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Bi-parameter paracontrolled approach to stochastic wave equations

Lec 1: July 06, 2023 (THU)

Part 1: Paracontrolled approach to rough DEs (RDEs)

Part 2: Bi-parameter paracontrolled approach to stochastic wave eqn.

EX:

SDE: $du = F(u) dB$ $B = \text{Brownian motion.}$

↳ easy to construct an Ito solution. (in $L^2(\Omega)$)

Q: How to give a pathwise meaning to the eqn & solns.

• 1-d stoch. nonlinear wave eqn (SNLW)

with a multiplicative space-time white noise

$$\partial_t^2 u - \partial_x^2 u = F(u) \xi \quad \text{on } \mathbb{T}_x \times \mathbb{R}_t.$$

$\xi = \xi(t, x) = \text{space-time white noise}$

$$\mathbb{E}[\xi(t_1, x_1) \xi(t_2, x_2)] = \delta(t_1 - t_2) \delta(x_1 - x_2)$$

• $\boxed{\xi \in C_t^{-\frac{1}{2}} C_x^{-\frac{1}{2}}}$ \leftarrow very rough.

• $\mathbb{E}[\xi(u) \xi(v)] = \langle u, v \rangle_{L^2_{t,x}}$

$\nearrow \Rightarrow \mathbb{E}[(\xi(u))^2] = \|u\|_{L^2_{t,x}}^2$

" $\int u(t, x) \xi(t, x) dt dx$ " = Wiener integral

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null coordinates:

$$x_1 = \frac{t+x}{\sqrt{2}}, \quad x_2 = \frac{t-x}{\sqrt{2}}$$

(SNLW) \Leftrightarrow

$$\partial_{x_1} \partial_{x_2} V = \frac{1}{2} F(t) \tilde{\xi}$$

$$V(x_1, x_2) = U\left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}\right)$$

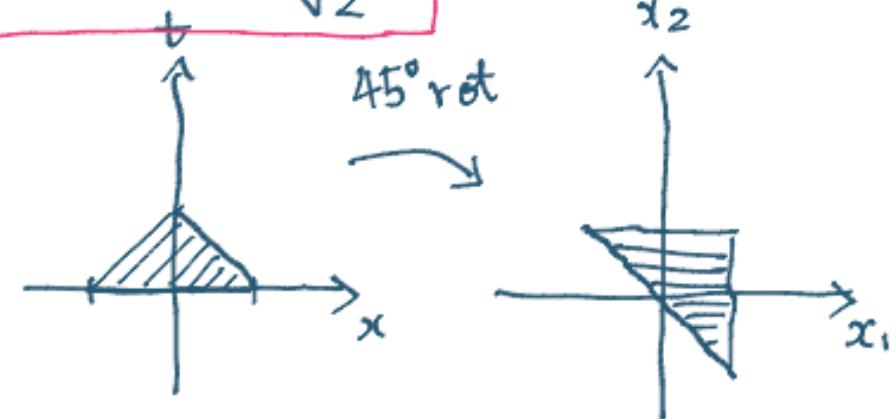
$$t = \frac{x_1 + x_2}{\sqrt{2}}, \quad x = \frac{x_1 - x_2}{\sqrt{2}}$$

$$\tilde{\xi}(x_1, x_2) = \tilde{\xi}\left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}\right)$$

$\Rightarrow \tilde{\xi}$ is also a space-time white noise.

Q: $\mathbb{E}[\tilde{\xi}(f) \tilde{\xi}(g)] = \langle f, g \rangle_{L^2_{x_1, x_2}} ?$

Define $F(t, x)$, $G(t, x)$ by $F(t, x) = f\left(\frac{t+x}{\sqrt{2}}, \frac{t-x}{\sqrt{2}}\right)$



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$$\text{RHS: } \langle f, g \rangle_{L^2_{x_1, x_2}} \stackrel{\text{Ch of var}}{=} \langle F, G \rangle_{L^2_{t, x}}$$

Similarly, $\langle \tilde{\gamma}, f \rangle = \langle \tilde{\gamma}, f \rangle_{L^2_{x_1, x_2}}$ (in the limiting sense)
 $= \langle \tilde{\gamma}, F \rangle_{L^2_{t, x}} = \tilde{\gamma}(F)$

$$(\text{LHS}) = \mathbb{E}[\tilde{\gamma}(F)\tilde{\gamma}(G)] = \langle F, G \rangle_{L^2_{t, x}} = \langle f, g \rangle_{L^2_{x_1, x_2}}$$

These problems are of importance in giving a pathwise meaning to stochastic integrals.

