quad , \quad \tau \in \mathbb{R}, k \in \mathbb{Z}_\lambda$$

[1] has the advantage of a clean differentiation rule on the Fourier side $\widehat{\partial_x f}(k) = ik\widehat{f}(k)$, but needs to deal with the factor $2\pi\lambda$ at the inversion.

Nonlinear interactions take on the Fourier side the form

$$\widehat{uv}(\tau,k) = \widehat{u} \star \widehat{v}(\tau,k) = \frac{1}{2\pi\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \widehat{u}(\tau_1,k_1)\widehat{v}(\tau-\tau_1,k-k_1)\,d\tau_1$$
$$= \frac{1}{2\pi\lambda} \sum_{k_1+k_2=k} \int_{\tau_1+\tau_2=\tau} \widehat{u}(\tau_1,k_1)\widehat{v}(\tau_2,k_2)\,d\tau_1$$

Corresponding to the linear Schrodinger evolution $i\partial_t u + \partial_x^2 u = 0$, we define

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} := \|\langle\tau\rangle^s\langle\tau+k^2\rangle^b\widehat{u}(\tau,k)\|_{L^2_\tau \ell^2_k}$$

For I a time interval, *the (time) restricted $X^{s,b}$-nom* is

$$\|u\|_{X^{s,b}(I\times\mathbb{T}_\lambda)} := \inf\{\|U\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)} : U_{|I} = u\}$$

In general, $\|u\|_{X^{s,b}}$ and $\|\overline{u}\|_{X^{s,b}} = \|\langle\tau\rangle^s\langle\tau-k^2\rangle^b\widehat{u}(\tau,k)\|_{L^2_\tau \ell^2_k}$ are not comparable. It is useful to introduce the conjugate space corresponding to the second norm, denoted either $\overline{X}^{s,b}$ (as in [Her06b]) or $X^{s,b}_{\tau=k^2}$ (as opposed to $X^{s,b} := X^{s,b}_{\tau=-k^2}$, see [Tao07, Sect. 2.6]).

As a matter of notation, $\alpha+$ denotes a quantity $\alpha+\varepsilon$ with $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ arbitrarily small and independent of any other constants present in the relation in which it appears.

2.3 Linear and bi-linear estimates

Along with the straightforward embeddings $X^{s_2,b_2} \hookrightarrow X^{s_1,b_1}$ for any $s_2 \geqslant s_1$ and $b_2 \geqslant b_1$ we also have:

Lemma 2.3.1 (Sobolev embeddings)**.**

1. *If $2 \leqslant p < \infty$ and $b \geqslant \frac{1}{2} - \frac{1}{p}$, then $X^{s,b}(\mathbb{R}\times\mathbb{T}) \hookrightarrow L^p_t H^s_x(\mathbb{R}\times\mathbb{T})$ with*

   (2.14) $$\|u\|_{L^p_t H^s_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})}$$

   Also, $X^{s,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}) \hookrightarrow L^\infty_t H^s_x(\mathbb{R}\times\mathbb{T})$ and

   (2.15) $$\|u\|_{L^\infty_t H^s_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{s,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T})}$$

2. *If $2 \leqslant p,q < \infty$, $b \geqslant \frac{1}{2} - \frac{1}{p}$, $s \geqslant \frac{1}{2} - \frac{1}{q}$, then $X^{s,b}(\mathbb{R}\times\mathbb{T}) \hookrightarrow L^p_t L^q_x(\mathbb{R}\times\mathbb{T})$ with*

   (2.16) $$\|u\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})}$$

   Also, $X^{\frac{1}{2}+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}) \hookrightarrow L^\infty_t H^s_x(\mathbb{R}\times\mathbb{T})$ and

   (2.17) $$\|u\|_{L^\infty_t L^\infty_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T})}$$

Lemma 2.3.2 (Strichartz estimates)**.**

(2.18) $$\|u\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T})}$$

(2.19) $$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{3}{8}}(\mathbb{R}\times\mathbb{T})}$$

(2.20) $$\|u\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T})}$$

For the proof of the above two lemmas, we refer to [Her06b, Prop. 2.2.3, Prop. 2.2.4].

Lemma 2.3.3. *Let $0 \leqslant \theta \leqslant 1$ and suppose $s = (1-\theta)s_0 + \theta s_1$, $b = (1-\theta)b_0 + \theta b_1$ for some $s_0 \leqslant s_1, b_0 \leqslant b_1$. Then, we have*

$$X^{s_0,b_0} \subset X^{s,b} \subset X^{s_1,b_1} \quad , \quad \|u\|_{X^{s,b}} \leqslant \|u\|_{X^{s_0,b_0}} \|u\|_{X^{s_1,b_1}}$$

For example, by interpolating the L^6-Strichartz estimate with the Sobolev embedding (2.16) (for $p = q = 6$, $b, s \geqslant \frac{1}{3}$), we also have

$$\|u\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0+,\frac{1}{2}-}(\mathbb{R}\times\mathbb{T})} \tag{2.21}$$

Remark 2.3.4. If u is smooth and 2π-periodic, then $u^\lambda(t,x) := \lambda^{-\frac{1}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is $2\pi\lambda$-periodic. We have the following

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{T})} = \lambda^{\frac{1}{2}-\frac{2}{q}-\frac{1}{r}} \|u^\lambda\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{T}_\lambda)}$$

and for $\lambda \geqslant 1$ and $s, b \geqslant 0$, we have $\langle \lambda k\rangle \leqslant \lambda\langle k\rangle$ and thus

$$\begin{aligned}
\|u\|^2_{X^{s,b}(\mathbb{R}\times\mathbb{T})} &\sim \lambda \int_{\mathbb{R}} \sum_{\xi\in\mathbb{Z}} \langle\xi\rangle^{2s} \langle\tau+\xi^2\rangle^{2b} \, |\lambda^{-3}\widehat{u^\lambda}(\lambda^{-2}\tau, \lambda^{-1}\xi)|^2 \, d\tau \\
&\sim \lambda\lambda^{-6}\lambda^3 \int_{\mathbb{R}} \frac{1}{\lambda} \sum_{\xi\in\mathbb{Z}} \langle\lambda\lambda^{-1}\xi\rangle^{2s} \langle\lambda^2(\lambda^{-2}\tau + \lambda^{-1}\xi\rangle^{2b} |\widehat{u^\lambda}(\lambda^{-2}\tau, \lambda^{-1}\xi)|^2 \frac{d\tau}{\lambda^2} \\
&= \lambda^{-2} \int_{\mathbb{R}} \frac{1}{\lambda} \sum_{k\in\mathbb{Z}_\lambda} \langle\lambda k\rangle^{2s} \langle\lambda^2(\tau+k^2)\rangle^{2b} |\widehat{u^\lambda}(\tau,k)|^2 \, d\tau \\
&\leqslant \lambda^{-2}\lambda^{2s}\lambda^{4b} \|u^\lambda\|^2_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)}
\end{aligned}$$

which gives

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} \lesssim \lambda^{-1+s+2b} \|u^\lambda\|_{X^{s,b}(\mathbb{R}\times\mathbb{T}_\lambda)}$$

Therefore, applying the Strichartz inequalities for u, we derive the scaled versions:

$$\|u^\lambda\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{0+} \|u^\lambda\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.22}$$

$$\|u^\lambda\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u^\lambda\|_{X^{0,\frac{3}{8}}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.23}$$

$$\|u^\lambda\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{0+} \|u^\lambda\|_{X^{0+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{2.24}$$

and the interpolated L^6-estimate:

$$\|u^\lambda\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{0+} \|u^\lambda\|_{X^{0+,\frac{1}{2}-}(\mathbb{R}\times\mathbb{T}_\lambda)}. \tag{2.25}$$

All the Sobolev embeddings as they appear in Lemma 2.3.1 hold on any $\mathbb{T}_\lambda$ without any powers of λ on the right hand sides. Also, all the estimates above hold with $\overline{X}^{s,b}$-norms on the right hand sides.

Lemma 2.3.5 (Gagliardo-Nirenberg type inequalities in the periodic setting)**.**

1. $\|f(|f|^2 - \mu(f))\|_{L^2(\mathbb{T}_\lambda)} \leqslant \|\partial_x f\|_{L^2(\mathbb{T})} \|f\|^2_{L^2(\mathbb{T})}$;
2. $\|f\|^3_{L^6(\mathbb{T}_\lambda)} \leqslant (\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)} + \frac{1}{2\pi\lambda}\|f\|_{L^2(\mathbb{T})}) \|f\|^2_{L^2(\mathbb{T})}$

Lemma 2.3.6 (bi-linear L^2-Strichartz estimate)**.** *Let $\eta \in C^\infty_0(\mathbb{R})$ a smooth time cut-off with compact support and $0 \leqslant \eta(t) \leqslant 1$. Suppose $u_1, u_2 \in \mathcal{S}_{per}$ are supported in the Fourier space in $\{|k_1| \sim N_1\}$ and $\{|k_2| \sim N_2\}$, respectively at all times t. Then*

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim C(\lambda, N_1) \|u_1\|_{X^{0,\frac{1}{2}+}} \|u_2\|_{X^{0,\frac{1}{2}+}}, \tag{2.26}$$

provided that either $N_1 \gg N_2$ or $N_1 \sim N_2$ and the two Fourier-space supports are on the same side of the *real line, and where*

$$C(\lambda, N_1) = \begin{cases} 1 & , \text{ if } N_1 \leqslant 1 \\ (\frac{1}{\lambda} + \frac{1}{N_1})^{\frac{1}{2}} & , \text{ if } N_1 > 1 \end{cases} \tag{2.27}$$

Proof. See [Win10, Prop. 2.1]. □

Remark 2.3.7. By the L^4-Strichartz estimate (2.19) and Hölder inequality, we have

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \leqslant \|u_1\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \|u_2\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \|u_1\|_{X^{0,\frac{3}{8}}} \|u_2\|_{X^{0,\frac{3}{8}}}. \tag{2.28}$$

We will only use the bi-linear estimate above in the regime $\lambda \lesssim N_1$, hence $C(\lambda, N_1) \sim \lambda^{-\frac{1}{2}}$. Hence, by interpolating (2.26) and (2.28), we have

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \lesssim \lambda^{-\frac{1}{2}+} \|u_1\|_{X^{0,\frac{1}{2}-}} \|u_2\|_{X^{0,\frac{1}{2}-}}. \tag{2.29}$$

2.4 Multilinear forms

For n even integer, we define the n-multilinear form of f associated to the multiplier $M_n : \mathbb{R}^n \to \mathbb{C}$ as

$$\Lambda_n(M_n; f) := \frac{1}{(2\pi\lambda)^{n-1}} \sum_{\substack{k_1,\dots,k_n \in \mathbb{Z}_\lambda \\ k_{12\dots n}=0}} M_n(k_1, k_2, ..., k_n) \widehat{f}(k_1) \widehat{\overline{f}}(k_2) \cdots \widehat{f}(k_{n-1}) \widehat{\overline{f}}(k_n)$$

As in [CKS+01], we use the short-hand notation $k_{12\dots n} := k_1 + k_2 + \dots k_n$ and $k_{1-2} := k_1 - k_2$, etc. Also, denote $\Gamma_n(\mathbb{T}_\lambda) := \{(k_1, \dots, k_n) \in (\mathbb{Z}_\lambda)^n : k_{12\dots n} = 0\}$; we endow $(\mathbb{Z}_\lambda)^n$ with the Dirac measure $\delta_0(k_1 + k_2 + \dots + k_n)$.

Hence, we can write

$$\int_{\mathbb{T}_\lambda} |v_x^2| dx = -\Lambda_2(k_1 k_2; v) \quad , \quad \operatorname{Im} \int_{\mathbb{T}_\lambda} |v|^2 v \overline{v}_x dx = -\frac{1}{4} \Lambda_4(k_{13-24}; v)$$

and we also have

$$\frac{1}{2}\mu(v) \int_{\mathbb{T}_\lambda} |v|^4 dx = \frac{1}{4\pi\lambda} \Lambda_2(1; v) \Lambda_4(1; v) = \frac{1}{2} \Lambda_6(\mathbb{1}_{\{k_{12}=0\}}; v)$$

(where $\mathbb{1}_X$ denotes the characteristic function of a set X). Therefore, (2.9) can be written using the multilinear forms as

$$E(v) = -\Lambda_2(k_1 k_2; v) + \frac{1}{8} \Lambda_4(k_{13-24}; v) + \frac{1}{2\lambda} \Lambda_2(1; v) \Lambda_4(1; v) \tag{2.30}$$

and also as

$$E(v) = -\Lambda_2(k_1 k_2; v) + \frac{1}{8} \Lambda_4(k_{13-24}; v) + \frac{1}{2} \Lambda_6(\mathbb{1}_{\{k_{12}=0\}}; v) \tag{2.31}$$

since $k_{12} = 0$ implies $k_{3456} = 0$ on $\Gamma_6(\mathbb{T}_\lambda)$.

2.5 The I-operator and modified energy

We define the Fourier multiplication operator

$$I : H^s(\mathbb{T}_\lambda) \to H^1(\mathbb{T}_\lambda)\ ,\ \widehat{If}(k) = m(k)\hat{f}(k)$$

where $m : \mathbb{R} \to [0,1]$ is an even, monotone on semiaxes, smooth function

$$m(\xi) = \begin{cases} 1 & ,\ \text{if } |\xi| \lesssim N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & ,\ \text{if } |\xi| \gg N \end{cases}$$

and $\frac{1}{2} \leqslant s < 1$.

We often use the following properties: $0 < m(\xi) \leqslant 1$, $\langle\xi\rangle m(\xi) \gtrsim 1$, the map $\xi \mapsto m(\xi)^2\xi^2$ is increasing on $(0,\infty)$ and $\xi \mapsto m(\xi)^2\xi$ is increasing for $\xi \gg N$. Hence, for regularities $\frac{1}{2} \leqslant s < 1$, we have

$$m(\xi)^2\langle\xi\rangle \gtrsim 1. \tag{2.32}$$

This indeed holds since for $|\xi| \lesssim N$ we have $m(\xi) \sim 1$ and $\langle\xi\rangle \geqslant 1$, while for $|\xi| \gg N$ we have $\langle\xi\rangle \sim |\xi|$ and $m(\xi)^2|\xi| = N\left(\frac{|\xi|}{N}\right)^{1-2s} \geqslant 1$.

We note that I is continuous and has a smoothing property:

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^1} \lesssim N^{1-s}\|u\|_{H^s} \tag{2.33}$$

$$\|Iu\|_{\dot{H}^1} \lesssim N^{1-s}\|u\|_{\dot{H}^s} \tag{2.34}$$

Indeed,

$$\sum_{k\lesssim N}\langle k\rangle^{2s}|\hat{u}(k)|^2 \lesssim \sum_{k\lesssim N}\langle k\rangle^2 m(k)^2|\hat{u}(k)|^2 \lesssim N^{2(1-s)}\sum_{k\lesssim N}\left(\frac{\langle k\rangle}{N}\right)^{2-2s}\langle k\rangle^{2s}m(k)^2|\hat{u}(k)|^2$$

and

$$\sum_{k\gg N}\langle k\rangle^2\frac{1}{\langle k\rangle^{2-2s}}|\hat{u}(k)|^2 \lesssim \sum_{k\gg N}\langle k\rangle^2 m(k)^2|\hat{u}(k)|^2 \lesssim \sum_{k\gg N}\langle k\rangle^2\left(\frac{N}{\langle k\rangle}\right)^{2-2s}|\hat{u}(k)|^2\ .$$

Using the interaction representation of the $X^{s,b}$-norm, we can write (2.33) in the form

$$\|u\|_{X^{s,b}} \lesssim \|Iu\|_{X^{1,b}} \lesssim N^{1-s}\|u\|_{X^{s,b}} \tag{2.33'}$$

With E as in (2.9), we consider the modified energy for $v \in H^s(\mathbb{T}_\lambda)$:

$$E_N(v) := E(Iv) \tag{2.35}$$

Heuristically, we know that $\partial_t E(v(t)) = 0$ for $v \in H^1(\mathbb{T}_\lambda)$ and we expect the modified energy to be "almost conserved" for $v \in H^s(\mathbb{T}_\lambda)$.

