PERIODIC L^4-STRICHARTZ ESTIMATE FOR KDV

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1. Introduction

In [2], Bourgain proved global well-posedness of the periodic KdV in $L^2(\mathbb{T})$:

$$u_t + u_{xxx} + uu_x = 0, \quad (x,t) \in \mathbb{T} \times \mathbb{R}. \tag{1.1}$$

The key ingredient in the proof of local well-posedness is the periodic L^4-Strichartz estimate:

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{1}{3}}(\mathbb{T}\times\mathbb{R})}, \tag{1.2}$$

where the $X^{s,b}$-norm is defined by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau}. \tag{1.3}$$

The result in [2] is in fact stated for time-periodic functions:

$$\|u\|_{L^4_{x,t}(\mathbb{T}^2)} \lesssim \|u\|_{X^{0,\frac{1}{3}}(\mathbb{T}^2)}. \tag{1.4}$$

Such a restriction, however, is not necessary. See Tao [3, 4].

Remark 1.1. In [1], Bourgain also proved the periodic L^4-Strichartz estimate for the Schrödinger equation:

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{3}{8}}(\mathbb{T}\times\mathbb{R})}, \tag{1.5}$$

Let us compare (1.5) with the classical result by Zygmund [5]:

$$\|e^{-t\partial_x^2} u_0\|_{L^4_{x,t}(\mathbb{T}^2)} \lesssim \|u_0\|_{L^2(\mathbb{T})}. \tag{1.6}$$

By the transference principle (Lemma 2.9 in [3]), (1.6) yields

$$\|u\|_{L^4_{x,t}(\mathbb{T}^2)} \lesssim \|u\|_{X^{0,b}(\mathbb{T}^2)}, \quad b > \tfrac{1}{2}. \tag{1.7}$$

From this, we can see that Bourgain's periodic L^4-Strichartz estimate (1.5) is a significant improvement of (1.6) and (1.7).

In the following, we present a heuristic argument to show how the regularity $b = \frac{1}{3}$ naturally arises. On the one hand, the $X^{s,b}$-space given by the norm (1.3) is adapted to the linear part of the KdV equation (called the Airy equation):

$$u_t + u_{xxx} = 0.$$

Namely, we can formally view three spatial derivatives as "equivalent" to one temporal derivative. On the other hand, by the Sobolev inequality, we have

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{\frac{1}{4},\frac{1}{4}}(\mathbb{T}\times\mathbb{R})}. \tag{1.8}$$

Then, by formally moving the spatial derivative $s = \frac{1}{4}$ in (1.8) to the temporal side, we obtain the temporal regularity $b = \frac{1}{3}$ in (1.2), since $\frac{1}{3} = \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}$. Of course, this is merely a heuristic argument showing why $b = \frac{1}{3}$ is the natural regularity in (1.2). The necessity of the regularity $b = \frac{1}{3}$ can be shown in Section 3, following the argument in [1, p.116].

The proof of (1.4) appeared in [2]. Upon a small modification, it is easy to see that the same proof works for (1.2). In [4, Proposition 6.4], Tao presented a short proof of (1.2) by estimating a certain $[3 : \mathbb{Z} \times \mathbb{R}]$-multiplier. In [3], a simple proof of the periodic L^4-Strichartz estimate (1.5) for the Schrödinger equation is presented (after Tzvetkov). In Section 2, we present a short proof of (1.2) based on this argument.

2. Proof of (1.2)

We basically follow the argument in [3, Proposition 2.13] with a small but important modification. First, write u as

$$u = \sum_{M, \text{ dyadic}} u_M$$

where u_M is the localization of u on $M \leq \langle \tau - n^3 \rangle < 2M$. Then, we have

$$\sum_{M, \text{ dyadic}} M^{\frac{2}{3}} \|u_M\|^2_{L^2_{x,t}} \leq \|u\|^2_{X^{0,\frac{1}{3}}}. \tag{2.1}$$

We also have

$$\|u\|^2_{L^4_{x,t}} = \|uu\|_{L^2_{x,t}} \leq \sum_{M, M', \text{ dyadic}} \|u_M u'_M\|_{L^2_{x,t}} \lesssim \sum_{m=0}^{\infty} \sum_{M, \text{ dyadic}} \|u_M u_{2^m M}\|_{L^2_{x,t}} \tag{2.2}$$

In the following, we use (n_1, τ_1) and (n_2, τ_2) as the frequencies for u_M and $u_{2^m M}$, respectively. We also use (n, τ) for the frequencies for the product $u_M u_{2^m M}$, i.e. we have $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$.