$\partial_t u = F(u) \tilde{\gamma}$ ✓ $\tilde{\gamma} = \tilde{\gamma}(t) = \text{temporal white noise.}$

$$\Rightarrow u(t) = u(0) + \boxed{\int_0^t F(u)(t') \tilde{\gamma}(t') dt'}.$$

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$$\boxed{\partial_{x_1} \partial_{x_2} V = F(V) \xi} \quad \text{on } \underline{[0,1] \times [0,1]}$$

$$\int dx_2 \Rightarrow \partial_{x_1} V(x_1, x_2) - \partial_{x_1} V(x_1, 0) = \int_0^{x_2} F(V)(x_1, y_2) \xi(x_1, y_2) dy_2$$

$$\Rightarrow V(x_1, x_2) = V(x_1, 0) + V(0, x_2) - V(0, 0)$$

$$+ \boxed{\int_0^{x_1} \int_0^{x_2} F(V)(y_1, y_2) \xi(y_1, y_2) dy_1 dy_2}$$

Q: How to give a pathwise meaning?

- SDE : pathwise soln via rough paths (Lyons '98, Gubinelli '04).
- SNLW : Ito soln (Walsh '86, Dalang '90's)
random field soln (Farré - Nualart '93)
- Chouk - Gubinelli '14 : rough sheet but not successful.

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Q: Pathwise well-posedness of SNLW?

⇐ completely open.

- Main goal of solving a nonlinear problem
 - = give a proper meaning to a product
(nonlinearity)

① Impose a structure

- Soln map : data (initial/boundary data & noise)
 - ↪ soln is ill-defined
- ② Decompose a soln map into 2 steps :
 - (i) Use stoch. analysis to construct an enhanced dataset \mathbb{E}
 - (ii) Use deterministic analysis to build a conti. map : $\mathbb{E} \mapsto u$.

- SDE: rough path (Lyons '98, Gubinelli '04)
- stoch. heat egn:
 - regularity structures (Hairer '14)
 - paracontrolled distributions (Gubinelli-Imkeller-Penkowski '15)
 - RG method (Kupiainen '16, Duch '21).
- stoch dispersive egn: Bourgain '96, Tzvetkov, Bring-Tz Oh mid '00's.
 - Gubinelli-Koch-Oh '18 -
+ Tolomeo
 - Bringmann + Deng-Nahmod-Yue.

:

• Part 1: SPE / ODE / RDE : $\partial_t u = F(u) \xi$ ⑧

- $\xi = 1-d$ white noise:

$$\xi(t) = \sum_{n \in \mathbb{Z}} g_n e^{int} \sim -\frac{1}{2} - \text{rough}$$

- $\{g_n\}_{n \in \mathbb{Z}}$ = indep. standard \mathbb{C} -valued Gaussian r.v.'s

$$\text{s.t. } g_{-n} = \overline{g_n} \quad \begin{cases} \mathbb{E}[|g_n|^2] = 1, \\ \mathbb{E}[g_n^k \overline{g_m}^l] = \underset{n=m}{1} \cdot \underset{k=l}{1} \cdot k! \end{cases}$$

FACT: $f \sim S_1, g \sim S_2$

The product fg is defined only when $S_1 + S_2 > 0$

- $\partial_t u = u \xi \Rightarrow u \sim \frac{1}{2} - = (-\frac{1}{2} -) + 1$ BUT $\xi \sim -\frac{1}{2} -$

\Rightarrow The product $u \xi$ is ill-defined.

• Analytical background: Grafakos vol 1, vol 2.
Bahouri - Chemin - Danchin.

- Littlewood-Paley projections / decomposition.