In multilinear forms, we have

$$\begin{aligned} E_N(v) = -\Lambda_2(k_1k_2m_1m_2;v) + \frac{1}{8}\Lambda_4(k_{13-24}m_1m_2m_3m_4;v) \\ + \frac{1}{2\lambda}\Lambda_2(m_1m_2;v)\Lambda_4(m_1m_2m_3m_4;v) \end{aligned} \tag{2.36}$$

Remark 2.5.1. In view of Remark 2.1.2, in the second generation of the I-method we refine the functional

$\mathcal{E}_N(v) := \mathcal{E}(Iv)$, which can be written as

$$\mathcal{E}(v) = -\Lambda_2(k_1k_2m_1m_2; v) + \frac{1}{8}\Lambda_4(k_{13-24}m_1m_2m_3m_4; v) \tag{2.37}$$

In what follows we need to derive the time-differentiation rule for a multilinear form $\Lambda_n(M_n; v)$ where v is a smooth solution of (2.10). Hence, we compute:

$$\begin{aligned}\widehat{v\overline{v}_x v}(k) &= \widehat{(v\overline{v}_x)} *_\lambda \widehat{v}(k) = \frac{1}{\lambda}\sum_{k_1+k_3=k} \widehat{v\overline{v}_x}(k_1)\widehat{v}(k_3) = \frac{1}{\lambda^2}\sum_{k_1+k_2+k_3=k} \widehat{v}(k_1)\widehat{\overline{v}_x}(k_2)\widehat{v}(k_3) \\ &= \frac{1}{\lambda^2}\sum_{k_1+k_2+k_3=k} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\end{aligned}$$

Then, by integration by parts and periodicity of v, we can write

$$\begin{aligned}2i\mathrm{Im}\int_{\mathbb{T}_\lambda} v\overline{v}_x dx &= \int_{\mathbb{T}_\lambda} v\overline{v}_x dx - \int_{\mathbb{T}_\lambda} \overline{v}v_x dx = \int_{\mathbb{T}_\lambda} v\overline{v}_x dx - |v|^2(\lambda) + |v|^2(0) + \int_{\mathbb{T}_\lambda} v\overline{v}_x dx \\ &= 2\int_{\mathbb{T}_\lambda} v\overline{v}_x dx = 2\,\widehat{v\overline{v}_x}(0) = 2\,\widehat{v} *_\lambda \widehat{\overline{v}_x}(0) = \frac{2}{\lambda}\sum_{k_1+k_2=0} \widehat{v}(k_1)ik2\widehat{\overline{v}}(k_2)\end{aligned}$$

and thus by symmetrization

$$\begin{aligned}\left(\left(2i\mathrm{Im}\int_{\mathbb{T}_\lambda} v\overline{v}_x dx\right) v\right)^{\widehat{}}(k) &= \frac{2}{\lambda^2}\sum_{\substack{k_1+k_2=0\\ k_3=k}} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) \\ &= \frac{1}{\lambda^2}\sum_{\substack{k_{123}=k\\ k_3=k}} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) + \frac{1}{\lambda^2}\sum_{\substack{k_{123}=k\\ k_1=k}} ik_2\widehat{v}(k_3)\widehat{\overline{v}}(k_2)\widehat{v}(k_1).\end{aligned}$$

This allows us to write using the inclusion-exclusion principle

$$\begin{aligned}\widehat{\mathcal{T}(v)}(k) &= \frac{1}{\lambda^2}\sum_{\substack{k_{123}=k\\ k_1\neq k\\ k_3\neq k}} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) + \frac{1}{\lambda^2}\sum_{\substack{k_{123}=k\\ k_1=k_3=k}} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) \\ &= \frac{1}{\lambda^2}\sum_{\substack{k_{123}=k\\ k_1\neq k\\ k_3\neq k}} ik_2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3) + \frac{1}{\lambda^2}i(-k)\,\widehat{v}(k)\widehat{\overline{v}}(-k)\widehat{v}(k).\end{aligned}$$

For the cvintilinear term, write $\mathcal{Q} = \mathcal{Q}_1(v) - 2\mathcal{Q}_2(v)$. We have

$$\begin{aligned}\widehat{\mathcal{Q}_1(v)}(k) &= \frac{1}{\lambda^4}\sum_{k_{12345}=k} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) - \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_5=k}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) \\ &= \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_5\neq k}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)\end{aligned}$$

and for $\mathcal{Q}_2$ we have

$$\fint_{\mathbb{T}_\lambda} |v|^2 dx = \frac{1}{\lambda}\widehat{v\overline{v}}(0) = \frac{1}{\lambda}\,\widehat{v} *_\lambda \widehat{\overline{v}}(0) = \frac{1}{\lambda^2}\sum_{k_{12}=0} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)$$

$$\left(\left(|v|^2 - \fint_{\mathbb{T}_\lambda} |v|^2 dx\right) v\right)^{\widehat{}}(k) = \frac{1}{\lambda^2}\sum_{k_{345}=k} \widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) - \frac{1}{\lambda^2}\sum_{\substack{k_{345}=k\\ k_5=k}} \widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)$$
$$= \frac{1}{\lambda^2}\sum_{\substack{k_{345}=k\\ k_5\neq k}} \widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)$$

We put these two together and obtain

$$\widehat{\mathcal{Q}_2(v)}(k) = \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_{12}=0\\ k_5\neq k}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)$$

We symmetrize:

$$2\widehat{\mathcal{Q}_2(v)}(k) = \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_{12}=0\\ k_5\neq k}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) \;+\; \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_{34}=0\\ k_5\neq k}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)$$

Hence, by the inclusion-exclusion principle

$$\widehat{\mathcal{Q}(v)}(k) = \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_5\neq k\\ k_{12}\neq 0\\ k_{34}\neq 0}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) \;+\; \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_5\neq k\\ k_{12}=k_{34}=0}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5)$$

Notice that the range in the second summation above is over an empty set since $k_{12345} = k$ and $k_{12} = k_{34} = 0$ imply $k_5 = k$, so if we denote $\Xi_k := \{k_1, ..., k_5 \in \mathbb{Z}_\lambda : k_{12345} = k, k_5 \neq k, k_{12} \neq 0, k_{34} \neq 0\}$ we can write

$$\widehat{\mathcal{Q}(v)}(k) = \frac{1}{\lambda^4}\sum_{\substack{k_{12345}=k\\ k_5\neq k\\ k_{12}\neq 0\\ k_{34}\neq 0}} \widehat{v}(k_1)\widehat{\overline{v}}(k_2)\widehat{v}(k_3)\widehat{\overline{v}}(k_4)\widehat{v}(k_5) = \Lambda_5(\mathbb{1}_{\Xi_k}; v)$$

Then, for v a smooth solution of (2.10), we have

$$v_t = iv_{xx} - \mathcal{T}(v) + \frac{i}{2}\mathcal{Q}(v)$$
$$\overline{v}_t = -i\overline{v}_{xx} - \mathcal{T}(\overline{v}) - \frac{i}{2}\mathcal{Q}(\overline{v})$$

(we used $\overline{\mathcal{T}(v)} = \mathcal{T}(\overline{v})$ and $\overline{\mathcal{Q}(v)} = \mathcal{Q}(\overline{v})$), and we can now derive the differentiation rule

$$\begin{aligned}
\partial_t \Lambda_n(M_n; v) &= \frac{1}{\lambda^{n-1}} \sum_{k_{12\ldots n}=0} \left[\widehat{\partial_t v}(k_1)\widehat{\overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) + \widehat{v}(k_1)\widehat{\partial_t \overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) + \ldots \right] \\
&= \frac{1}{\lambda^{n-1}} \sum_{k_{12\ldots n}=0} \left[-ik_1^2\widehat{v}(k_1)\widehat{\overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) + \widehat{v}(k_1)ik_2^2\widehat{\overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) - \ldots \right] \\
&\quad - \frac{1}{\lambda^{n-1}} \sum_{k_{12\ldots n}=0} \left[\widehat{\mathcal{T}(v)}(k_1)\widehat{\overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) + \widehat{v}(k_1)\widehat{\mathcal{T}(\overline{v})}(k_2)\cdots\widehat{\overline{v}}(k_n) + \ldots \right] \\
&\quad + \frac{i}{2\lambda^{n-1}} \sum_{k_{12\ldots n}=0} \left[\widehat{\mathcal{Q}(v)}(k_1)\widehat{\overline{v}}(k_2)\cdots\widehat{\overline{v}}(k_n) - \widehat{v}(k_1)\widehat{\mathcal{Q}(\overline{v})}(k_2)\cdots\widehat{\overline{v}}(k_n) + \ldots \right]
\end{aligned}$$

obtaining the following:

Lemma 2.5.2. *If v is a (formal) solution of (2.10) and M_n a multiplier of order n, then*

$$\begin{aligned}
\partial_t \Lambda_n(M_n; v) = i\Lambda_n &\left(M_n \sum_{j=1}^{n} (-1)^j k_j^2; v \right) \\
&- i\Lambda_{n+2} \left(\sum_{j=1}^{n} \mathbb{X}_j^2(M_n) k_{j+1} (\mathbb{1}_{\Theta_{n+2}^j} + \mathbb{1}_{\Upsilon_{n+2}^j}); v \right) \\
&+ \frac{i}{2}\Lambda_{n+4} \left(\sum_{j=1}^{n} (-1)^{j-1} \mathbb{X}_j^4(M_n) \mathbb{1}_{\Sigma_{n+4}^j}; v \right)
\end{aligned} \tag{2.38}$$

where $\mathbb{X}_j^l$ is the elongation operator (at position j, of length l),

$$\Theta_{n+2}^j := \{(k_1, ..., k_{n+2}) \in \Gamma_{n+2} : k_j + k_{j+1} \neq 0 \ , \ k_{j+1} + k_{j+2} \neq 0\} \tag{2.39}$$

$$\Upsilon_{n+2}^j := \{(k_1, ..., k_{n+2}) \in \Gamma_{n+2} : k_j = k_{j+2} = -k_{j+1}\} \tag{2.40}$$

and

$$\Sigma_{n+4}^j := \{(k_1, ..., k_{n+4}) \in \Gamma_{n+4} : k_j + k_{j+1} \neq 0 \ , \ k_{j+2} + k_{j+3} \neq 0 \ , \ k_j + k_{j+1} + k_{j+2} + k_{j+3} \neq 0\} \tag{2.41}$$

for $1 \leqslant j \leqslant n$.

We extend the sets appearing in the above differentiation rule for indices $j > n$ with the understanding that $j+1, j+2$, etc. are running circularly over $\{1, 2, ..., n+2\}$. It is important to notice that Θ_4^j and Υ_4^j are independent of j, so we denote them Θ_4 and Υ_4 respectively. Also,

$$\begin{aligned}
\Sigma_6^1 = \Sigma_6^3 = \Sigma_6^5 = \{\mathbf{k} \in \Gamma_6 : k_{12} \neq 0, k_{34} \neq 0, k_{56} \neq 0\} \\
\Sigma_6^2 = \Sigma_6^4 = \Sigma_6^6 = \{\mathbf{k} \in \Gamma_6 : k_{23} \neq 0, k_{45} \neq 0, k_{61} \neq 0\}
\end{aligned} \tag{2.42}$$

Proposition 2.5.3 (modified energy increment)**.**

$$E_N(v(T+\delta)) - E_N(v(T)) = \int_T^{T+\delta} [\Lambda_4(M_4; v) + \Lambda_6(M_6; v) + \Lambda_8(M_8; v) + \Lambda_{10}(M_{10}; v)] \, dt \tag{2.43}$$

where the multipliers $M_j : \Gamma_j \to \mathbb{C}$ *are given by*

$$
\begin{aligned}
M_4(\mathbf{k}) &:= C_1 m_1 m_2 m_3 m_4 k_{12} k_{13} k_{14} + C_2 \left(m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2\right) \mathbb{1}_{\Theta_4}(\mathbf{k}) \\
M_6(\mathbf{k}) &:= C_3 \Big((m_1^2 k_1^2 + m_3^2 k_3^2 + m_5^2 k_5^2) \mathbb{1}_{\Sigma_6^1}(\mathbf{k}) - (m_1^2 k_1^2 + m_3^2 k_3^2 + m_5^2 k_5^2) \mathbb{1}_{\Sigma_6^2}(\mathbf{k}) \Big) \\
&\quad + C_4 \Big(m_{123} m_4 m_5 m_6 k_{46} k_2 (\mathbb{1}_{\Theta_6^1} + \mathbb{1}_{\Upsilon_6^1})(\mathbf{k}) - m_{234} m_5 m_6 m_1 k_{15} k_3 (\mathbb{1}_{\Theta_6^2} + \mathbb{1}_{\Upsilon_6^2})(\mathbf{k}) \\
&\qquad + m_{345} m_6 m_1 m_2 k_{26} k_4 (\mathbb{1}_{\Theta_6^3} + \mathbb{1}_{\Upsilon_6^3})(\mathbf{k}) - m_{456} m_1 m_2 m_3 k_{13} k_5 (\mathbb{1}_{\Theta_6^4} + \mathbb{1}_{\Upsilon_6^4})(\mathbf{k}) \Big) \\
&\quad + C_5 m_1 m_2 m_3 m_4 m_5 m_6 k_{34} k_{45} \mathbb{1}_{k_{12}=0}(\mathbf{k}) \\
M_8(\mathbf{k}) &:= C_6 (m_{12345} m_6 m_7 m_8 k_{123457} \mathbb{1}_{\Sigma_8^1} - m_1 m_{23456} m_7 m_8 k_{17} \mathbb{1}_{\Sigma_8^2} \\
&\qquad + m_1 m_2 m_{34567} m_8 k_{134567} \mathbb{1}_{\Sigma_8^3} - m_1 m_2 m_3 m_{45678} k_{13} \mathbb{1}_{\Sigma_8^4}) \\
&\quad + C_7 (m_{123}^2 m_5 m_6 m_7 m_8 k_2 \mathbb{1}_{\{k_{1234}=0\}} (\mathbb{1}_{\Theta_8^1} + \mathbb{1}_{\Upsilon_8^1}) + m_1^2 m_5 m_6 m_7 m_8 k_3 \mathbb{1}_{\{k_{1234}=0\}} (\mathbb{1}_{\Theta_8^2} + \mathbb{1}_{\Upsilon_8^2}) \\
&\qquad + m_1^2 m_{345} m_6 m_7 m_8 k_4 \mathbb{1}_{\{k_{12}=0\}} (\mathbb{1}_{\Theta_8^3} + \mathbb{1}_{\Upsilon_8^3}) + m_1^2 m_3 m_{456} m_7 m_8 k_5 \mathbb{1}_{\{k_{12}=0\}} (\mathbb{1}_{\Theta_8^4} + \mathbb{1}_{\Upsilon_8^4}) \\
&\qquad + m_1^2 m_3 m_4 m_{567} m_8 k_6 \mathbb{1}_{\{k_{12}=0\}} (\mathbb{1}_{\Theta_8^5} + \mathbb{1}_{\Upsilon_8^5}) + m_1^2 m_3 m_4 m_5 m_{678} k_7 \mathbb{1}_{\{k_{12}=0\}} (\mathbb{1}_{\Theta_8^6} + \mathbb{1}_{\Upsilon_8^6})) \\
M_{10}(\mathbf{k}) &:= C_8 (m_{12345} m_6 m_7 m_8 m_9 m_{10} \mathbb{1}_{\{k_{123456}=0\}} \mathbb{1}_{\Sigma_{10}^1} - m_1 m_{23456} m_7 m_8 m_9 m_{10} \mathbb{1}_{\{k_{123456}=0\}} \mathbb{1}_{\Sigma_{10}^2} \\
&\qquad + m_1 m_2 m_{34567} m_8 m_9 m_{10} \mathbb{1}_{\{k_{12}=0\}} \mathbb{1}_{\Sigma_{10}^3} - m_1 m_2 m_3 m_{45678} m_9 m_{10} \mathbb{1}_{\{k_{12}=0\}} \mathbb{1}_{\Sigma_{10}^4} \\
&\qquad + m_1^2 m_3 m_4 m_{56789} m_{10} \mathbb{1}_{\{k_{12}=0\}} \mathbb{1}_{\Sigma_{10}^5} - m_1 m_2 m_3 m_4 m_5 m_{6789[10]} \mathbb{1}_{\{k_{12}=0\}} \mathbb{1}_{\Sigma_{10}^6})
\end{aligned}
$$

with constants $C_1 = C_2 = -\frac{i}{2}$, $C_3 = -\frac{i}{6}$, $C_4 = \frac{i}{4}, C_5 = i$, $C_6 = \frac{i}{8}, C_7 = -\frac{i}{2}$, $C_8 = \frac{i}{4}$. *Moreover, if* $|k_j| \ll N$ *for all* j, *then the multipliers vanish.*