First, consider the case when $|n| \gtrsim (2^m M)^{\frac{1}{3}}$. Then, it suffices to show that

$$\sum_{M, \text{ dyadic}} \|u_M u_{2^m M}\|_{L^2_{x,t}} \lesssim 2^{-\varepsilon m} \sum_{M, \text{ dyadic}} M^{\frac{2}{3}} \|u_M\|^2_{L^2_{x,t}} \tag{2.3}$$

for all $m \geq 0$, where $\varepsilon > 0$ is some constant to be chosen later. Now, we claim that (2.3) follows once we show

$$\|u_M u_{2^m M}\|_{L^2_{x,t}} \lesssim 2^{-\varepsilon m} M^{\frac{1}{3}} \|u_M\|_{L^2_{x,t}} (2^m M)^{\frac{1}{3}} \|u_{2^m M}\|_{L^2_{x,t}}. \tag{2.4}$$

Indeed, by Cauchy-Schwarz inequality with (2.4), we have

$$\text{LHS of (2.3)} \lesssim 2^{-\varepsilon m} \bigg(\sum_M M^{\frac{2}{3}} \|u_M\|^2_{L^2_{x,t}} \bigg)^{\frac{1}{2}} \bigg(\sum_M (2^m M)^{\frac{2}{3}} \|u_M\|^2_{L^2_{x,t}} \bigg)^{\frac{1}{2}} \leq \text{RHS of (2.3)}.$$

In the following, we show (2.4) when $|n| \gtrsim (2^m M)^{\frac{1}{3}}$. Without loss of generality, assume that $\|u_M\|_{L^2_{x,t}} = \|u_{2^m M}\|_{L^2_{x,t}} = 1$. By Cauchy-Schwarz and Hölder inequalities with Fubini, we have

$$\begin{aligned}\|u_M u_{2^m M}\|_{L^2_{x,t}} &= \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u}_M(n_1,\tau_1)\widehat{u}_{2^m M}(n_2,\tau_2)d\tau_1 \right\|_{\ell^2_n L^2_\tau} \\ &\le \sup_{n,\tau} \bigg(\sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} 1\, d\tau_1 \bigg)^{\frac{1}{2}} \|u_M\|_{L^2_{x,t}} \|u_{2^m M}\|_{L^2_{x,t}} \\ &= \sup_{n,\tau} \bigg(\sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} 1\, d\tau_1 \bigg)^{\frac{1}{2}}.\end{aligned}$$

Hence, (2.4) follows once we show

$$\sup_{n,\tau} \sum_{n=n_1+n_2} \int_{\substack{\tau=\tau_1+\tau_2 \\ \tau_1=n_1^3+O(M) \\ \tau_2=n_2^3+O(2^m M)}} 1\, d\tau_1 \lesssim 2^{(\frac{2}{3}-2\varepsilon)m} M^{\frac{4}{3}} \tag{2.5}$$

Fix n and τ. Under the given constraints, we have $\tau = n_1^3 + n_2^3 + O(2^m M)$. With $n = n_1 + n_2$, this yields

$$\begin{aligned} &\tau = n^3 - 3nn_1n_2 + O(2^m M) \\ \implies\ & 3n\Big(n_1 - \frac{n}{2}\Big)^2 = \tau - \frac{1}{4}n^3 + O(2^m M) \\ \implies\ & \Big(n_1 - \frac{n}{2}\Big)^2 = C(n,\tau) + O\big((2^m M)^{\frac{2}{3}}\big), \end{aligned}$$

since $|n| \gtrsim (2^m M)^{\frac{1}{3}}$ by assumption. This in particular implies that there are at most $O\big(2^{\frac{m}{3}} M^{\frac{1}{3}}\big)$ possible values for n_1 (and hence for n_2). Then, (2.5) follows from integrating in τ_1 and this observation. Note that we can take $\varepsilon = \frac{1}{6}$.

Next, we consider the case when $|n| \ll (2^m M)^{\frac{1}{3}}$. This is where the argument is different from that in [3]. As before, our goal is to prove (2.4) under $\|u_M\|_{L^2_{x,t}} = \|u_{2^m M}\|_{L^2_{x,t}} = 1$. Our goal is to show (2.4):

$$\|u_M u_{2^m M}\|_{L^2_{x,t}} \lesssim 2^{(\frac{1}{3}-\varepsilon)m} M^{\frac{2}{3}} \tag{2.6}$$