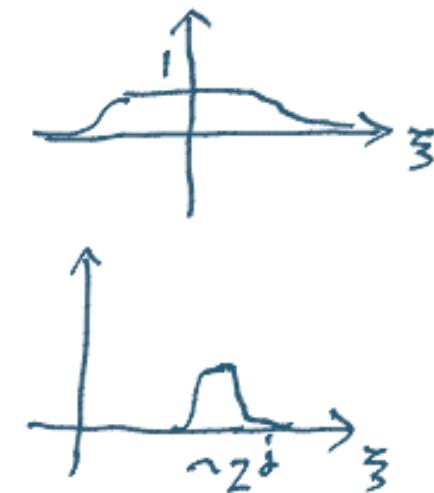
$\phi: \mathbb{R} \rightarrow [0, 1]$ smooth bump function supp on $[-\frac{8}{5}, \frac{8}{5}]$
s.t. $\phi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$

$$\text{Set } \varphi_0(\xi) = \phi(|\xi|)$$

and for $j \in \mathbb{N}$,

$$\varphi_j(\xi) = \phi\left(\frac{|\xi|}{2^j}\right) - \phi\left(\frac{|\xi|}{2^{j-1}}\right)$$

$$\Rightarrow \sum_{j=0}^{\infty} \varphi_j(\xi) \equiv 1.$$



- Define the LP projection P_j as the Fourier multiplier operator with multiplier φ_j

$$\widehat{P_j f}(n) = \varphi_j(n) \widehat{f}(n).$$

$$f = \sum_{j=0}^{\infty} p_j f \quad (\text{LP decomposition})$$

freq supp $|n| \sim 2^j$

Note: Let $K_j = \mathcal{F}_{\mathbb{R}^d}^{-1}(\varphi_j)$

$$K_j(x) = 2^{jd} K(2^j x), \quad K = \mathcal{F}_{\mathbb{R}^d}^{-1}(\varphi_0(2 \cdot))$$

$$\text{On } \mathbb{T}^d, \quad K_j^{\text{per}}(x) = \sum_{n \in \mathbb{Z}^d} K_j(x - 2\pi n) \leftarrow \text{periodization of } K_j$$

\Rightarrow
Poisson summation formula

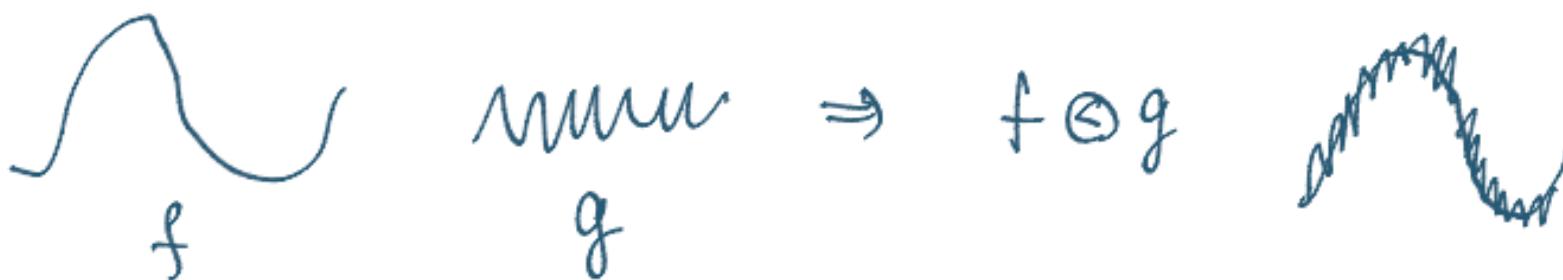
$$\widehat{K_f^{\text{per}}}(n) = \mathcal{F}_{\mathbb{R}^d}(K_f^{\text{per}})(n) = \varphi_f(n).$$

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• Paraproduct decomposition (Bony '81).

$$\begin{aligned}
 fg &= \sum_{j=0}^{\infty} P_j f \sum_{k=0}^{\infty} P_k g \\
 &= \underbrace{\sum_{j < k-2} P_j f \cdot P_k g}_{\text{high frequency interaction}} + \underbrace{\sum_{|j-k| \leq 2} P_j f \cdot P_k g}_{\text{intermediate frequency interaction}} + \underbrace{\sum_{k < j-2} P_j f \cdot P_k g}_{\text{low frequency interaction}} \\
 &=: f \circledcirc g + f \ominus g + f \oslash g
 \end{aligned}$$

• $f \circledcirc g$ = paraproduct of g by f
 \Leftarrow freq of $f \Leftarrow$ freq of g



- para products are always well defined.