Proof. Using (2.31), we have

$$
\begin{aligned}
E_N(v(t)) &= -\Lambda_2(k_1 k_2; Iv(t)) + \frac{1}{8}\Lambda_4(k_{13-24}; Iv(t)) + \frac{1}{2}\Lambda_6(\mathbb{1}_{\{k_{12}=0\}}; Iv(t)) \\
&= -\Lambda_2(m_1 m_2 k_1 k_2; v(t)) + \frac{1}{8}\Lambda_4(m_1 m_2 m_3 m_4 k_{13-24}; v(t)) \\
&\quad + \frac{1}{2}\Lambda_6(m_1 m_2 m_3 m_4 m_5 m_6 \mathbb{1}_{\{k_{12}=0\}}; v(t))
\end{aligned}
$$

In what follows, we omit writing $v(t)$ in Λ expressions. For the first and last term, we know that $k_2 = -k_1$ and thus $m_2 = m(k_2) = m(k_1) = m_1$, while for the second term $k_{24} = -k_{13}$, and therefore the modified energy expression simplifies to

$$
E_N(v(t)) = \Lambda_2(m_1^2 k_1^2) + \frac{1}{4}\Lambda_4(m_1 m_2 m_3 m_4 k_{13}) + \frac{1}{2}\Lambda_6(m_1^2 m_3 m_4 m_5 m_6 \mathbb{1}_{\{k_{12}=0\}}) \tag{2.44}
$$

Then we compute the δ-increment of the modified energy via

$$
E_N(v(T+\delta)) - E_N(v(T)) = \int_T^{T+\delta} \partial_t E_N(v(t))\, dt \ ,
$$

for which we use the differentiation rule (2.38) to obtain

$$
\begin{aligned}
\partial_t \Lambda_2(m_1^2 k_1^2) &= i\Lambda_2(m_1^2 k_1^2(-k_1^2 + k_2^2)) \\
&\quad - i\Lambda_4(m_{123}^2 k_{123}^2 k_2 (\mathbb{1}_{\Theta_4^1} + \mathbb{1}_{\Upsilon_4^1}) + m_1^2 k_1^2 k_3 (\mathbb{1}_{\Theta_4^2} + \mathbb{1}_{\Upsilon_4^2})) \\
&\quad + \frac{i}{2}\Lambda_6(m_{12345}^2 k_{12345}^2 \mathbb{1}_{\Sigma_6^1} - m_1^2 k_1^2 \mathbb{1}_{\Sigma_6^2})
\end{aligned}
$$

Since $k_{12} = 0$, the Λ_2's multiplier above is 0.

Also,

$$\begin{aligned}\partial_t\Lambda_4(m_1m_2m_3m_4k_{13}) = {} & i\Lambda_4(m_1m_2m_3m_4k_{13}(-k_1^2+k_2^2-k_3^2+k_4^2))\\
& - i\Lambda_6(m_{123}m_4m_5m_6k_{1235}k_2(\mathbb{1}_{\Theta_6^1}+\mathbb{1}_{\Upsilon_6^1})\\
& \qquad + m_1m_{234}m_5m_6k_{15}k_3(\mathbb{1}_{\Theta_6^2}+\mathbb{1}_{\Upsilon_6^2})\\
& \qquad + m_1m_2m_{345}m_6k_{1345}k_4(\mathbb{1}_{\Theta_6^3}+\mathbb{1}_{\Upsilon_6^3})\\
& \qquad + m_1m_2m_3m_{456}k_{13}k_5(\mathbb{1}_{\Theta_6^4}+\mathbb{1}_{\Upsilon_6^4})\\
& + \frac{i}{2}\Lambda_8(m_{12345}m_6m_7m_8k_{123457}\mathbb{1}_{\Sigma_8^1}\\
& \qquad - m_1m_{23456}m_7m_8k_{17}\mathbb{1}_{\Sigma_8^2}\\
& \qquad + m_1m_2m_{34567}m_8k_{134567}\mathbb{1}_{\Sigma_8^3}\\
& \qquad - m_1m_2m_3m_{45678}k_{13}\mathbb{1}_{\Sigma_8^4})\end{aligned}$$

We take

$$M_4 := -i\left(m_4^2k_4^2k_2+m_1^2k_1^2k_3\right)\mathbb{1}_{\Theta_4}+\frac{i}{4}m_1m_2m_3m_4k_{13}(-k_1^2+k_2^2-k_3^2+k_4^2)$$

We symmetrize the first term and it becomes $-\frac{i}{2}\left(m_1^2k_1^2k_3+m_2^2k_2^2k_4+m_3^2k_3^2k_1+m_4^2k_4^2k_2\right)\mathbb{1}_{\Theta_4}$, while the second term reduces to $-\frac{i}{2}m_1m_2m_3m_4k_{12}k_{13}k_{14}$ (due to $-k_1^2+k_2^2-k_3^2+k_4^2=-2k_{12}k_{14}$).

Lastly, the third term of the modified energy gives

$$\begin{aligned}\partial_t\Lambda_6(m_1^2m_3m_4m_5m_6\mathbb{1}_{\{k_{12}=0\}}) = {} & i\Lambda_6(m_1^2m_3m_4m_5m_6\mathbb{1}_{\{k_{12}=0\}}(-k_3^2+k_4^2-k_5^2+k_6^2))\\
& - i\Lambda_8(m_{123}^2m_5m_6m_7m_8k_2\mathbb{1}_{\{k_{1234}=0\}}(\mathbb{1}_{\Theta_8^1}+\mathbb{1}_{\Upsilon_8^1})\\
& \qquad + m_1^2m_5m_6m_7m_8k_3\mathbb{1}_{\{k_{1234}=0\}}(\mathbb{1}_{\Theta_8^2}+\mathbb{1}_{\Upsilon_8^2})\\
& \qquad + m_1^2m_{345}m_6m_7m_8k_4\mathbb{1}_{\{k_{12}=0\}}(\mathbb{1}_{\Theta_8^3}+\mathbb{1}_{\Upsilon_8^3})\\
& \qquad + m_1^2m_3m_{456}m_7m_8k_5\mathbb{1}_{\{k_{12}=0\}}(\mathbb{1}_{\Theta_8^4}+\mathbb{1}_{\Upsilon_8^4})\\
& \qquad + m_1^2m_3m_4m_{567}m_8k_6\mathbb{1}_{\{k_{12}=0\}}(\mathbb{1}_{\Theta_8^5}+\mathbb{1}_{\Upsilon_8^5})\\
& \qquad + m_1^2m_3m_4m_5m_{678}k_7\mathbb{1}_{\{k_{12}=0\}}(\mathbb{1}_{\Theta_8^6}+\mathbb{1}_{\Upsilon_8^6}))\\
& + \frac{i}{2}\Lambda_{10}(m_{12345}^2m_7m_8m_9m_{10}\mathbb{1}_{\{k_{123456}=0\}}\mathbb{1}_{\Sigma_{10}^1}\\
& \qquad - m_1^2m_7m_8m_9m_{10}\mathbb{1}_{\{k_{123456}=0\}}\mathbb{1}_{\Sigma_{10}^2}\\
& \qquad + m_1^2m_{34567}m_8m_9m_{10}\mathbb{1}_{\{k_{12}=0\}}\mathbb{1}_{\Sigma_{10}^3}\\
& \qquad - m_1^2m_3m_{45678}m_9m_{10}\mathbb{1}_{\{k_{12}=0\}}\mathbb{1}_{\Sigma_{10}^4}\\
& \qquad + m_1^2m_3m_4m_{56789}m_{10}\mathbb{1}_{\{k_{12}=0\}}\mathbb{1}_{\Sigma_{10}^5}\\
& \qquad - m_1^2m_3m_4m_5m_{6789[10]}\mathbb{1}_{\{k_{12}=0\}}\mathbb{1}_{\Sigma_{10}^6})\end{aligned}$$

We take

$$\begin{aligned}M_6 := {} & \frac{i}{2}\left(m_6^2k_6^2\mathbb{1}_{\Sigma_6^1}-m_1^2k_1^2\mathbb{1}_{\Sigma_6^2}\right)\\
& - \frac{i}{4}\Big(m_{123}m_4m_5m_6(-k_{46})k_2(\mathbb{1}_{\Theta_6^1}+\mathbb{1}_{\Upsilon_6^1})+m_{234}m_5m_6m_1k_{15}k_3(\mathbb{1}_{\Theta_6^2}+\mathbb{1}_{\Upsilon_6^2})\\
& \qquad +m_{345}m_6m_1m_2(-k_{26})k_4(\mathbb{1}_{\Theta_6^3}+\mathbb{1}_{\Upsilon_6^3})+m_{456}m_1m_2m_3k_{13}k_5(\mathbb{1}_{\Theta_6^4}+\mathbb{1}_{\Upsilon_6^4})\Big)\\
& + \frac{i}{2}m_1^2m_3m_4m_5m_6(-k_3^2+k_4^2-k_5^2+k_6^2)\mathbb{1}_{\{k_{12}=0\}}\end{aligned}$$

Here, we just symmetrize the first term to

$$\frac{i}{6}\sum_{j=1}^{6}(-1)^j m_j^2 k_j^2 \mathbb{1}_{\Sigma_6^{j+1}} = -\frac{i}{6}(m_1^2k_1^2 + m_3^2k_3^2 + m_5^2k_5^2)\mathbb{1}_{\Sigma_6^1} + \frac{i}{6}(m_2^2k_2^2 + m_4^2k_4^2 + m_6^2k_6^2)\mathbb{1}_{\Sigma_6^2}$$

and since $-k_3^2 + k_4^2 - k_5^2 + k_6^2 = 2k_{34}k_{45}$ when $k_{12} = 0$ on Γ_6, the third term becomes

$$im_1m_2m_3m_4m_5m_6k_{34}k_{45}\mathbb{1}_{\{k_{12}=0\}}.$$

For the 8th and 10th order multipliers we just collect the terms from the above calculations.

□

2.6 Local well-posedness in $H^1(\mathbb{T}_\lambda)$ of the I-system

If $v \in H^s(\mathbb{T}_\lambda)$ satisfies the (2.8) equation, then Iv satisfies

$$(Iv)_t - i(Iv)_{xx} = -I\mathcal{T}(v) + \frac{i}{2}I\mathcal{Q}(v) \tag{2.45}$$

with $\mathcal{T}$ and $\mathcal{Q}$ as in (2.10). Note that while I commutes with ∂_t, ∂_x^2, it does not commute with the nonlinear operators $\mathcal{T}$ and $\mathcal{Q}$. As such $E(Iv)$, is not necessarily conserved.

Proposition 2.6.1. *Let $w_0 \in H^1(\mathbb{T}_\lambda)$ and $\delta > 0$. There exist $D_1 \sim \delta^{-\alpha}$ (for some $\alpha > 0$) and $D_2 \geqslant 1$ such that if $\|w_0\|_{\dot{H}^1(\mathbb{T}_\lambda)} \leqslant D_1$, then*

$$\|w\|_{X^{1,\frac{1}{2}}([0,\delta]\times\mathbb{T}_\lambda)} \leqslant D_2 \tag{2.46}$$

Proof. Here we sketch the proof of Takaoka [Tak99]. We look at the gauged version (DNLS_1) and adopt a perturbative approach:

$$\partial_t v = i\partial_x^2 v + \mathcal{N}(v)$$

where $\mathcal{N}(v) = \mathcal{T}(v) + \mathcal{Q}(v)$, $\mathcal{T}(v) := -iv^2\partial_x v$, $\mathcal{Q}(v) = -\frac{1}{2}|v|^4 v$. We set up a fixed point problem for the associated Cauchy problem with initial condition $u_{|t=0} = u_0 \in H^s(\mathbb{R})$:

$$\eta(t)v(t) = \eta(t)S(t)v_0 - i\eta(t)\int_0^t S(t-t')\eta(t'/T)N(v)(t')dt'$$

where η is a compactly supported, smooth cutoff in time, $\eta(t) = 1$ for $t \in [-1, 1]$. Note that on $-T \leqslant t \leqslant T$, the above is equivalent with the integral formulation of (DNLS_1). The cutoff in front of the Duhamel term is needed for the $X^{s,b}$ estimate, while the one in the integrand is used to gain the smallness T^{0+} which will allow to close a contraction mapping argument.

We consider the map $\Gamma : X^{s,b} \to X^{s,b}$ (depending on T and v_0) given by

$$\Gamma(v)(t) := \eta(t)S(t)v_0 - i\eta(t)\int_0^t S(t-t')\eta(t'/T)N(v)(t')dt'$$

Following the energy estimate [Tao07, Prop. 2.12, p. 103], we have

$$\|\Gamma(v)\|_{X^{s,b}} \lesssim_{\eta,b} \|u_0\|_{X^{s,b}} + \|N(v)\|_{X^{s,b-1}} \tag{2.47}$$

$$\|\Gamma(v) - \Gamma(\tilde{v})\|_{X^{s,b}} \lesssim \|N(v) - N(\tilde{v})\|_{X^{s,b-1}} \tag{2.48}$$

Lemma 2.6.2.

$$\|u_1u_2\partial_x\overline{u_3}\|_{X^{s,b-1}} \lesssim \|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{s,b}}$$

$$\|u_1u_2u_3u_4u_5\| \lesssim \prod_{j=1}^{5}\|u_j\|$$

The first estimate allows to conclude that Γ is well-defined, while the second one is needed to prove that Γ is a contraction in a ball of $X^{s,b}$.

□

2.7 Estimating the Λ_4-term of the modified energy increment

For $\mathbf{k} = (k_1, k_2, k_3, k_4) \in \Gamma_4(\mathbb{T}_\lambda)$, we denote $N_j = |k_j|$, $N_{ij} = |k_i + k_j|$ and $\{N_j^*\}$ denotes the reordering of $\{N_j\}$ so that $N_1^* \geqslant N_2^* \geqslant N_3^* \geqslant N_4^*$. However, from $k_1 = -(k_2 + k_3 + k_4)$ we deduce that we cannot have $N_2^* \ll N_1^*$, so $N_2^* \sim N_1^*$. Also, we denote $m_j = m(k_j) = m(N_j)$ for $1 \leqslant j \leqslant 4$, and for $(\tau_1, \tau_2, \tau_3, \tau_4) \in \Gamma_4(\mathbb{R})$, we introduce the modulation notation:

$$\sigma_j := \tau_j + k_j^2 \quad , \quad j = 1, 3 \tag{2.49}$$

$$\sigma_j := \tau_j - k_j^2 \quad , \quad j = 2, 4 \tag{2.50}$$

Note that

$$\begin{aligned} \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 &= \tau_{1234} + k_1^2 - k_2^2 + k_3^2 - k_4^2 = (k_1 - k_2)k_{12} + (k_3 - k_4)k_{34} \\ &= k_{12}(k_1 - k_2 - k_3 + k_4) = k_{12}(k_{14} - k_{23}) \\ &= 2k_{12}k_{14} \end{aligned} \tag{2.51}$$

Let $\{\sigma_j^*\}$ denote the reordering of $\{|\sigma_j|\}$ so that $\sigma_1^* \geqslant \sigma_2^* \geqslant \sigma_3^* \geqslant \sigma_4^*$. It follows that

$$|k_{12}k_{14}| \lesssim \sigma_1^*. \tag{2.52}$$

We recall that on Θ_4 we have $k_{12} \neq 0$ and $k_{14} \neq 0$, while on Υ_4 we have $k_1 = k_3 = -k_2 = -k_4$ (and consequently both M_4' and M_4'' vanish). Thus

$$M_4 = C_1 M_4' + C_2 M_4'' \mathbb{1}_{\Theta_4} \quad , \qquad \text{where } C_1 = C_2 = -\frac{i}{2} \tag{2.53}$$

and

$$M_4'(\mathbf{k}) = m_1 m_2 m_3 m_4 k_{12} k_{13} k_{14} \quad , \quad M_4''(\mathbf{k}) = m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2. \tag{2.54}$$

We want to establish the following estimate

$$\left| \int_T^{T+\delta} \Lambda_4(M_4; v(t))\, dt \right| \lesssim N^{-\beta+} \|Iv\|^4_{X^{1,\frac{1}{2}}([T,T+\delta]\times\mathbb{T}_\lambda)} \quad , \ v \in \mathcal{S}_{\text{per}} \tag{2.55}$$

for some $\beta > 0$. Since M_4 vanishes when $N_1^* \ll N$, by dyadic decomposition, it suffices to show that for some $\varepsilon > 0$,

$$\left| \int_T^{T+\delta} \Lambda_4(M_4 \mathbb{1}_{N_1^* \sim 2^\kappa}; v_1(t), \overline{v_2(t)}, v_3(t), \overline{v_4(t)})\, dt \right| \lesssim \frac{1}{2^{\kappa\varepsilon}} N^{-\beta+} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{1}{2}}([T,T+\delta]\times\mathbb{T}_\lambda)} \tag{2.56}$$

for all $\kappa \geqslant m$, where $m \in \mathbb{N}$ is such that $2^m \sim N$ and $v_j \in \mathcal{S}_{\text{per}}$.