By Minkowski inequality followed by Cauchy-Schwarz inequality (note that $\tau_1 = n_1^3 + O(M)$), we have

$$\begin{aligned}\|u_M u_{2^m M}\|_{L^2_{x,t}} &= \left\| \sum_{n=n_1+n_2} \int \widehat{u}_M(n_1,\tau_1)\widehat{u}_{2^m M}(n_2,\tau-\tau_1)d\tau_1 \right\|_{\ell^2_n L^2_\tau} \\ &\le \left\| \sum_{n=n_1+n_2} \int |\widehat{u}_M(n_1,\tau_1)|\, \|\widehat{u}_{2^m M}(n_2,\cdot)\|_{L^2_\tau} d\tau_1 \right\|_{\ell^2_n} \\ &\le M^{\frac{1}{2}} \left\| \sum_{n=n_1+n_2} \|\widehat{u}_M(n_1,\cdot)\|_{L^2_{\tau_1}} \|\widehat{u}_{2^m M}(n_2,\cdot)\|_{L^2_\tau} \right\|_{\ell^2_n} \\ &\le 2^{\frac{m}{6}} M^{\frac{2}{3}} \|u_M\|_{L^2_{x,t}} \|u_{2^m M}\|_{L^2_{x,t}}.\end{aligned}$$

With $\varepsilon = \frac{1}{6}$, this proves (2.6), and hence (1.2).

3. Necessity of the regularity $b = \frac{1}{3}$

In this section, we show that the regularity $b = \frac{1}{3}$ in (1.2) is indeed necessary. Consider the function

$$u(x,t) = \sum_{|n|\leq N} \int_{|\tau|\leq N^3} e^{i(nx+\tau t)} d\tau.$$

Namely, we have $\widehat{u}(n,\tau) = \chi_N(n)\chi_{N^3}(\tau)$, where χ_N is the characteristic function of the interval $[-N, N]$.

First, we establish the lower bound on the L^4-norm of u.

$$\|u\|^4_{L^4_{x,t}} = \|u^2\|^2_{L^2_{x,t}} = \|\widehat{v}(n,\tau)\|^2_{\ell^2_n L^2_\tau} = \sum_{|n|\leq 2N} \int_{|\tau|\leq 2N^3} |\widehat{v}(n,\tau)|^2 d\tau, \tag{3.1}$$

where

$$\widehat{v}(n,\tau) := \widehat{u^2}(n,\tau) = \big(\chi_N *_{\mathbb{Z}} \chi_N\big)(n)\big(\chi_{N^3} *_{\mathbb{R}} \chi_{N^3}\big)(\tau).$$

Note that the first convolution is with respect to the counting measure on $\mathbb{Z}$ and the second with respect to the Lebesgue measure on $\mathbb{R}$. Thus, we have

$$\begin{cases} \big(\chi_N *_{\mathbb{Z}} \chi_N\big)(n) \geq N, & \text{for } |n| \leq N, \\ \big(\chi_{N^3} *_{\mathbb{R}} \chi_{N^3}\big)(\tau) \geq N^3, & \text{for } |\tau| \leq N^3. \end{cases} \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\|u\|_{L^4_{x,t}} \gtrsim N^3. \tag{3.3}$$

Next, we compute the $X^{0,\frac{1}{3}}$-norm of u.

$$\begin{aligned} \|u\|^2_{X^{0,\frac{1}{3}}} &= \sum_{|n|\leq N} \int_{|\tau|\leq N^3} \langle \tau - n^3\rangle^{\frac{2}{3}} d\tau = \sum_{|n|\leq N} \sum_{m=-N^3}^{N^3-1} \int_0^1 \langle m + \alpha - n^3\rangle^{\frac{2}{3}} d\alpha \\ &\sim \sum_{|n|\leq N} \sum_{m=-N^3}^{N^3-1} \int_0^1 \langle m - n^3\rangle^{\frac{2}{3}} d\alpha = \sum_{|n|\leq N} \sum_{m=-N^3}^{N^3-1} \langle m - n^3\rangle^{\frac{2}{3}}. \end{aligned}$$

For fixed $n \in [-N, N]$, there is one-to-one correspondence between $m \in [-N^3, N^3-1]$ $k \in [-N^3 - n^3, N^3 - 1 - n^3]$ under $m - n^3 = k$. Hence, we have

$$\|u\|_{X^{0,\frac{1}{3}}} \sim \bigg(\sum_{|n|\leq N} \sum_{k=-N^3-n^3}^{N^3-1-n^3} \langle k\rangle^{\frac{2}{3}}\bigg)^{\frac{1}{2}} \sim \bigg(N \sum_{k=0}^{N^3} \langle k\rangle^{\frac{2}{3}}\bigg)^{\frac{1}{2}} \sim N^3. \tag{3.4}$$

The necessity of the regularity $b = \frac{1}{3}$ follows from (3.3) and (3.4).

References

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