$$f \sim s_1, \quad g \sim s_2$$

$$\Rightarrow f \circledast g \sim \min(s_2, s_1 + s_2).$$

- $f \ominus g$ = resonant product

\Leftarrow defined (in general) when $s_1 + s_2 > 0$

$$(\Rightarrow f \ominus g \sim s_1 + s_2)$$

- Besov spaces

$$B_{p,q}^s$$

$$\|f\|_{B_{p,q}^s} = \left\| 2^{sj} \|P_j f\|_{L_{\alpha}^p} \right\|_{l_j^q(\mathbb{Z}_{\geq 0})}$$

$$\cdot p=q=2 : B_{2,2}^s = H^s$$

$$\cdot \text{H\"older-Besov space} : C^s = B_{\infty,\infty}^s$$

- Lipschitz space \mathcal{L}^s , $0 < s < 1$

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$$\|f\|_{\mathcal{L}^s} = \|f\|_{L^\infty} + \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^s}$$

On \mathbb{T}^d , we view f as a periodic func on \mathbb{R}^d
 \Rightarrow We can assume $0 < |x_1 - x_2| \leq \pi$

Then,

$$\mathcal{C}^s = B_{\infty, \infty}^s = \mathcal{L}^s, \quad 0 < s < 1.$$

- Product estimates: (BCD) $p = q = \infty$.

Lemma 1: (i) $s_1 + s_2 > 0$.

$$\|f \otimes g\|_{\mathcal{C}^{s_1+s_2}} \lesssim \|f\|_{\mathcal{C}^{s_1}} \|g\|_{\mathcal{C}^{s_2}}$$

(ii) $\|f \circledast g\|_{\mathcal{C}^{s_2}} \lesssim \|f\|_{L^\infty} \|g\|_{\mathcal{C}^{s_2}}. \text{ (Note: } L^\infty \subset \mathcal{C}^0\text{.)}$

(iii) $s_1 < 0$. $\|f \circledast g\|_{\mathcal{C}^{s_1+s_2}} \lesssim \|f\|_{\mathcal{C}^{s_1}} \|g\|_{\mathcal{C}^{s_2}}$.

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(iv) $s_1 < 0 < s_2$ and $s_1 + s_2 > 0$.

$$\|fg\|_{\ell^{s_1}} \lesssim \|f\|_{\ell^{s_1}} \|g\|_{\ell^{s_2}}$$

(↑ comes from $f \odot g$.)

(v) $s > 0$.

$$\|fg\|_{\ell^s} \lesssim \|f\|_{\ell^s} \|g\|_{\ell^s}.$$

⇐ See Lec notes from 2021.

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$$du = F(u) dB$$

or $\partial_t u = F(u) \xi$. $\Rightarrow u(t) = u(0) + \int_0^t \underbrace{F(u)(t')}_{\partial_t B} \xi(t') dt'$.

Q: What is an integration?

$$(f, g) \mapsto I(f, g)(t) = \int_0^t f(t') \partial_t g(t') dt'.$$

① Differential calculus pt of view:

$$I(f, g) \text{ is the unique soln to } \partial_t I(f, g) = f \partial_t g$$

$$I(f, g)(0) = 0.$$

② Finite increment pt of view.

$I = I(f, g)$ is defined by

$$\begin{cases} I(t_1) - I(t_2) = f(t_2) (g(t_1) - g(t_2)) + \underbrace{o(|t_1 - t_2|)}_{\text{unif in } t_1, t_2} \\ I(0) = 0 \end{cases}$$

NOT unique.

- * is satisfied for $g \in C^1$, $f \in C$

- Young integral if $s_1 + s_2 > 1$ if

$$f \in C^{s_1}, g \in C^{s_2} \quad \text{or} \quad f \in V_c^{1/s_1}, g \in V_c^{1/s_2}$$

\Rightarrow rough path when $s_1 + s_2 < 1$.

③ Fourier analytic pt of view

$$I(f, g)(t) = \int_0^t \underbrace{f(t') \partial_t g(t')}_{\text{↑ define this product.}} dt' \quad f \in C^{s_1}, g \in C^{s_2}$$

↑ define this product.