We fix κ, and noticing that

$$\|Iv_1\|_{X^{1,\frac{1}{2}}} = \|\langle \tau_1 \rangle^1 \langle \tau_1 + k_1^2 \rangle^{\frac{1}{2}} m(k_1) \widehat{v_1}(\tau_1, k_1)\|_{L^2_{\tau_1}\ell^2_{k_1}},$$

$$\|I\overline{v_2}\|_{X^{1,\frac{1}{2}}} = \|\langle \tau_2 \rangle^1 \langle \tau_2 + k_2^2 \rangle^{\frac{1}{2}} m(k_2) \widehat{\overline{v_2}}(-\tau_2, -k_2)\|_{L^2_{\tau_2}\ell^2_{k_2}} = \|\langle \tau_2 \rangle \langle -\tau_2 + k_2^2 \rangle^{\frac{1}{2}} m(k_2) \widehat{v_2}(\tau_2, k_2)\|_{L^2_{\tau_2}\ell^2_{k_2}},$$

we introduce $w_j(t, x)$ $(1 \leqslant j \leqslant 4)$ defined by

$$\widehat{w_j}(\tau_j, k_j) = \langle k_j \rangle \langle \sigma_j \rangle^{\frac{1}{2}}\, m_j\, \widehat{v_j}(\tau_j, k_j) \tag{2.57}$$

Hence, (2.56) becomes

$$\left|\int_T^{T+\delta} \Lambda_4(\widetilde{M_4}\mathbb{1}_{N_1^*\sim 2^\kappa}; w_1(t), w_2(t), w_3(t), w_4(t))\, dt\right| \lesssim \left(\frac{1}{2^\kappa}\right)^\varepsilon N^{-\beta+} \prod_{j=1}^4 \|w_j\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.58}$$

where

$$\widetilde{M_4} = \frac{M_4}{\prod_{j=1}^4 m_j \langle k_j\rangle\langle\sigma_j\rangle^{\frac{1}{2}}} = C_1\widetilde{M_4'} + C_2\widetilde{M_4''}$$

The $X^{s,b}$ spaces don't behave nicely with respect to sharp cut-offs in time, therefore we need to write $\mathbb{1}_{[T,T+\delta]} = a(t) + b(t)$ with one of them smooth and the other rough but with small support. Using the shorthand notation $\widetilde{M_{4,\kappa}}$ instead of $\widetilde{M_4}\mathbb{1}_{N_1^*\sim 2^\kappa}$ (and $M_{4,\kappa}$ instead of $M_4\mathbb{1}_{N_1^*\sim 2^\kappa}$), we write

(2.59)

$$\begin{aligned}
&\int_T^{T+\delta} \Lambda_4(\widetilde{M_{4,\kappa}}; w_1(t), w_2(t), w_3(t), w_4(t))\, dt \\
&\quad= \int_{\mathbb{R}} \mathbb{1}_{[T,T+\delta]}(t)\Lambda_4(\widetilde{M_{4,\kappa}}; w_1(t), w_2(t), w_3(t), w_4(t))\, dt \\
&\quad= \int_{\mathbb{R}} \Lambda_4(\widetilde{M_{4,\kappa}}; a(t)w_1(t), w_2(t), w_3(t), w_4(t))dt \;+\; \int_{\mathbb{R}} \Lambda_4(\widetilde{M_{4,\kappa}}; b(t)w_1(t), w_2(t), w_3(t), w_4(t))dt
\end{aligned}$$

where $a(t) = \mathbb{1}_{[T,T+\delta]} * \eta_{2^{-100\kappa}}(t)$ is the smoothed out version [2] of the indicator function of $[T, T+\delta]$ and $b(t)$ is defined by

$$\mathbb{1}_{[T,T+\delta]}(t) = a(t) + b(t). \tag{2.60}$$

We have that

$$(i) \quad \|a(t)\|_{H_t^{\frac{1}{2}+}} \lesssim 2^{(0+)\kappa} \qquad ; \qquad (ii) \quad \|b(t)\|_{L^2_t} \lesssim 2^{-50\kappa} \tag{2.61}$$

Indeed, let $\varepsilon \geqslant 0$. Since a is supported on frequencies $\lesssim 2^{100\kappa}$, by Plancherel we have

$$\||\partial_t|^{\frac{1}{2}+\varepsilon} a(t)\|_{L^2_t} \lesssim 2^{(100\kappa)(\frac{1}{2}+\varepsilon)}(2^{-100\kappa})^{\frac{1}{2}} \sim 2^{(100\varepsilon)\kappa}\ ,$$

and we easily have $\|a(t)\|_{L^2_t} \lesssim (\delta + 2^{-100\kappa}) \lesssim 2^{(100\varepsilon)\kappa}$ provided κ is large enough. Thus $\|a(t)\|_{H_t^{\frac{1}{2}+\varepsilon}} \lesssim 2^{(100\varepsilon)\kappa}$. Note that the graph of $|b(t)|$ consists of two "bumps" of height at most 1 centered at T and $T+\delta$ concentrated on intervals of length $\sim 2^{-100\kappa}$, and therefore $\|b(t)\|^2_{L^2_t} \lesssim 2^{-100\kappa}$.

Assumptions and shorthand notation. Due to the multi-linearity of expressions of the form

$$\int_{\mathbb{R}} \Lambda_4(M, w_1(t), w_2(t), w_3(t), w_4(t))\, dt = \int_{\tau_{1234}=0} \frac{1}{\lambda^3} \sum_{k_{1234}=0} M(\mathbf{k})\widehat{w_1}(\tau_1, k_1)\widehat{w_2}(\tau_2, k_2)\widehat{w_3}(\tau_3, k_3)\widehat{w_4}(\tau_4, k_4)$$

we can assume without loss of generality that the time-space Fourier transforms $\widehat{w_j}$ (and equivalently $\widehat{v_j}$ and $\widehat{\overline{v_j}}$ provided (2.57) is in place) are real and nonnegative. Also, we'll use the shorthand notation $\int_*$ instead of $\int_{\tau_{1234}=0} d\tau_1 d\tau_2 d\tau_3 \sum_{k_{1234}=0}$.

The $b(t)$-term of (2.59) has fast decay in N as quantified by the following lemma.

Lemma 2.7.1. *We have $|M_{4,\kappa}| \lesssim 2^{3\kappa}$ and therefore*

$$\left|\int_{\mathbb{R}} \Lambda_4(M_4; b(t)v_1(t), \overline{v_2}(t), v_3(t), \overline{v_4}(t))dt\right| \lesssim N^{-47+} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac{1}{2}}} \tag{2.62}$$

[2] $\eta_\varepsilon(t) = \varepsilon^{-1}\eta(t/\varepsilon)$ is the L^1-invariant ε-scaling of $\eta \in \mathcal{S}(\mathbb{R})$

Proof. We have $|M_4'(\mathbf{k})| \lesssim |k_{12}|\,|k_{13}|\,|k_{14}| \lesssim (N_1^*)^3$ and since $|m_1^2 k_1^2 k_3| \leqslant N_1^2 N_3 \leqslant (N_1^*)^3$ (and its symmetric analogues), we get $|M_{4,\kappa}(\mathbf{k})| \lesssim 2^{3\kappa}$ for all $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$. Here, we assume without loss of generality that $\widehat{bv_1}, \widehat{\overline{v_2}}$ are real and nonnegative.

$$\begin{aligned}
\mathrm{LHS}_{(2.62)} &= \left| \int_{\mathbb{R}} \frac{1}{\lambda^3} \sum_{k_{1234}=0} M_{4,\kappa}(\mathbf{k}) b(t) \widehat{v_1}(t,k_1) \widehat{\overline{v_2}}(t,k_2) \widehat{v_3}(t,k_3) \widehat{\overline{v_4}}(t,k_4)\, dt \right| \\
&= \frac{1}{\lambda^3} \left| \int_* M_{4,\kappa}(\mathbf{k}) \widehat{bv_1}(\tau_1,k_1) \widehat{\overline{v_2}}(\tau_2,k_2) \widehat{v_3}(\tau_3,k_3) \widehat{\overline{v_4}}(\tau_4,k_4) \right| \\
&\lesssim \frac{2^{3\kappa}}{\lambda^3} \int_* \widehat{bv_1}(\tau_1,k_1) \widehat{\overline{v_2}}(\tau_2,k_2) \widehat{v_3}(\tau_3,k_3) \widehat{\overline{v_4}}(\tau_4,k_4) \\
&\lesssim 2^{3\kappa} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} |b(t) v_1(t,x)|\, |v_2(t,x)|\, |v_3(t,x)|\, |v_4(t,x)|\, dx dt
\end{aligned}$$

By applying the $L^2_{t,x} L^6_{t,x} L^6_{t,x} L^6_{t,x}$-Hölder inequality, then the Strichartz inequalities (Lemma 2.3.2(1) and (2.21)), and finally the smoothing property (2.33) of I in the form $\|v_j\|_{X^{0,b}} \lesssim \|Iv_j\|_{X^{1-s,b}}$, we further get

$$\begin{aligned}
\mathrm{LHS}_{(2.62)} &\lesssim 2^{3\kappa} \|bv_1\|_{L^2_{t,x}} \|v_2\|_{L^6_{t,x}} \|v_3\|_{L^6_{t,x}} \|v_4\|_{L^6_{t,x}} \\
&\leqslant 2^{3\kappa} \|b(t)\|_{L^2_t} \|v_1\|_{L^\infty_t L^2_x} \|v_2\|_{L^6_{t,x}} \|v_3\|_{L^6_{t,x}} \|v_4\|_{L^6_{t,x}} \\
&\lesssim 2^{-47\kappa} \|v_1\|_{X^{0,\frac12+}} \|v_2\|_{X^{0+,\frac12}} \|v_3\|_{X^{0+,\frac12}} \|v_4\|_{X^{0+,\frac12}} \\
&\lesssim \frac{1}{N^{47-}} \frac{1}{2^{47-(\kappa-m)}} \|v_1\|_{X^{0,\frac12+}} (N_1^{*0+})^3 \|v_2\|_{X^{0,\frac12}} \|v_3\|_{X^{0,\frac12}} \|v_4\|_{X^{0,\frac12}} \\
&\lesssim N^{-47+} \frac{1}{2^{(\kappa-m)(47-)}} \prod_{j=1}^4 \|Iv_j\|_{X^{1,\frac12+}}
\end{aligned}$$

Summing over $\kappa \gtrsim m$, yields (2.62). ◻

The analysis of the $a(t)$-term of (2.59) is more involved and requires better point-wise estimates on the multiplier $M_{4,\kappa}$. We now recall the estimates that work in the non-periodic setting.

Lemma 2.7.2 ([CKS+01, Lemma 6.1])**.** *We have*

1. *If $N_3^* \sim N_1^*$, then* $|M_{4,\kappa}(\mathbf{k})| \lesssim N^{-1} \left(\frac{N}{2^\kappa}\right)^{\frac{1}{10}} \langle k_{12} k_{14} \rangle^{\frac12} \prod_{j=1}^4 \langle k_j \rangle m_j$
2. *If $N_3^* \ll N_1^*$, then* $|M_{4,\kappa}(\mathbf{k})| \lesssim N^{-1} \left(\frac{N}{2^\kappa}\right)^{\frac{1}{10}} N_1^* \prod_{j=1}^4 \langle k_j \rangle m_j.$

The first item above can be used in the periodic setting, however the second estimate is too loose and since in the periodic case the bilinear improvement to Strichartz's estimate (see [CKS+01, Lemma 7.1]) doesn't hold, we cannot write the estimate on the Λ_4 term by a simple parallel to the non-periodic case.

Remark 2.7.3. If $N_3^* \sim N_1^*$, by using the pointwise estimate of Lemma 2.7.2 and $\langle k_{12} k_{14} \rangle \lesssim \langle \sigma_1^* \rangle$, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \Lambda_4(\widetilde{M_{4,\kappa}}; w_1(t), w_2(t), w_3(t), w_4(t)) dt \right| \\
&\lesssim N^{-1} \left(\frac{N}{N_1^*}\right)^{\frac{1}{10}} \int_{\tau_{1234}=0} \frac{1}{\lambda^3} \sum_{k_{1234}=0} \frac{\langle k_{12} k_{14} \rangle^{\frac12}}{\prod_{j=1}^4 \langle \sigma_j^* \rangle^{\frac12}} \prod_{j=1}^4 \widehat{w_j}(\tau_j,k_j)\, d\tau_1 d\tau_2 d\tau_3 \qquad (2.63) \\
&\lesssim N^{-1} \left(\frac{N}{N_1^*}\right)^{\frac{1}{10}} \int_* \widehat{w_1}(\tau_1,k_1) \prod_{j=2}^4 \frac{\widehat{w_j}(\tau_j,k_j)}{\langle \sigma_j^* \rangle^{\frac12}}
\end{aligned}$$

Then, taking u_j so that $\widehat{u_j}(\tau_j,k_j)=\frac{\widehat{w_j}(\tau_j,k_j)}{\langle\sigma_j^*\rangle^{\frac12}}$, we further have

$$\int_* \widehat{w_1}(\tau_1,k_1)\prod_{j=2}^{4}\frac{\widehat{w_j}(\tau_j,k_j)}{\langle\sigma_j^*\rangle^{\frac12}}\sim\int_{\mathbb{R}}\int_{\mathbb{T}_\lambda} w_1u_2u_3u_4\,dx\,dt \tag{2.64}$$

It turns out that the decay obtained in the periodic case for the Λ_4 term of the increment of $E(Iv)$ is not as fast as in the real-line case: we get $N^{-1/2+}$ rate of decay, as compared to N^{-1+} (see [CKS+01, Lemma 6.1]). We also show that this rate of decay of the Λ_4-term of (2.83) is sharp, see Remark 2.7.7 and Remark 2.7.8 below.

Lemma 2.7.4 (refined pointwise estimate in the periodic setting)**.**
For any $0<\varepsilon<\frac12$ *and* $\mathbf{k}\in\Gamma_4(\mathbb{T}_\lambda)$, *we have*

$$|\widetilde{M_{4,\kappa}}(\mathbf{k})|\lesssim N^{-\frac12+\varepsilon}\frac{1}{2^{\kappa\varepsilon}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}$$

This result is the prerequisite of Lemma 2.7.5 below. We point out the conceptual strategy in this proof: namely, we use $\langle\sigma_1^*\rangle^{\frac12}$ to cancel derivatives from the numerator of $\widetilde{M}_4$. Also, we point out that we use the crude lower bounds $\langle N_3^*\rangle\geqslant 1,\langle N_4^*\rangle\geqslant 1$ which renders an estimate that is non-optimal in λ as compared to what we obtain in Lemma **??**.

Proof. Case 1: $N_3^*\sim N_1^*$. We have $|M'_{4,\kappa}|\leqslant\langle\sigma_1^*\rangle N_1^*m_1m_2m_3m_4$ and so

$$|M'_{4,\kappa}|\lesssim\frac{\langle\sigma_1^*\rangle^{\frac12}N_1^*}{\langle N_1^*\rangle^3\langle N_4^*\rangle}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\lesssim\frac{N_1^{*2}}{\langle N_1^*\rangle^3}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\lesssim\frac{1}{N^{1-\varepsilon}N_1^{*\varepsilon}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}$$

To estimate $M''_{4,\kappa}$ we need to separate the analysis between the cases when all modes have comparable sizes or not.