- product esti $\Rightarrow f \cdot \partial_t g$ is well defined if $s_1 + (s_2 - 1) > 0$

Q: What to do if $s_1 + s_2 < 1$?

$$\text{ex: } \int_0^t u(t') \xi(t') dt'. \quad \xi \in C^{\frac{-1}{2}-} \Rightarrow u \in C^{\frac{1}{2}-}$$

$$(-\frac{1}{2}-) + (\frac{1}{2}-) < 0$$

- Impose a structure.

$$s_2 - 1 \quad s_1$$

Paracontrolled distribution (GIP '15)

We say f is paracontrolled by a given reference distribution g if

$$f = f' \odot g + R \quad \text{smoother}$$

f' = Gubinelli derivative.

$$f' \in \mathcal{C}^s, g \in \mathcal{C}^s, R \in \mathcal{C}^{2s}$$

- high freq behavior of $f \approx$ that of g .
- small scale behavior

Controlled path: f is controlled by g if

$$f(t_1) - f(t_2) = \frac{f'(t_2)}{\uparrow} (g(t_1) - g(t_2)) + R_{t_1, t_2} \quad \text{smooth}$$

• Gubinelli '04

Gubinelli derivative

• Consider $\partial_t u = u \xi$. ($F(u) = u$) $\int_0^t f(t') dt' = \int_0^t f(t') dt'$ ⑯

• paracontrolled ansatz: $u = u' \odot \underbrace{\xi}_{\frac{1}{2}-} + R$
 \Rightarrow solve a system for (u', R)

Paraccontrolled ansatz: $u = X + Y$, where

$$\begin{aligned} \partial_t X &= (X + Y) \odot \xi \sim -\frac{1}{2}- && \cdot \text{ Masmoudi-Weber, CMP'17} \\ \partial_t Y &= (X + Y) \boxtimes \xi & \boxtimes = \odot + \boxplus & \cdot G-Koch-Oh, JEMS'23. \end{aligned}$$

- $X \sim \frac{1}{2}- = (-\frac{1}{2}-) + 1$
 - If we ignore \boxplus , then $Y \sim 1-$ $\leftarrow \frac{1}{2}- \sim 0-$
- $$\Rightarrow Y \boxtimes \xi \sim (1-) + (-\frac{1}{2}-) = \frac{1}{2}- > 0$$

Q: $X \boxtimes \xi$? $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0$.

- X has a structure. (say $X(0) = 0$)

$$X = f((x+y) \odot z).$$

$$\Rightarrow X \ominus z = f((x+y) \odot z) \ominus z$$

give a meaning to RHS.

$$f((x+y) \odot z) = (x+y) \odot f(z) + C_f(x+y, z)$$

$$C_f(f, g) = f(f \odot g) - f \odot f(g)$$

$$\|C_f(f, g)\|_{e^{s_1+s_2+1}} \lesssim \|f\|_{e^{s_1}} \|g\|_{e^{s_2}}, \quad s_1 < 1$$

$$\underline{C_f(x+y, z) \ominus z} \sim (1-) + (-\frac{1}{2}-) = \frac{1}{2}- > 0,$$

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$$[(X+Y) \odot f(\zeta)] \ominus \zeta$$

$$= (X+Y)(f(\zeta) \ominus \zeta) + \text{com}_1(X+Y, f(\zeta), \zeta) \sim \frac{1}{2} -$$

• $\text{com}_1(f, g, h) = (f \odot g) \ominus h - f(g \ominus h)$

Lem 2.4 in GIP
Prop A.9 in MW

$$\| \text{com}_1(f, g, h) \|_{\ell^{S_1+S_2+S_3}} \lesssim \| f \|_{\ell^{S_1}} \| g \|_{\ell^{S_2}} \| h \|_{\ell^{S_3}}$$

• $S_1 < 1, S_2 + S_3 < 0, S_1 + S_2 + S_3 > 0$

Claim: $f(\zeta) \ominus \zeta$ can be defined via stock analysis

$$\begin{matrix} \frac{1}{2} - & -\frac{1}{2} - \\ \text{and } f(\zeta) \ominus \zeta & \sim 0 - \end{matrix}$$

$$\Rightarrow \boxed{\begin{matrix} (X+Y)(f(\zeta) \ominus \zeta) & \sim 0 - \\ \frac{1}{2} - & 0 - \end{matrix}}$$