Subcase 1".1: $N_4^*\sim N_1^*$. We write $M''_{4,\kappa}=f(0)-f(k_{12})+g(0)-g(k_{12})$, where

$$f(h)=m(k_1-h)^2(k_1-h)^2(k_3+h)\quad,\quad g(h)=m(k_3+h)^2(k_3+h)^2(k_1-h).$$

By the mean-value theorem, we get

$$|m_1^2k_1^2k_3+m_2^2k_2^2k_4|=|f(0)-f(k_{12})|\leqslant|k_{12}|\,|f'(h_f)|\quad,\quad|m_3^2k_3^2k_1+m_4^2k_4^2k_2|=|g(0)-g(k_{12})|\leqslant|k_{12}|\,|g'(h_g)|$$

for some h_f,h_g between 0 and k_{12}. We have

$$|f'(h_f)|\lesssim|m'(k_1-h_f)|\,m(k_1-h_f)|k_1-h_f|^2|k_3+h_f|+m(k_1-h_f)^2|k_1-h_f||k_3+h_f|+m(k_1-h_f)^2|k_1-h_f|^2$$

and since $|m'(h)|\lesssim\frac{1}{|h|}m(h)$ for any h and $m(h)^2|h|\lesssim m(N_1)^2N_1$ for $h=O(N_1)$, we deduce

$$|f'(h_f)|\lesssim m(k_1-h_f)^2|k_1-h_f||k_3+h_f|+m(k_1-h_f)^2|k_1-h_f|^2\lesssim m(N_1)^2N_1^2$$

Similarly, $|g'(h_g)|\lesssim m(N_1)^2N_1^2$ and therefore $|M''_{4,\kappa}|\lesssim m(N_1)^2N_1^2|k_{12}|$. Without loss of generality we can assume $|k_{12}|\geqslant|k_{14}|$ (otherwise we work with $f(0)-f(k_{14}=m_1^2k_1^2k_3+m_4^2k_4^2k_2$ and the analogue for the g-terms, so that the argument is identical). We thus estimate $\sigma_1^*\gtrsim|k_{12}|$ and since $m(N_1)N_1\gtrsim1$,

$$|\widetilde{M''_{4,\kappa}}|\lesssim\frac{|k_{12}|}{m(N_1)^2N_1^2}\frac{1}{\prod_{j=1}^4\langle\sigma_j\rangle^{\frac12}}\lesssim\frac{1}{m(N_1)^2N_1^2}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\lesssim\frac{1}{m(N_1)N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}.$$

Subcase 1".2: $N_4^*\ll N_1^*$. Using the fact that $\xi\mapsto m(\xi)^2\xi^2$ is increasing, we easily estimate $|M''_{4,\kappa}|\lesssim$

$m(N_1)^2N_1^3$ and $\sigma_1^* \gtrsim |k_{12}|\,|k_{14}| \sim N_1^2$. Therefore

$$|\widetilde{M''_{4,\kappa}}| \lesssim \frac{m(N_1)^2N_1^3}{m(N_1)^3N_1^3m(N_4)\langle N_4\rangle N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}} \lesssim \frac{1}{m(N_1)N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}$$

Common to the two subcases, due to $N_1 \gtrsim N$ we have $m(N_1)N_1 \sim N^{1-s}N_1^{s-\varepsilon}N_1^{\varepsilon} \gtrsim N^{1-\varepsilon}N_1^{*\varepsilon}$. Therefore, we get

$$|\widetilde{M''_{4,\kappa}}| \lesssim N^{-1+\varepsilon}\frac{1}{N_1^{*\varepsilon}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}$$

Case 2: $N_3^* \ll N_1^*$. As in *Case 1*, we estimate separately the contributions of $\widetilde{M'_{4,\kappa}}$ and $\widetilde{M''_{4,\kappa}}$, for each distinguishing between the situations of the largest two frequencies having the same or different parity.

Subcase 2'.1: $N_1 \sim N_3 \gg N_2, N_4$. Then $|k_{12}| \sim N_1$, $|k_{14}| \sim N_1$ and $|k_{13}| = |k_{24}| \leqslant N_2 + N_4 \ll N_3$, and therefore $\langle\sigma_1^*\rangle \gtrsim |k_{12}k_{14}| \sim N_1^2$. Consequently,

(2.65)
$$\begin{aligned}|\widetilde{M'_{4,\kappa}}| &= \frac{|k_{12}k_{13}k_{14}|}{\prod_{j=1}^4\langle k_j\rangle\langle\sigma_j\rangle^{\frac12}} \lesssim \frac{N_1^2|k_{24}|}{N_1\langle N_2\rangle N_3\langle N_4\rangle}\frac{1}{\prod_{j=1}^4\langle\sigma_j\rangle^{\frac12}} \lesssim \frac{|k_{24}|}{\langle N_2\rangle\langle N_4\rangle}\frac{1}{N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}} \lesssim \frac{1}{N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\\ &\lesssim \frac{1}{N^{1-\varepsilon}}\frac{1}{N_1^{*\varepsilon}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\end{aligned}$$

Subcase 2'.2: $N_2 \sim N_4 \gg N_1, N_3$. It is analogous to the subcase *2'.1*.

Subcase 2'.3: $N_1 \sim N_2 \gg N_3, N_4$. Then $|k_{12}| = |k_{34}| \leqslant N_3 + N_4 \ll N_1$, $|k_{14}| \sim N_1$, $|k_{13}| \sim N_1$, and therefore $\langle\sigma_1^*\rangle \gtrsim N_1|k_{34}|$. It follows that

$$\begin{aligned}|\widetilde{M'_{4,\kappa}}| &\lesssim \frac{N_1^2|k_{34}|}{N_1N_2\langle N_3\rangle\langle N_4\rangle}\frac{1}{N_1^{\frac12}|k_{34}|^{\frac12}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}} \lesssim \frac{|k_{34}|^{\frac12}}{\langle N_3\rangle\langle N_4\rangle}\frac{1}{N_1^{\frac12}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\\ &\lesssim \frac{1}{N^{\frac12-\varepsilon}}\frac{1}{N_1^{*\varepsilon}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}\end{aligned} \qquad (2.66)$$

Subcase 2'.4: $N_1 \sim N_4 \gg N_2, N_3$. It is analogous to the subcase *2'.3*.

Subcase 2".1: $N_1 \sim N_3 \gg N_2, N_4$. We can write $M_4''^{13} := m(k_1)^2k_1^2k_3 + m(k_3)^2k_3^2k_1 = f(0) - f(k_{24})$, where $f(h) = m(k_1+h)^2(k_1+h)^2(k_3+h)$. By the Mean-Value Theorem, we have $|f(0) - f(k_{24})| \leqslant |k_{24}|\,|f'(h)|$ for some h between 0 and k_{24} (hence $|h| \leqslant |k_{24}| \ll N_1$). Then $|m'(k_1+h)| \lesssim \frac{1}{|k_1+h|}m(k_1+h) \sim \frac{1}{N_1}m(N_1)$, , and thus

$$|f'(h)| \lesssim \frac{1}{N_1}m(N_1)^2N_1^2N_3 + m(N_1)^2N_1N_3 + m(N_1)^2N_1^2 \lesssim m(N_1)^2N_1^2$$

For $M_4''^{24} := m(k_2)^2k_2^2k_4 + m(k_4)^2k_4^2k_2 = g(0) - g(k_{24})$, we take $g(h) = m(k_2-h)^2(k_2-h)^2(k_4-h)$. We have $|g(0) - g(k_{24})| \leqslant |k_{24}|\,|g'(h)| \lesssim$ for some h between 0 and k_{24}, and since $|m'(k_2-h)| \lesssim \frac{1}{|k_2-h|}m(k_2-h)$, $|k_2-h| \sim N_1$, $k_4 - h| \ll N_1$, we also have

$$|g'(h)| \lesssim |m'(k_2-h)|m(k_2-h)|k_2-h|^2|k_4-h| + m(N_1)^2N_1|k_4-h| + m(N_1)^2N_1^2 \lesssim m(N_1)^2N_1^2.$$

Therefore $|M''_{4,\kappa}| \leqslant |f(0) - f(k_{24})| + |g(0) - g(k_{24})| \lesssim |k_{24}|m(N_1)^2N_1^2$ and using $\sigma_1^* \gtrsim |k_{12}k_{14}| \sim N_1^2$ we get

$$|\widetilde{M''_{4,\kappa}}| = \left|\frac{M''_{4,\kappa}}{\prod_{j=1}^4 m_j\langle k_j\rangle\langle\sigma_j\rangle^{\frac12}}\right| \lesssim \frac{|k_{24}|m_1^2N_1^2}{m_1^2m_2m_4N_1^2\langle N_2\rangle\langle N_4\rangle}\frac{1}{\prod_{j=1}^4\langle\sigma_j^*\rangle^{\frac12}} \lesssim \frac{|k_{24}|}{m_2m_4\langle N_2\rangle\langle N_4\rangle N_1}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac12}}$$

If $N_2 \gtrsim N$ and $N_4 \ll N$, then $|k_{24}| \sim N_2$, $m_2 \sim \left(\frac{N}{N_2}\right)^{1-s}$, $m_4 = 1$ and so

$$\frac{|k_{24}|}{m_2 m_4 \langle N_2\rangle\langle N_4\rangle N_1} \lesssim \frac{{N_2}^{1-s}}{N^{1-s}\langle N_4\rangle N_1^{1-\varepsilon} N_1^{\varepsilon}} \lesssim \frac{N_1^{1-s}}{N^{1-s} N_1^{1-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \sim \frac{N_1^{\varepsilon-s}}{N^{1-s}} \frac{1}{N_1^{*\varepsilon}} \lesssim \frac{N^{\varepsilon-s}}{N^{1-s}} \frac{1}{N_1^{*\varepsilon}} \sim N^{-1+\varepsilon} \frac{1}{N_1^{*\varepsilon}}$$

The case $N_4 \gtrsim N$ and $N_2 \ll N$ is analogous due to the symmetry in k_2, k_4 of the left hand side above. If $N_2 \gtrsim N$ and $N_4 \gtrsim N$, without loss of generality we may assume $N_2 \geqslant N_4$ and therefore we have $|k_{24}| \lesssim N_2$, $m_2 \sim \left(\frac{N}{N_2}\right)^{1-s}$, $m_4 \sim \left(\frac{N}{N_4}\right)^{1-s}$ and so

$$\begin{aligned}\frac{|k_{24}|}{m_2 m_4 \langle N_2\rangle\langle N_4\rangle N_1} &\lesssim \frac{N_2 N_2^{1-s} N_4^{1-s}}{N^{2-2s} N_2 N_4 N_1^{1-\varepsilon}} \frac{1}{N_1^{\varepsilon}} \lesssim \frac{N_2^{1-s} N_4^{-s}}{N^{2-2s} N_1^{1-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \lesssim \frac{N_1^{1-s} N^{-s}}{N^{2-2s} N_1^{1-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \\ &\lesssim \frac{N_1^{\varepsilon-s}}{N^{2-s}} \frac{1}{N_1^{*\varepsilon}} \lesssim \frac{N^{\varepsilon-s}}{N^{2-s}} \frac{1}{N_1^{*\varepsilon}} = N^{-2+\varepsilon} \frac{1}{N_1^{*\varepsilon}}\end{aligned}$$

Subcase 2".2: $N_1 \sim N_2 \gg N_3, N_4$. We can write $M_4''^{12} := m(k_1)^2 k_1^2 k_3 + m(k_2)^2 k_2^2 k_4 = f(0) - f(k_{34})$, where $f(h) = m(k_1+h)^2 (k_1+h)^2 (k_3-h)$. By the Mean-Value Theorem, we have $|f(0) - f(k_{34})| \leqslant |k_{34}|\,|f'(h)|$ for some h between 0 and k_{34} (hence $|h| \leqslant |k_{34}| \ll N_1$). We estimate $|m'(k_1+h)| \lesssim \frac{1}{|k_1+h)|} m(k_1+h) \sim \frac{1}{N_1} m(N_1)$ and thus

$$|f'(h)| \lesssim |m'(k_1+h)| m(N_1) N_1^2 N_3 + m(N_1)^2 N_1^2 \lesssim m(N_1)^2 N_1 N_3 + m(N_1)^2 N_1^2 \lesssim m(N_1)^2 N_1^2.$$

Using $\sigma_1^* \gtrsim |k_{34}| N_1$, it follows that

$$|\widetilde{M_{4,\kappa}''^{12}}| = \left| \frac{M_4''^{12}}{\prod_{j=1}^4 m_j \langle k_j\rangle\langle\sigma_j\rangle^{\frac12}} \right| \lesssim \frac{m_1^2 N_1^2 |k_{34}|}{m_1^2 m_3 m_4 N_1^2 \langle N_3\rangle\langle N_4\rangle} \frac{1}{\prod_{j=1}^4 \langle\sigma_j\rangle^{\frac12}} \lesssim \frac{|k_{34}|^{\frac12}}{m_3 m_4 \langle N_3\rangle\langle N_4\rangle N_1^{\frac12}} \frac{1}{\prod_{j=2}^4 \langle\sigma_j^*\rangle^{\frac12}}$$

If $N_3 \gtrsim N$ and $N_4 \ll N$, then $|k_{34}| \sim N_3$, $m_3 \sim \left(\frac{N}{N_3}\right)^{1-s}$, $m_4 = 1$ and so

$$\frac{|k_{34}|^{\frac12}}{m_3 m_4 \langle N_3\rangle\langle N_4\rangle N_1^{\frac12}} \lesssim \frac{N_3^{\frac12 - s}}{N^{1-s}\langle N_4\rangle} \frac{1}{N^{\frac12-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \lesssim \frac{N^{\frac12-s}}{N^{1-s} N^{\frac12-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \sim N^{-1+\varepsilon} \frac{1}{N_1^{*\varepsilon}}.$$

The case $N_4 \gtrsim N$ and $N_3 \ll N$ is analogous due to the symmetry in k_3, k_4 of the left hand side above. If $N_3 \gtrsim N$ and $N_4 \gtrsim N$, without loss of generality we may assume $N_3 \geqslant N_4$ and therefore we have $|k_{34}| \lesssim N_3$, $m_3 \sim \left(\frac{N}{N_3}\right)^{1-s}$, $m_4 \sim \left(\frac{N}{N_4}\right)^{1-s}$ and so

$$\frac{|k_{34}|^{\frac12}}{m_3 m_4 \langle N_3\rangle\langle N_4\rangle N_1^{\frac12}} \lesssim \frac{N_3^{\frac12+1-s} N_4^{1-s}}{N^{2-2s}\langle N_3\rangle\langle N_4\rangle N_1^{\frac12}} \lesssim \frac{N_3^{\frac12-s} N_4^{-s}}{N^{2-2s} N_1^{\frac12}} \lesssim \frac{N^{\frac12-s} N^{-s}}{N^{2-2s} N^{\frac12-\varepsilon}} \frac{1}{N_1^{*\varepsilon}} \sim N^{-2+\varepsilon} \frac{1}{N_1^{*\varepsilon}}$$

Therefore

$$|\widetilde{M_{4,\kappa}''^{12}}| \lesssim N^{-1+\varepsilon} \frac{1}{2^{\varepsilon\kappa}} \frac{1}{\prod_{j=2}^4 \langle\sigma_j^*\rangle^{\frac12}}$$

For $M_{4,\kappa}''^{34} := m(k_3)^2 k_3^2 k_1 + m(k_4)^2 k_4^2 k_2 = g(0) - g(k_{34})$, where $g(h) = m(k_3-h)^2 (k_3-h)^2 (k_1+h)$, we have

$$|g'(h)| \lesssim |m'(k_3-h)| m(k_3-h) |k_3-h|^2 N_1 + m(k_3-h)^2 |k_3-h| N_1 + m(k_3-h)^2 |k_3-h|^2 \lesssim m(N_4^*)^2 N_3^* N_1^*$$

and thus

$$|\widetilde{M_{4,\kappa}''^{34}}| \leqslant \frac{|k_{34}|\,|g'(h)|}{\prod_{j=1}^4 m_j \langle k_j\rangle\langle\sigma_j\rangle^{\frac12}} \lesssim \frac{|k_{34}|^{\frac12} m(N_4^*)}{m(N_1^*) m(N_2^*) m(N_3^*) N_2^* \langle N_4^*\rangle {N_1^*}^{\frac12}} \frac{1}{\prod_{j=2}^4 \langle\sigma_j^*\rangle^{\frac12}}.$$

We further estimate

$$\frac{|k_{34}|^{\frac{1}{2}}m(N_4^*)}{m(N_1^*)m(N_2^*)m(N_3^*)N_2^*N_4^*N_1^{*\frac{1}{2}}} \lesssim \frac{N_3^{*\frac{1}{2}}\cdot 1}{\left(\frac{N}{N_1^*}\right)^{2-2s}\left(\frac{N}{N_3^*}\right)^{1-s}N_1^{*\frac{3}{2}}} \lesssim \frac{N_1^{*\frac{1}{2}-2s}N_3^{*\frac{3}{2}-s}}{N^{3-3s}} \lesssim \frac{N_1^{*\frac{3}{2}-s}}{N_1^{*\frac{5}{2}-s}} \lesssim \frac{1}{N^{1-\varepsilon}N_1^{*\varepsilon}}.$$

which gives

$$|\widetilde{M''_{4,\kappa}}^{34}| \lesssim N^{-1+\varepsilon}\frac{1}{2^{\varepsilon\kappa}}\frac{1}{\prod_{j=2}^4\langle\sigma_j^*\rangle^{\frac{1}{2}}}\ .$$

□

The contribution of the $a(t)$-term will follow from the following lemma, the algebra property of the $H^{\frac{1}{2}}(\mathbb{T}_\lambda)$ space and the estimate on $\|a(t)\|_{H_t^{\frac{1}{2}+}}$.