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$$\text{Summary: } \partial_t u = u \tilde{\zeta}$$

$$\partial_t X = (X + Y) \odot \tilde{\zeta} \sim -\frac{1}{2}-$$



$$\partial_t Y = (X + Y) \odot \tilde{\zeta} \underset{0-}{+} Y \ominus \tilde{\zeta} \underset{\frac{1}{2}-}{+}$$

$$\left. \begin{array}{l} X \sim \frac{1}{2}- \\ Y \sim 1- \end{array} \right\}$$

$$+ C_g(X + Y, \tilde{\zeta}) \ominus \tilde{\zeta} \sim \frac{1}{2}-$$

$$+ \text{Com}_1(X + Y, f(\tilde{\zeta}), \tilde{\zeta}) \sim \frac{1}{2}-$$

$$+ (X + Y) (f(\tilde{\zeta}) \ominus \tilde{\zeta}) \sim 0-$$

· Decomposition of the ill-defined soln map:

$$X(0) = 0$$

$$(u(0), \tilde{\zeta}) \xrightarrow[\text{stoch analysis}]{{}^1} \text{enhanced dataset } \Xi = (Y(0), \tilde{\zeta}, f(\tilde{\zeta}) \ominus \tilde{\zeta})$$

$$\xrightarrow[\text{deterministic analysis}]{{}^2 \leftarrow \text{continuous}} (X, Y) \longmapsto u = X + Y$$

$$\text{Step ①: } \tilde{\zeta}(t) = \sum_{n \in \mathbb{Z}} q_n e^{int}$$

$$\| \| P_j \tilde{\zeta} \| \|_{L_t^{\infty} L_p(\Omega)}^{\text{Sobolev}} \lesssim 2^{\varepsilon j} \| \| P_j \tilde{\zeta} \| \|_{L_t^r} \|_{L_p(\Omega)} \quad \text{for some } r = r(\varepsilon) \gg 1$$

or Bernstein's Ineq

$$\| P_j f \|_{L_p} \lesssim 2^{d(\frac{1}{q} - \frac{1}{p})j} \| P_j f \|_{L_q} \quad 1 \leq q \leq p \leq \infty$$

$$= 2^{\varepsilon j} \left\| \left\| \sum_{|m| \sim 2^j} q_m e^{int} \right\|_{L_t^r} \right\|_{L_p(\Omega)}$$

Mink. int ineq

$$(p \geq r) \leq 2^{\varepsilon j} \left\| \underbrace{\left\| \sum_{|m| \sim 2^j} q_m e^{int} \right\|_{L_p(\Omega)}}_{\sim \sqrt{p}} \right\|_{L_t^r} \lesssim 2^{(\frac{1}{2} + \varepsilon)j}$$

$$\sim \sqrt{p} \left\| \cdot \right\|_{L^2(\Omega)}$$

$$\sim \sqrt{p} \left(\sum_{|m| \sim 2^j} \mathbb{E}[|q_m|^2] \right)^{1/2} \sim \sqrt{p} \underline{2^{\frac{1}{2}j}}$$

(23)

$$\Rightarrow \left\| \sup_j 2^{\left(\frac{1}{2}-2\varepsilon\right)j} \|P_j \tilde{z}\|_{L_t^\infty} \right\|_{L^p(\Omega)}$$

$$\leq \underbrace{\sum_{j=0}^{\infty} 2^{-\varepsilon j}}_{\lesssim 1} \cdot \underbrace{2^{\frac{1}{2}-\varepsilon)j}}_{\lesssim 1} \\ \leq \underbrace{\sum_{j=0}^{\infty} 2^{-\varepsilon j}}_{\lesssim 1} \left\| \underbrace{2^{\frac{1}{2}-\varepsilon)j}}_{\lesssim 1} \|P_j \tilde{z}\|_{L_t^\infty} \right\|_{L^p(\Omega)} \lesssim 1$$

$$\Rightarrow \tilde{z} \in C^{-\frac{1}{2}-2\varepsilon} \text{ a.s.}$$

$$\cdot f(\xi) \ominus \xi \rightarrow I(\xi) \ominus \xi$$

where $I[f] = \sum_{n \neq 0} \frac{f(n)}{in} e^{int}$

$$(I(\xi) \ominus \xi)(t) = \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{n=n_1+n_2 \\ |n_1| \sim |n_2| \\ n_1 \neq 0}} \underbrace{g_{n_1} g_{n_2}}_{in_1} \right) e^{int}$$

$$= \sum_{n \in \mathbb{Z}} \sum_{\substack{n=n_1+n_2 \\ |n_1| \sim |n_2| \\ n_1 \neq 0}} \frac{g_{n_1} g_{n_2} - \mathbf{1}_{n_1+n_2=0}}{in_1} e^{int}$$

renormalization

2nd order Wiener chaos

$\epsilon \chi^2$

symmetry is crucial

$$\left(\Leftarrow \sum_{n_1} \frac{1}{in_1} = \lim_{N \rightarrow \infty} \sum_{0 < |n_1| \leq N} \frac{1}{in_1} = \lim_{N \rightarrow \infty} \sum_{0 < n < N} \left(\frac{1}{in_1} - \frac{1}{in_n} \right) = 0 \right)$$

$$\cdot \parallel \parallel P_j(I(\tilde{z}) \oplus \tilde{z}) \parallel_{L_t^r} \parallel_{L^p(\Omega)}, \quad r \gg 1$$

$P \geq r$
 \leq
Mink

$$\parallel \parallel P_j(I(\tilde{z}) \oplus \tilde{z}) \parallel_{L^p(\Omega)} \parallel_{L_t^r}$$

$\lesssim P \parallel \sim \parallel_{L^2(\Omega)}$

$\{g_n, g_{n_2} - \mathbf{1}_{n_1+n_2}\}$ is an orthogonal family in $L^2(\Omega)$

$$\parallel \sim \parallel_{L^2(\Omega)} = \sum_{|n| \approx n_2} \sum_{\substack{n_i \neq 0 \\ |n_i| \geq 2^j}} \frac{1}{n_i^2} \quad \left(\begin{array}{l} \Leftarrow \frac{1}{n_1^2} + \frac{1}{n_1 n_2} \sim \frac{1}{n_1^2} \\ \text{under } |n_1| \sim |n_2| \end{array} \right)$$

$$\lesssim 1$$

$$|n_i| \gtrsim |n| \sim 2^j$$

$$\Rightarrow I(\tilde{z}) \oplus \tilde{z} \in \mathcal{C}^{-2^j}, \text{ a.s.}$$

self-renormalizing

b/c $n = n_1 + n_2$

Lec 2 : July 07, 2023 (FRI)

①

- Last time, by introducing two commutators and substituting the X -eqn in $X \ominus \bar{z}$, we derived a system for X and Y .



We "assumed" that X solves the X -eqn.

In order to make this rigorous, we first need to regularize \bar{z} by a mollification or a freq. truncation and consider $\partial + U_N = U_N \bar{z}_N$ [↑] needs to be symmetric.

$$\bar{z}_N = P_{\leq N} \bar{z} \quad P_{\leq N} = \text{freq trunc onto } \{|m| \leq N\}$$

- Then, by writing $U_N = X_N + Y_N$, we derive a system for X_N and Y_N as in Lec 1.

- Then, take a limit as $N \rightarrow \infty$ and show

$$\textcircled{1} (\bar{z}_N, g(\bar{z}_N) \ominus \bar{z}_N) \rightarrow (\bar{z}, g(\bar{z}) \ominus \bar{z}), \textcircled{2} (X_N, Y_N) \rightarrow (X, Y)$$

- In ①, we use stoch. analysis.
- In ②, we use deterministic analysis and show that the soln map to the system depends continuously on enhanced data sets $\Xi = (\nu_0, \Sigma_1, \Sigma_2)$

②

$$\begin{matrix} \cap \\ \mathcal{C}^{1,-} \end{matrix} \quad \begin{matrix} \cap \\ \mathcal{C}^{0,-} \end{matrix}$$

- Back to Besov spaces

→ bounded on L_x^p , wif in j

$$\|f\|_{B_{p,q}^s} = \|2^{sj} \|P_j f\|_{L_x^p}\|_{l_j^q} \quad 1 \leq p \leq \infty$$

- Sobolev space $W^{s,p}$, $H^s = W^{s,2}$

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_p = \|\tilde{\mathcal{F}}(\langle n \rangle^s \hat{f}(n))\|_p$$

$$\langle n \rangle = \sqrt{1 + |n|^2}$$

(3)

$$\|f\|_{B_{p,\infty}^s} = \sup_j 2^{sj} \|P_j f\|_{L_x^p}$$

Bernstein

$$\sim \sup_j \|P_j \langle \nabla \rangle^s f\|_{L_x^p}$$

See Tao's book.
App. A.

$$\lesssim \|\langle \nabla \rangle^s f\|_{L_x^p} = \|f\|_{W^{s,p}}$$

$$= \left\| \sum_j \langle \nabla \rangle^s P_j f \right\|_{L_x^p}$$

$$\leq \sum_j \|\langle \nabla \rangle^s P_j f\|_{L_x^p} \stackrel{\text{Bern}}{\sim} \sum_j 2^{sj} \|P_j f\|_{L_x^p}$$

$$= \|f\|_{B_{p,1}^s}$$

summable in j

$$\lesssim \|f\|_{B_{p,\infty}^{s+\varepsilon}}, \quad \forall \varepsilon > 0.$$

$$\Rightarrow \|f\|_{B^s_{p,\infty}} \lesssim \|f\|_{W^{s,p}} \lesssim \|f\|_{B^s_{p,1}} \\ \lesssim \|f\|_{B^{s+\varepsilon}_{p,\infty}}$$
④