Lemma 2.7.5.

$$\left|\int_{\mathbb{R}}\Lambda_4(M_{4,\kappa};v_1(t),\overline{v_2}(t),v_3(t),\overline{v_4}(t))dt\right| \lesssim \lambda^{0+}N^{-\frac{1}{2}+}\frac{1}{2^{\varepsilon\kappa}}\prod_{j=1}^4\|Iv_j\|_{X^{1,\frac{1}{2}}} \tag{2.67}$$

Proof. With the functions defined by (2.57), the above (2.67) is equivalent with

$$\left|\int_{\mathbb{R}}\Lambda_4(\widetilde{M_{4,\kappa}};w_1(t),w_2(t),w_3(t),w_4(t))dt\right| \lesssim \lambda^{0+}N^{-\frac{1}{2}+}\frac{1}{2^{\varepsilon\kappa}}\prod_{j=1}^4\|w_j\|_{L^2_{x,t}}. \tag{2.68}$$

Without loss of generality we assume that $\widehat{w_j}$ are real and nonnegative, and also that $\sigma_j^*=\sigma_j$, $1\leqslant j\leqslant 4$. Then, using Lemma 2.7.4 and considering u_j defined by $\hat{u_j}=\frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac{1}{2}}}$ $(j=2,3,4)$, we have

$$\begin{aligned}
\mathrm{LHS}_{(2.68)} &\lesssim \frac{1}{\lambda^3}\int_*|\widetilde{M_{4,\kappa}}|\,\widehat{w_1}(\tau_1,k_1)\widehat{w_2}(\tau_2,k_2)\widehat{w_3}(\tau_3,k_3)\widehat{w_4}(\tau_4,k_4)\\
&\lesssim N^{-\frac{1}{2}+\varepsilon}\frac{1}{2^{\kappa\varepsilon}}\frac{1}{\lambda^3}\int_*\widehat{w_1}(\tau_1,k_1)\widehat{u_2}(\tau_2,k_2)\widehat{u_3}(\tau_3,k_3)\widehat{u_4}(\tau_4,k_4)\\
&\sim N^{-\frac{1}{2}+\varepsilon}\frac{1}{2^{\kappa\varepsilon}}\left|\int_{\mathbb{R}}\int_{\mathbb{T}_\lambda}w_1(t,x)u_2(t,x)u_3(t,x)u_4(t,x)\,dx\,dt\right|
\end{aligned} \tag{2.69}$$

Applying the $L^2_{x,t}L^6_{x,t}L^6_{x,t}L^6_{x,t}$–Hölder inequality and then three times the L^6–Strichartz inequality (2.21), we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}}\int_{\mathbb{T}_\lambda}w_1(t,x)u_2(t,x)u_3(t,x)u_4(t,x)\,dx\,dt\right| &\leqslant \|w_1\|_{L^2_{x,t}}\|u_2\|_{L^6_{x,t}}\|u_3\|_{L^6_{x,t}}\|u_4\|_{L^6_{x,t}}\\
&\lesssim \lambda^{3(0+)}\|w_1\|_{L^2_{x,t}}\|u_2\|_{X^{0+,\frac{1}{2}}}\|u_3\|_{X^{0+,\frac{1}{2}}}\|u_4\|_{X^{0+,\frac{1}{2}}}\\
&\lesssim \lambda^{0+}N_1^{*3\delta}\|w_1\|_{L^2_{x,t}}\|u_2\|_{X^{0,\frac{1}{2}}}\|u_3\|_{X^{0,\frac{1}{2}}}\|u_4\|_{X^{0,\frac{1}{2}}}\\
&\sim \lambda^{0+}2^{3\delta\kappa}\|w_1\|_{L^2_{x,t}}\|w_2\|_{L^2_{x,t}}\|w_3\|_{L^2_{x,t}}\|w_4\|_{L^2_{x,t}}
\end{aligned} \tag{2.70}$$

Choosing $\varepsilon=4\delta$ and some $0<\delta\ll 1$, (2.69) and (2.69) imply

$$\mathrm{LHS}_{(2.68)} \lesssim \lambda^{0+}N^{-\frac{1}{2}+}\frac{1}{2^{\delta\kappa}}\prod_{j=1}^4\|w_j\|_{L^2_{x,t}}.$$

□

Proposition 2.7.6 (multilinear estimate for Λ_4)**.**

$$\left|\int_T^{T+\delta} \Lambda_4(M_4; v(t))\, dt\right| \lesssim \lambda^{0+} N^{-\frac{1}{2}+} \|Iv\|^4_{X^{1,\frac{1}{2}}([T,T+\delta]\times\mathbb{T}_\lambda)} \tag{2.71}$$

Proof. Since $2^\kappa = N_1^* \geqslant N_j^*$, by summing over all dyadics $N_1^* \geqslant N_2^* \geqslant N_3^* \geqslant N_4^*$, it follows from Proposition **??**, that

□

Remark 2.7.7. The decay $N^{-\frac{1}{2}}$ is optimal for the Λ_4-term in the periodic setting, as compared to the analogous result on $\mathbb{R}$ where we can get $\beta = 1$.

Indeed, let the dyadic numbers $N_j = |k_j|$, $1 \leqslant j \leqslant 4$ in the following relation:

$$N_1 \sim N_2 \gtrsim N \gg N_3 \gg N_4$$

and consider the functions f_j defined by

$$\widehat{f}_j(k_j, \tau_j) = \mathbb{1}_{(-1)^{j-1}N_j}(k_j)\mathbb{1}_{|\sigma_j|\leqslant 1}(\tau_j) \quad , \quad j = 2, 3, 4$$

and taking into account that $k_{1234} = 0$ which forces k_1 and k_2 to have opposite signs and by (2.51), $|\sigma_1 - 2k_{12}k_{14}| = |\sigma_2 + \sigma_3 + \sigma_4|$, we define f_1 as

$$\widehat{f}_1(k_1, \tau_1) = \mathbb{1}_{N_1}(k_1)\mathbb{1}_{|\sigma_1 - 2N_{12}N_{14}|\leqslant 1}(\tau_1).$$

Remark 2.7.8. We cannot expect cancelation in the M_4 multiplier due to the constants $C_1 = 4i$ and $C_2 = -\frac{1}{2}i$. Indeed, consider the case when

$$N_1 \sim N_2 \gg N \ , \ N_{12} = N_{34} \sim 1 \text{ and } N_3 \lesssim N$$

Then $m_3 = m_4 = 1$ and

$$4M_4' \sim 4m_1^2 N_1^2 N_3 \quad , \quad \frac{1}{2}M_4'' = m_1^2 N_1^2 N_3 + N_1 N_3^2 \ .$$

Since $m_1^2 N_1^2 N_3 \sim N^{2-2s} N_1^{2s} N_3$ and $2s > 1$, it is clear that $m_1^2 N_1^2 N_3 \gg N_1 N_3^2$.

2.8 Global well-posedness using the first generation almost conserved energy

Here we present a generic/prototype argument for obtaining the GWP of (DNLS). We assume that $N^{-\frac{1}{2}+}$ is the fastest decay of the increment of $E_N(v)$, i.e. the remaining terms involving $\Lambda_6, \Lambda_8, \Lambda_{10}$ in (2.83) have estimates with the same rates of decay in N as the Λ_4 term:

$$\left|\int_T^{T+\delta} \Lambda_n(M_n; v(t))\, dt\right| \lesssim N^{-\frac{1}{2}+} \|Iv\|^4_{X^{1,\frac{1}{2}}([T,T+\delta]\times\mathbb{T}_\lambda)} \quad \text{for } n = 4, 6, 8, 10. \tag{2.72}$$

Here we proceed with the proof of the global well-posedness of the gauged DNLS (2.8) corresponding to this decay rate.

Theorem 2.8.1. *Let $s > \frac{3}{4}$ and let $w_0 \in H^s(\mathbb{T})$ with $\|w_0\|_{L^2_x(\mathbb{T})}$ small enough. If $T > 0$ and w is a solution of (2.10) on $[0,T]$, then there exists $C = C(\|w_0\|_{H^s(\mathbb{T})}, T) > 0$ such that*

$$\sup_{t\in[0,T]} \|w(t)\|_{\dot{H}^s(\mathbb{T})} \leqslant C.$$

Proof. By (2.34), we have $\|\partial_x I w_0^\lambda\|_{L^2(\mathbb{T}_\lambda)} = \|I w_0^\lambda\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim N^{1-s}\|w_0^\lambda\|_{H^s(\mathbb{T}_\lambda)}$, where $w_0^\lambda(x) = \lambda^{-\frac{1}{2}} w_0(\lambda^{-1}x)$. We have $\widehat{w_0^\lambda}(k) = \lambda^{\frac{1}{2}}\widehat{w_0}(\lambda k)$ and

$$\|w_0^\lambda\|_{\dot{H}^s(\mathbb{T}_\lambda)} = \lambda^{\frac{1}{2}}\|\,|k|^s w_0(\lambda k)\|_{\ell^2_k} = \|\,|\frac{1}{\lambda}\xi|^s w_0(\xi)\|_{\ell^2_\xi} = \lambda^{-s}\|\,|\xi|^s w_0(\xi)\|_{\ell^2_\xi} = \lambda^{-s}\|w_0\|_{\dot{H}^s(\mathbb{T})}$$

Thus

$$\|\partial_x I w_0^\lambda\|_{L^2_x(\mathbb{T}_\lambda)} \leqslant c\frac{N^{1-s}}{\lambda^s}\|w_0\|_{\dot{H}^s(\mathbb{T})} \tag{2.73}$$

(while $\|I w_0^\lambda\|_{L^2(\mathbb{T}_\lambda)} \leqslant \|w_0^\lambda\|_{L^2(\mathbb{T}_\lambda)} = \|w_0\|_{L^2(\mathbb{T})}$). By choosing $\lambda > 0$ so that $\frac{N^{1-s}}{\lambda^s} \sim 1$, or equivalently

$$\lambda \sim N^{\frac{1-s}{s}} \tag{2.74}$$

we have $1 \lesssim \lambda \lesssim N$ (provided $\frac{1}{2} \leqslant s \leqslant 1$) and $\|I w_0^\lambda\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim 1$.

With $C_1 > 0$ to be chosen later, suppose that $E(Iw^\lambda(t_0)) \leqslant 2C_1$. Then, by Gagliardo-Nirenberg inequality, we have $\|Iw^\lambda(t_0)\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim E(Iw^\lambda(t_0))$. By Proposition 2.6.1, for $\delta = 1$ there exists $D_1 > 0$ and $D_2 \geqslant 1$ such that

$$\|Iw^\lambda(t_0)\|_{H^1(\mathbb{T}_\lambda)} \leqslant D_1 \;\Rightarrow\; \|Iw^\lambda\|_{X^{1,\frac{1}{2}}([t_0,t_0+1]\times\mathbb{T}_\lambda)} \leqslant D_2 \,.$$

With these preparations, we can begin arguing by induction. Fix $T > 0$ and take $j_* = \lceil T\lambda^2 \rceil$. Hence

$$j_* \sim T\lambda^2. \tag{2.75}$$

If C_1 is chosen so that $E(Iw_0^\lambda(0)) \leqslant C_1 \leqslant 2C_1$, by the observation above we get $\|Iw^\lambda\|_{X^{1,\frac{1}{2}}([0,\delta]\times\mathbb{T}_\lambda)} \leqslant D_2$. Then, by Proposition ?? and Lemma ?? we have

$$E(Iw^\lambda(\delta)) \leqslant E(Iw^\lambda(0)) + c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10} \,.$$

We impose that $c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10} \leqslant C_1$.

For the next step, note that we have $E(Iw^\lambda(\delta)) \leqslant 2C_1$ and by the above argument $\|Iw^\lambda\|_{X^{1,\frac{1}{2}}([\delta,2\delta]\times\mathbb{T}_\lambda)} \leqslant D_2$. The modified energy increment is estimated by

$$E(Iw^\lambda(2\delta)) \leqslant E(Iw^\lambda(\delta)) + c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10} \leqslant E(Iw^\lambda(0)) + 2c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10}.$$

We impose that $2c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10} \leqslant C_1$.

Inductively, we get

$$E(Iw^\lambda(j_*\delta)) \leqslant E(Iw^\lambda(0)) + j_*c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10}.$$

We impose that $j_*c\lambda^{0+}N^{-\frac{1}{2}+}D_2^{10} \leqslant C_1$. In particular, since $j_*\delta > T\lambda^2$ we get $E(Iw^\lambda(T\lambda^2)) \leqslant 2C_1$ and by Gagliardo-Nirenberg inequality, we thus have $\|Iw^\lambda\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim C_1$

Now note that the choice of C_1 asks for $j_*\lambda^{0+}N^{-\frac{1}{2}+} \sim 1$, or equivalently

$$j_* \sim \lambda^{0-}N^{\frac{1}{2}-} \tag{2.76}$$

From (2.74), (2.75) and (2.76), we get $T \sim N^{(2+)\frac{s-1}{s}}N^{\frac{1}{2}-}$ which can be taken arbitrarily large if and only if $\frac{5}{2} - \frac{2}{s} > 0$, or equivalently $s > \frac{4}{5}$.

□

Remark 2.8.2. By using the bi-linear estimate (2.29) in the proof of Lemma 2.7.5 we can get $\lambda^{-\frac{1}{2}+}N^{-\frac{1}{2}+}$ decay on the increments of the almost conserved energy, hence obtaining global well-posedness in $H^{\frac{3}{4}+}(\mathbb{T})$.

We go here only through the numerology involved. Essentially we want to cover the interval $[0, \lambda^2 T]$

with j intervals of length δ, hence $j\delta \sim \lambda^2 T$ where δ is given by the local well-posedness theory for the I-system.

From the inductive step, we require that $j\lambda^{-\frac{1}{2}+}N^{-\frac{1}{2}+} \sim C_1$, hence $j \sim \lambda^{\frac{1}{2}-}N^{\frac{1}{2}-}$.

So $\lambda^2 T \sim j \sim \lambda^{\frac{1}{2}-}N^{\frac{1}{2}-}$ and thus $T \sim N^{(\frac{3}{2}+)\frac{s-1}{s}}N^{\frac{1}{2}-}$ which goes to infinity if and only if $2 - \frac{3}{2s} > 0$ or equivalently $s > \frac{3}{4}$.

Notice that even this regularity threshold is not as good as the threshold obtained with the first generation of the I-method (see [CKS+01]) in the non-periodic setting where was obtained $s > \frac{2}{3}$.

Theorem 2.8.3. *The gauged DNLS (2.10) is globally well-posed in $H^s(\mathbb{T})$ for $s > \frac{4}{5}$, provided that the initial data has mass smaller than $2\sqrt{2}$.*

2.9 The I-method in a nutshell

1. Prescribe $T > 0$ (arbitrary time of existence to be reached by the solutions of DNLS, or equivalently of a gauged DNLS) and let $u_0 \in H^s(\mathbb{T})$ be an initial datum with $M(u_0) < \delta$.

   Note that we need to apply the gauge, i.e. $v(t) := \mathcal{G}_1(u(t))$ in order to deal with the cubic derivative-nonlinearity. The goal is to establish (2.1), i.e. to show that $\sup_{t\in[0,T]} \|v(t)\|_{H^s_x(\mathbb{T})}$ is finite.

2. Apply the smoothing operator I (with parameter N) to obtain the I-system. Even though $v_0 \notin H^1$, the smoother object Iv_0 belongs to H^1.

3. Use the natural scaling of the equation, namely $v^\lambda(t,x) = \lambda^{-1/2}v(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ to ensure that $\|Iv\|_{\dot{H}^1} \leqslant D$, where D is a fixed threshold that ensures the same time of existence (i.e. ~ 1) for the solution Iv of the I-system.

4. At each iteration, invoke the local well-posedness of the I-system and note that the modified energy E_N does not increase much and, in fact, in collaboration with its control of the $\dot{H}^1$-norm of Iv, maintains $\|Iv\|_{\dot{H}^1} \leqslant D$ at all times.

5. Impose a doubling condition on the growth of E_N on $[0,T]$ and determine the condition on the choice of N that allows T to be as large as prescribed. For such a choice of N, the I-operator acts as an identity on $[0,T]$, and thus $E_N(v(t)) = E(v(t))$ for $t \in [0,T]$. Since $\|v\|_{\dot{H}^1}$ is controlled by $E(v)$ and $E(v)$ stays finite, we certainly have (2.1).

6. Finally, use the bi-continuity of the gauge transform $\mathcal{G}_1$ to migrate the result to the original (DNLS).

2.10 Refinement of the I-method

We modify further the expression of the (almost conserved) energy by taking

$$\mathcal{E}_N^2(v) := \mathcal{E}(Iv) + \Lambda_4(\sigma_4; v) \tag{2.77}$$

where the "correction" multiplier σ_4 is taken so that when we compute $\partial_t \mathcal{E}^2(v)$, the lowest order term is $\Lambda_6(M_6^2; v)$ (i.e. a 6th order as opposed to a 4th order multilinear form that appeared in the first generation). We recall that

$$\partial_t \mathcal{E}(Iv(t)) = \sum_{j=\{4,6,8\}} \Lambda_j(M_j; v(t))$$

with M_j's given by Proposition 2.5.3. By (2.38), we have

$$\partial_t \Lambda_4(\sigma_4; v(t)) = i\Lambda_4(\sigma_4 \sum_{j=1}^{4} (-1)^j k_j^2; v(t)) - i\Lambda_6(\sum_{j=1}^{4} \mathbb{X}_j^2(\sigma_4) k_{j+1}(\mathbb{1}_{\Theta_6^j} + \mathbb{1}_{\Upsilon_6^j})(\mathbf{k}); v(t))$$

Set $\alpha_4(\mathbf{k}) := k_1^2 - k_2^2 + k_3^2 - k_4^2$ and define $\sigma_4(\mathbf{k})$ so that $M_4 - i\sigma_4\alpha_4 = 0$ or equivalently

$$\sigma_4 := -i\frac{M_4}{\alpha_4} \tag{2.78}$$

Notice that for $\mathbf{k} \in \Gamma_4$ we have $\alpha_4(\mathbf{k}) = 2k_{12}k_{14}$ and therefore

$$\sigma_4(\mathbf{k}) = -\frac{1}{4}(\sigma_4' + \sigma_4'')$$

where

$$\sigma_4'(\mathbf{k}) := m_1 m_2 m_3 m_4 k_{13} \tag{2.79}$$

$$\sigma_4''(\mathbf{k}) := \frac{m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2}{k_{12} k_{14}} \mathbb{1}_{\Theta_4} \tag{2.80}$$

Therefore, by (2.77) and (2.44), our second generation modified energy is

$$\mathcal{E}_N^2(v) = \Lambda_2(m_1^2 k_1^2) - \frac{1}{4}\Lambda_4(\sigma_4'')$$

From this construction, we have that

$$\partial_t \mathcal{E}_N^2(v(t)) = \sum_{j=\{6,8\}} \Lambda_j(M_j^2; v(t))$$

where we can easily track down the multipliers

$$M_6^2 = M_6 - i\sum_{j=1}^{4} \mathbb{X}_j^2(\sigma_4) k_{j+1}(\mathbb{1}_{\Theta_6^j} + \mathbb{1}_{\Upsilon_6^j}) \tag{2.81}$$

$$M_8^2 = M_8 + \frac{i}{2}\sum_{j=1}^{4} (-1)^{j-1} \mathbb{X}_j^4(\sigma_4) \mathbb{1}_{\Sigma_8^j} \tag{2.82}$$

and so we obtain the analogue of Proposition 2.5.3 as follows.

Proposition 2.10.1. *Suppose $v(t) \in H^s(\mathbb{T}_\lambda)$ for all t and let $\delta > 0$. Then:*

$$\mathcal{E}_N^2(v(T+\delta)) - \mathcal{E}_N^2(v(T)) = \int_T^{T+\delta} \left[\Lambda_6(M_6^2; v) + \Lambda_8(M_8^2; v)\right] dt \tag{2.83}$$

where the multipliers $M_j^2 : \Gamma_j \to \mathbb{C}$ are given by (2.81) and (2.82).

2.11 Estimating increments of the second generation almost conserved energy

Lemma 2.11.1. *For every $\mathbf{k} = (k_1, \ldots, k_n) \in \Gamma_n(\mathbb{T}_\lambda)$, we have*

1. $|M_6^2| \lesssim (m(N_1)N_1)^2$, *and if in addition $N_3 \ll N$, then $|M_6^2| \lesssim N_1 N_3$;*
2. $|M_8^2| \lesssim m(N_1)^2 N_1$,

where $N_1, \ldots, N_j$ denote a decreasing rearrangement (i.e. $N_1 \geqslant N_2 \geqslant \ldots$) of $n_1 = |k_1|, \ldots, n_j = |k_j|$. If $N_1 \ll N$, then both multipliers M_6^2 and M_8^2 vanish.

Proof. See Lemma 6.4 and Lemma 6.6 in [CKS+02]. □

Proposition 2.11.2.

$$\left| \int_{\mathbb{R}} \Lambda_6(M_6; v_1, \overline{v_2}, v_3, \overline{v_4}, v_5, \overline{v_6}) \right| \lesssim \lambda^{-1+} N^{-1+} \prod_{j=1}^{6} \|I v_j\|_{X^{1,\frac{1}{2}}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.84}$$

Proof. We introduce the functions w_j defined by

$$\widehat{w_j}(\tau_j, k_j) = m(k_j)\langle k_j\rangle\langle\sigma_j\rangle^{\frac{1}{2}} \widehat{v_j}(\tau_j, k_j), \text{ with } \sigma_j := \tau_j + k_j^2 \text{ , if } j \text{ is odd}$$
$$\widehat{w_j}(\tau_j, k_j) = m(k_j)\langle k_j\rangle\langle\sigma_j\rangle^{\frac{1}{2}} \widehat{\overline{v_j}}(\tau_j, k_j), \text{ with } \sigma_j := \tau_j - k_j^2 \text{ , if } j \text{ is even}$$

It follows that $\|I v_j\|_{X^{1,\frac{1}{2}}} = \|w_j\|_{L^2_{t,x}}$ for all j. Also, it is useful to have

$$\widehat{f_j} := \frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac{1}{2}}} \quad , \quad \widehat{g_j} := \frac{\widehat{w_j}}{\langle k_j\rangle^{\frac{1}{2}}\langle\sigma_j\rangle^{\frac{1}{2}}}$$

so that $\|f_j\|_{X^{0,\frac{1}{2}}} = \|w_j\|_{L^2_{t,x}}$ and $\|g_j\|_{X^{\frac{1}{2},\frac{1}{2}}} = \|w_j\|_{L^2_{t,x}}$ for all j.

We use Littlewood-Paley decomposition:

$$w_j = \sum_{n_j} P_{n_j} w_j$$

and in the sequel we assume $w_j = P_{n_j} w_j$. Without loss of generality, we can assume that for all j's, $\widehat{w_j}$ is real-valued and non-negative. To assume summability over all dyadics $N_1 \geqslant N_2 \geqslant \ldots \geqslant N_6$ we usually ensure a factor of

$$\frac{1}{N_1^{0+}} \quad \left(\lesssim \prod_{j=1}^{6} \frac{1}{N_j^{0+}} \right)$$

on the right hand side of the estimates. However, in one of the sub-cases that we discuss below, we need to make use of a trick similar to one used by Bourgain (see sub-case (a.2) below).

As in the Euclidean setting, we distinguish three cases:

(a) $N_4 \gtrsim N$

(b) $N_3 \gtrsim N \gg N_4$

(c) $N_2 \gtrsim N \gg N_3$

This is a partition over of the entire frequency space taking into account that we must have $N_1 \sim N_2 \gtrsim N$ (while in the case $N_1 \ll N$, we have no contribution since M_6 vanishes). We denote

$$I_6(w_1,\ldots,w_6) := \int_* \frac{M_6}{\prod_{j=1}^6 m(n_j)\langle n_j\rangle} \prod_{j=1}^6 \frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac12}} \quad , \quad \text{where } \int_* := \int_{\tau_{123456}=0} \frac{1}{(2\pi\lambda)^5} \sum_{k_{123456}=0}$$

$$I_6^{(a)}(w_1,\ldots,w_6) := \int_* \frac{M_6 \mathbb{1}_{N_4\gtrsim N}}{\prod_{j=1}^6 m(n_j)\langle n_j\rangle} \prod_{j=1}^6 \frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac12}} \quad , \quad \text{etc.}$$

Case (a): It is enough to estimate

$$\int_* \frac{1}{\prod_{j=3}^6 m(N_j)\langle N_j\rangle} \prod_{j=1}^6 \frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac12}} \tag{2.85}$$

We have $m(N_4) \geqslant m(N_3)$, $N_4 \gtrsim N$ and

$$m(N_3)^2 N_3 = N_3^{\varepsilon} N^{2-2s} N_3^{2s-1-\varepsilon} = N_3^{\varepsilon}\left(\frac{N_3}{N}\right)^{2s-1-\varepsilon} N^{1-\varepsilon} \gtrsim N^{1-} N_3^{0+},$$

as well as $m(N_j)\langle N_j\rangle^{\frac12} \gtrsim 1$ for $j=5,6$. Therefore

$$I_6^{(a)} \lesssim N^{-2+}\frac{1}{N_3^{0+}} \int_* \widehat{f_1}\widehat{f_2}\widehat{f_3}\widehat{f_4}\widehat{g_5}\widehat{g_6} \lesssim N^{-2+}\frac{1}{N_3^{0+}} \int_{\mathbb{R}}\int_{\mathbb{T}_\lambda} \prod_{j=1}^4 f_j(t,x) \prod_{j=5,6} g_j(t,x)\,dx\,dt.$$

By Hölder, we get

$$I_6^{(a)} \lesssim N^{-2+}\frac{1}{N_3^{0+}} \|f_1\|_{L^4_{t,x}}\|f_2\|_{L^4_{t,x}}\|f_3\|_{L^{4+}_{t,x}}\|f_4\|_{L^4_{t,x}}\|g_5\|_{L^p_{t,x}}\|g_6\|_{L^p_{t,x}} \ ,$$

where $p \gg 1$ is such that $\frac{1}{4+}+\frac34+\frac2p = 1$. Using the interpolated L^4-Strichartz estimate, we have

$$\|f_3\|_{L^{4+}_{t,x}} \lesssim \lambda^{0+}\|f_3\|_{X^{0+,\frac38+}} \sim \lambda^{0+}N_3^{0+}\|f_3\|_{X^{0,\frac38+}} \leqslant \lambda^{0+}N_3^{0+}\|f_3\|_{X^{0,\frac12}} = \lambda^{0+}N_3^{0+}\|w_3\|_{L^2_{t,x}},$$

and using the interpolated $L^\infty_{t,x}$-Sobolev inequality, we have

$$\|g_j\|_{L^p_{t,x}} \lesssim \|g_j\|_{X^{\frac12,\frac12}} \leqslant \|w_j\|_{L^2_{t,x}} \quad , \ j=5,6.$$

Therefore

$$I_6^{(a)} \lesssim \lambda^{0+}N^{-2+}\frac{1}{N_3^{0+}}\prod_{j=1}^6 \|w_j\|_{L^2_{t,x}}. \tag{2.86}$$

Sub-case (a.1) $N_3 \sim N_1$: In this case we can directly sum over all N_j's and we get

$$\sum_{N1\geqslant N_2\geqslant\ldots\geqslant N_6} I_6^{(a.1)}(P_{N_1}w_1, P_{N_2}w_2,\ldots,P_{N_6}w_6) \lesssim \lambda^{0+}N^{-2+}\prod_{j=1}^6 \|w_j\|_{L^2_{t,x}}.$$

Sub-case (a.2) $N_3 \ll N_1$: In this case, $k_1, k_2 \in I_1$ with $|I_1| \sim N_1$. We fix N_3 and we split I_1 into dyadic subintervals of size $\sim N_3$. So

$$I_1 = \bigsqcup_\ell I_\ell \ , \quad |I_\ell| \sim N_3$$

Hence, we sum over $N_1 \sim N_2$ and obtain

$$\sum_{N_1 \sim N_2 \gg N_3} I_6^{(a.2)}(P_{N_1}w_1, P_{N_2}w_2, \dots, P_{N_6}w_6) \leqslant \sum_{\ell \geqslant 1} I_6^{(a.1)}(P_{I_\ell}w_1, P_{I_\ell}w_2, P_{N_3}w_3, \dots, P_{N_6}w_6)$$
$$\lesssim \lambda^{0+} N^{-2+} \frac{1}{N_3^{0+}} \left(\sum_\ell \|P_{I_\ell}w_1\|_{L^2_{t,x}} \|P_{I_\ell}w_2\|_{L^2_{t,x}} \right) \prod_{j=3}^{6} \|P_{N_j}w_j\|_{L^2_{t,x}}.$$

We apply Cauchy-Schwartz and we get

$$\sum_\ell \|P_{I_\ell}w_1\|_{L^2_{t,x}} \|P_{I_\ell}w_2\|_{L^2_{t,x}} \leqslant \left(\sum_\ell \|P_{I_\ell}w_1\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} \left(\sum_\ell \|P_{I_\ell}w_2\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} = \prod_{j=1,2} \|P_{N_j}w_j\|_{L^2_{t,x}} \leqslant \prod_{j=1,2} \|w_j\|_{L^2_{t,x}}.$$

We can now sum over the remaining dyadics $N_3 \geqslant \ldots \geqslant N_6$ to obtain

$$\sum_{N_1 \geqslant N_2 \geqslant \ldots \geqslant N_6} I_6^{(a.2)}(P_{N_1}w_1, P_{N_2}w_2, \dots, P_{N_6}w_6) \lesssim \lambda^{0+} N^{-2+} \prod_{j=1}^{6} \|w_j\|_{L^2_{t,x}}.$$

Case (b): As in the previous case, it is enough to estimate (2.85). Here, we have

$$m(N_3)N_3 = N_3^{\varepsilon} N^{1-s} N_3^{s-\varepsilon} = N_3^{\varepsilon} \left(\frac{N_3}{N} \right)^{s-\varepsilon} N^{1-\varepsilon} \gtrsim N^{1-} N_3^{0+}$$

and $m(N_j)\langle N_j \rangle^{\frac{1}{2}} \gtrsim 1$ for $j = 4, 5, 6$. Therefore

$$I_6^{(b)} \lesssim N^{-1+} \frac{1}{N_3^{0+}} \int_* \widehat{f}_1 \widehat{f}_2 \widehat{f}_3 \widehat{f}_4 \widehat{g}_5 \widehat{g}_6$$

Notice that among the largest three frequencies, namely k_1, k_2 and k_3, precisely two have the same sign, say k_1 and k_2. We apply Hölder's inequality and get

$$I_6^{(b)} \lesssim N^{-1+} \frac{1}{N_3^{0+}} \|f_1 f_2\|_{L^2_{t,x}} \|f_3 f_4\|_{L^{2+}_{t,x}} \|g_5\|_{L^p_{t,x}} \|g_6\|_{L^p_{t,x}},$$

where $\frac{1}{2} + \frac{1}{2+} + \frac{2}{p} = 1$. Note that by interpolation, then Hölder's inequality and the interpolated L^6-Strichartz estimate, we have

$$\begin{aligned} \|f_3 f_4\|_{L^{2+}_{t,x}} &\lesssim \|f_3 f_4\|^{1-\theta}_{L^2_{t,x}} \|f_3 f_4\|^{\theta}_{L^3_{t,x}} \leqslant \|f_3 f_4\|^{1-\theta}_{L^2_{t,x}} \|f_3\|^{\theta}_{L^6_{t,x}} \|f_4\|^{\theta}_{L^6_{t,x}} \lesssim \|f_3 f_4\|^{1-\theta}_{L^2_{t,x}} \|f_3\|^{\theta}_{X^{0+,\frac{1}{2}}} \|f_4\|^{\theta}_{X^{0+,\frac{1}{2}}} \\ &\lesssim N_3^{0+} N_4^{0+} \|f_3 f_4\|^{1-\theta}_{L^2_{t,x}} \|f_3\|^{\theta}_{X^{0,\frac{1}{2}}} \|f_4\|^{\theta}_{X^{0,\frac{1}{2}}} \end{aligned} \tag{2.87}$$

where $0 < \theta \ll 1$ is defined by $\frac{1}{2+} = \frac{1-\theta}{2} + \frac{\theta}{3}$. Then, using the bi-linear Strichartz estimate (2.29) and the $L^p_{t,x}$-Sobolev embedding, we obtain

$$I_6^{(b)} \lesssim N^{-1+} \frac{1}{N_3^{0+}} \lambda^{-\frac{1}{2}+} \|f_1\|_{X^{0,\frac{1}{2}}} \|f_2\|_{X^{0,\frac{1}{2}}} \lambda^{-\frac{1}{2}+} \|f_3\|^{1-\theta}_{X^{0,\frac{1}{2}}} \|f_4\|^{1-\theta}_{X^{0,\frac{1}{2}}} \|f_3\|^{\theta}_{X^{0,\frac{1}{2}}} \|f_4\|^{\theta}_{X^{0,\frac{1}{2}}} \|g_5\|_{X^{\frac{1}{2},\frac{1}{2}}} \|g_6\|_{X^{\frac{1}{2},\frac{1}{2}}}$$
$$\lesssim N^{-1+} \lambda^{-1+} \frac{1}{N_3^{0+}} \prod_{j=1}^{4} \|f_j\|_{X^{0,\frac{1}{2}}} \prod_{j=5,6} \|g_j\|_{X^{\frac{1}{2},\frac{1}{2}}} = N^{-1+} \lambda^{-1+} \frac{1}{N_3^{0+}} \prod_{j=1}^{6} \|w_j\|_{L^2_{t,x}}$$

We can now proceed as in the sub-cases (a.1) and (a.2) to perform the summation over all dyadic frequency sizes and obtain

$$\sum_{N_1 \geqslant N_2 \geqslant \ldots \geqslant N_6} I_6^{(b)}(P_{N_1}w_1, P_{N_2}w_2, \dots, P_{N_6}w_6) \lesssim \lambda^{-1+} N^{-1+} \prod_{j=1}^{6} \|w_j\|_{L^2_{t,x}}$$

Case (c): Here we can use the refined estimate given by Lemma 2.11.1. Hence it is enough to estimate

$$\int_* \frac{1}{m(N_1)m(N_2)N_2 m(N_3)\prod_{j=4}^6 m(N_j)\langle N_j\rangle} \prod_{j=1}^{6} \frac{\widehat{w_j}(\tau_j,k_j)}{\langle\sigma_j\rangle^{\frac12}}$$

We have $m(N_3)\sim 1$, $N_2\sim N_1$ and therefore

$$m(N_1)m(N_2)N_2 \sim N_1^{\varepsilon}N^{2-2s}N_1^{2s-1-\varepsilon} = N_1^{\varepsilon}\left(\frac{N_1}{N}\right)^{2s-1-\varepsilon} N^{1-\varepsilon} \gtrsim N^{1-}N_1^{0+},$$

and $m(N_4)\langle N_4\rangle \gtrsim 1$, $m(N_j)\langle N_j\rangle^{\frac12}\gtrsim 1$ for $j=5,6$. Therefore,

$$I_6^{(c)} \lesssim N^{-1+}\frac{1}{N_1^{0+}}\int_* \widehat{f}_1\widehat{f}_2\widehat{f}_3\widehat{f}_4\widehat{g}_5\widehat{g}_6 \leqslant N^{-1+}\frac{1}{N_1^{0+}}\|f_1f_3\|_{L^{2+}_{t,x}}\|f_2f_4\|_{L^2_{t,x}}\|g_5\|_{L^p_{t,x}}\|g_6\|_{L^p_{t,x}},$$

where $p\gg 1$ is defined by $\frac{1}{2+}+\frac12+\frac{2}{p}=1$. Proceeding as in (2.87) and then using the bi-linear Strichartz estimate (2.29) and the $L^p_{t,x}$-Sobolev embedding, we obtain

$$I_6^{(c)}(P_{N_1}w_1, P_{N_2}w_2,\dots,P_{N_6}w_6) \lesssim \lambda^{-1+}N^{-1+}\frac{1}{N_1^{0+}}\prod_{j=1}^{6}\|w_j\|_{L^2_{t,x}}$$

In this case, the factor $\frac{1}{N_1^{0+}}$ is enough to ensure the summability over all dyadic sizes, hence

$$\sum_{N_1\geqslant N_2\geqslant\dots N_6} I_6^{(c)}(P_{N_1}w_1, P_{N_2}w_2,\dots,P_{N_6}w_6) \lesssim \lambda^{-1+}N^{-1+}\prod_{j=1}^{6}\|w_j\|_{L^2_{t,x}}$$

Combining the three cases, we obtain (2.84). □

Proposition 2.11.3.

$$\left|\int_T^{T+\delta}\Lambda_6(M_6;v_1,\overline{v_2},v_3,\overline{v_4},v_5,\overline{v_6})\right| \lesssim \lambda^{-1+}N^{-1+}\prod_{j=1}^{6}\|Iv_j\|_{X^{1,\frac12}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.88}$$

Proof. Follows from the Proposition 2.11.2 after we decompose the sharp time cut-off $\mathbb{1}_{[T,T+\delta]}$ as in Section 2.7, via (2.60). □

Proposition 2.11.4.

$$\left|\int_T^{T+\delta}\Lambda_8(M_8;v_1,\overline{v_2},v_3,\overline{v_4},v_5,\overline{v_6},v_7,\overline{v_8})\right| \lesssim \lambda^{-1+}N^{-1+}\prod_{j=1}^{8}\|Iv_j\|_{X^{1,\frac12}(\mathbb{R}\times\mathbb{T}_\lambda)} \tag{2.89}$$

Proof. As for Proposition 2.11.3, it is enough to carry the estimate ignoring the time cut-off $\mathbb{1}_{[T,T+\delta]}$. We keep the same notation as in the proof of Theorem 2.11.2 for w_j, f_j and g_j. Taking into account the point-wise estimate $|M_8|\lesssim m(N_1)^2N_1$, it is enough to estimate

$$\int_* \frac{1}{N_1\prod_{j=3}^{8} m(N_j)\langle N_j\rangle}\prod_{j=1}^{8}\frac{\widehat{w_j}}{\langle\sigma_j\rangle^{\frac12}}$$

We distinguish two cases: (a) $N_3\gtrsim N$ and (b) $N_3\ll N$. In both, we will use $\frac{1}{N_1}\lesssim N^{-1+}\frac{1}{N_1^{0+}}$ to sum over all dyadic pieces.

Case (a): We have $\langle N_3\rangle \sim N_3$, $m(N_3) \sim N\left(\frac{N_3}{N}\right)^s \gtrsim N$ and $m(N_j)\langle N_j\rangle^{\frac{1}{2}} \gtrsim 1$. Therefore

$$I_8^{(a)} \lesssim N^{-2+}\frac{1}{N_1^{0+}}\int_* \frac{1}{\prod_{j=4}^8 \langle N_j\rangle^{\frac{1}{2}}}\prod_{j=1}^8 \frac{\widehat{w_j}}{\langle \sigma_j\rangle^{\frac{1}{2}}} = N^{-2+}\frac{1}{N_1^{0+}}\int_{\mathbb{R}}\int_{\mathbb{T}_\lambda}\prod_{j=1}^3 f_j(t,x)\prod_{j=4}^8 g_j(t,x)\,dx\,dt$$
$$\leqslant N^{-2+}\frac{1}{N_1^{0+}}\prod_{j=1}^3 \|f_j\|_{L^4_{t,x}}\prod_{j=4}^8 \|g_j\|_{L^p_{t,x}}$$

where p is given by $\frac{3}{4}+\frac{5}{p}=1$. By the L^4-Strichartz estimate and the Sobolev embedding $X^{s,b}\subset L^{20}_{t,x}$ for any $s,b\geqslant \frac{1}{2}-\frac{1}{20}$, we have

$$I_8^{(a)} \lesssim \frac{N^{-2+}}{N_1^{0+}}\prod_{j=1}^3 \|f_j\|_{X^{0,\frac{3}{8}}}\prod_{j=4}^8 \|g_j\|_{X^{\frac{9}{20},\frac{9}{20}}} \leqslant \frac{N^{-2+}}{N_1^{0+}}\prod_{j=1}^3 \|f_j\|_{X^{0,\frac{1}{2}}}\prod_{j=4}^8 \|g_j\|_{X^{\frac{1}{2},\frac{1}{2}}} = N^{-2+}\frac{1}{N_1^{0+}}\prod_{j=1}^8 \|w_j\|_{L^2_{t,x}}.$$

Case (b): We have $m(N_j)\sim 1$ for all $j\geqslant 3$ and

$$I_8^{(b)} \lesssim N^{-1+}\frac{1}{N_1^{0+}}\int_*\prod_{j=1}^4 \widehat{f}_j(\tau_j,k_j)\prod_{j=5}^8 \widehat{g}_j(\tau_j,k_j) = N^{-1+}\frac{1}{N_1^{0+}}\int_{\mathbb{R}}\int_{\mathbb{T}_\lambda}\prod_{j=1}^4 f_j(t,x)\prod_{j=5}^8 g_j(t,x)\,dx\,dt$$
$$\leqslant N^{-1+}\frac{1}{N_1^{0+}}\|f_1f_3\|_{L^{2+}_{t,x}}\|f_2f_4\|_{L^2_{t,x}}\prod_{j=5}^8 \|g_j\|_{L^p_{t,x}}$$

where $p\gg 1$ is defined by $\frac{1}{2+}+\frac{1}{2}+\frac{4}{p}=1$. By interpolation, Hölder's inequality and the interpolated L^6-Strichartz estimate, we have

$$\|f_1f_3\|_{L^{2+}_{t,x}} \lesssim \|f_1f_3\|_{L^2_{t,x}}^{1-\theta}\|f_1f_3\|_{L^3_{t,x}}^{\theta} \lesssim \lambda^{(-\frac{1}{2}+)(1-\theta)}\|f_1\|_{X^{0,\frac{1}{2}}}^{1-\theta}\|f_3\|_{X^{0,\frac{1}{2}}}^{1-\theta}\|f_1\|_{L^6_{t,x}}^{\theta}\|f_3\|_{L^6_{t,x}}^{\theta}$$
$$\lesssim \lambda^{-\frac{1}{2}+}\|f_1\|_{X^{0,\frac{1}{2}}}^{1-\theta}\|f_3\|_{X^{0,\frac{1}{2}}}^{1-\theta}\|f_1\|_{X^{0+,\frac{1}{2}}}^{\theta}\|f_3\|_{X^{0+,\frac{1}{2}}}^{\theta} \lesssim N_1^{0+}\lambda^{-\frac{1}{2}+}\|f_1\|_{X^{0,\frac{1}{2}}}\|f_3\|_{X^{0,\frac{1}{2}}}.$$

Also,

$$\|f_2f_4\|_{L^2_{t,x}} \lesssim \lambda^{-\frac{1}{2}+}\|f_2\|_{X^{0+,\frac{1}{2}}}\|f_4\|_{X^{0+,\frac{1}{2}}} \lesssim \lambda^{-\frac{1}{2}+}N_1^{0+}\|f_2\|_{X^{0,\frac{1}{2}}}\|f_4\|_{X^{0,\frac{1}{2}}},$$
$$\|g_j\|_{L^p_{t,x}} \lesssim \|g_j\|_{X^{\frac{1}{2}-,\frac{1}{2}-}} \leqslant \|g_j\|_{X^{\frac{1}{2},\frac{1}{2}}}.$$

Hence

$$I_8^{(b)} \lesssim N^{-1+}\lambda^{-1+}\frac{1}{N_1^{0+}}\prod_{j=1}^4 \|f_j\|_{X^{0,\frac{1}{2}}}\prod_{j=5}^8 \|g_j\|_{X^{\frac{1}{2},\frac{1}{2}}} = N^{-1+}\lambda^{-1+}\frac{1}{N_1^{0+}}\prod_{j=1}^8 \|w_j\|_{L^2_{t,x}}$$

□

2.12 Global well-posedness using the second generation almost conserved energy

The scheme of the I-method from Section 2.9 (implemented in the proof of Theorem 2.8.1) still carries on, provided that the new almost conserved quantity controls the $\dot{H}^1$-norm of Iv. Hence, in order to obtain our main result, we are left to establish

Proposition 2.12.1. *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $v \in H^1(\mathbb{T}_\lambda)$ with $\|v\|^2_{L^2(\mathbb{T}_\lambda)} < \delta$, we have*

$$\|Iv\|^2_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim \mathcal{E}^2_N(v) + \frac{1}{\varepsilon}\mu(v)^5 \tag{2.90}$$

with the implicit constant independent of λ. Moreover $\delta \nearrow 2\sqrt{2}$ as $\varepsilon \searrow 0$.

Proof. Taking into account Proposition 2.1.3 (applied to Iv), it suffices to show that $\mathcal{E}^2_N(v)$ is a perturbation of $\mathcal{E}^1_N(v) = \mathcal{E}(Iv)$. More precisely, we show that

$$|\mathcal{E}^2_N(v) - \mathcal{E}^1_N(v)| \lesssim O(N^{-\alpha})\|Iv\|^4_{H^1(\mathbb{T}_\lambda)}$$

for some $\alpha > 0$. By (2.77), we need to estimate $\Lambda_4(\sigma_4; v)$, hence (2.90) will follow once we prove

$$|\Lambda_4(\sigma_4; v_1, \overline{v_2}, v_3, \overline{v_4})| \lesssim O(N^{-\alpha}) \prod_{j=1}^{4} \|Iv_j\|_{H^1(\mathbb{T}_\lambda)}.$$

We can treat the multipliers σ_4' and σ_4'' separately. See [CKS+02, Lemma 3.8].

□

Theorem 2.12.2. *The gauged DNLS (2.10) is globally well-posed in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$, provided that the mass of the initial data is smaller than $2\sqrt{2}$.*

Proof. We can adapt the proof of Theorem 2.8.1 (see also Remark 2.8.2) to establish the global existence criterion (2.1), but now with the better decay estimates on the increments of $\mathcal{E}^2_N$. What differs is the numerology involved when choosing the parameter N. Here, we obtain $\lambda^2 T \sim \lambda^{1-} N^{1-}$ with the same choice of $\lambda \sim N^{\frac{1-s}{s}}$. Hence, T can be chosen arbitrarily large if and only if $2 - \frac{1}{s} > 0$. □

Remark 2.12.3. Future possible continuations of this project are in the following directions.

(1) In the non-periodic setting, the endpoint regularity was obtained recently in [MWX11]. It is of interest to study whether the analogue of this result holds in the periodic setting.

(2) The mass threshold $2\sqrt{2}$ (to be compared with 2π in the Euclidean setting) is not sharp due to the unoptimal constant in the Gagliardo-Nirenberg inequality (see Lemma 2.3.5). A second goal would be to research the literature and check if optimal constants for Gagliardo-Nirenberg inequality on compact domains have been found.

(3) Recently, the global well-posedness in $H^1(\mathbb{R})$ result was improved [Wu13, Wu14] in the sense of relaxing Assumption (A). In [MO15], we show that this improvement also carries in the periodic setting for $H^1(\mathbb{T})$-solutions. In both the periodic and non-periodic settings, it is believed that the smallness of mass condition (A) (upon which global well-posedness is proved) can also be relaxed for H^s regularities, where $\frac{1}{2} < s < 1$.

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