In particular,

$$\boxed{\|f\|_{e^s} \lesssim \|f\|_{W^{s,\infty}} \lesssim \|f\|_{e^{s+\varepsilon}}, \quad \forall \varepsilon > 0}$$

• Pf of Lemma 1 (product estimates)

(i) $s_1 + s_2 > 0$. WTS $\|f \ominus g\|_{e^{s_1+s_2}} \lesssim \|f\|_{e^{s_1}} \|g\|_{e^{s_2}}.$

$$P_k(f \ominus g) = \sum_{l=-2}^2 \sum_{j \geq k-10} P_k(P_f f \cdot P_{j+l} g)$$

freq supp $\approx 2^j$ $(2 \cdot 2^j + 2 \cdot 2^{j+2})$

⑤

$$\Rightarrow \|P_R(f \oplus g)\|_{L^\infty} \leq \sum_{i=-2}^2 \| \cancel{P_R} \sum_{j \geq k-10} P_j f \cdot P_{j+1} g \|_{L^\infty}$$

$\overbrace{2^{(s_1+s_2)R}}$
 $\overbrace{2^{(s_1+s_2)(k-j)}}$
 $\overbrace{2^{s_1 i}}$
 $\overbrace{2^{s_2(j+i)}}$

summable in $j \geq k-10 \Rightarrow \approx 1.$

$$\lesssim \sum_{i=-2}^2 2^{s_1 i} \|P_j f\|_{L^\infty} \cdot 2^{s_2(j+i)} \|P_{j+i} g\|_{L^\infty}$$

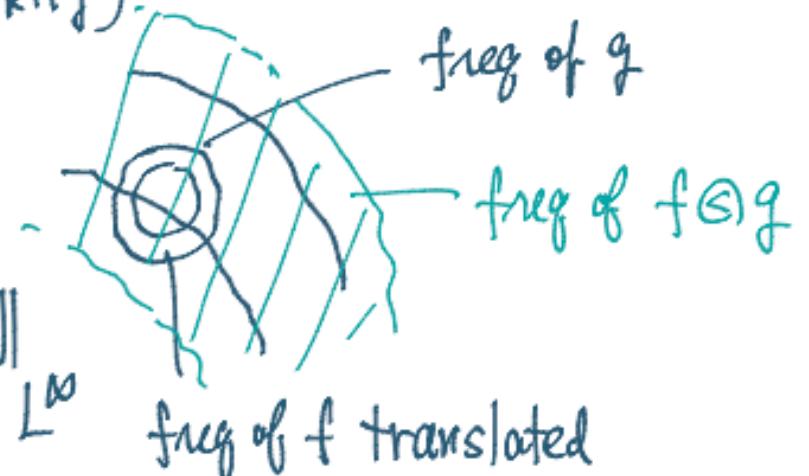
$$\lesssim \|f\|_{C^{s_1}} \|g\|_{C^{s_2}}.$$

\Rightarrow take sup in $k.$

⑥

$$(ii) \text{ WTS } \|f \otimes g\|_{\ell^{S_2}} \lesssim \|f\|_{L^\infty} \|g\|_{\ell^{S_2}}.$$

$$P_k(f \otimes g) = \sum_{i=-2}^2 \sum_{j < k+i-2} P_k(P_j f \cdot P_{k+i} g)$$



$$\begin{aligned} \|P_k(f \otimes g)\|_{L^\infty} &\lesssim \sum_{i=-2}^2 \|S_{k+i-2}(f) \cdot P_{k+i} g\|_{L^\infty} \\ &\leq \|f\|_{L^\infty} \|g\|_{\ell^{S_2}} \end{aligned}$$

$$S_k(f) = \sum_{j < k} P_j f.$$

S_k is bdd on L_x^∞ , unif in k

(7)

(iii) $s_1 < 0$.

WTS

$$\|f \otimes g\|_{\ell^{s_1+s_2}} \lesssim \|f\|_{\ell^{s_1}} \|g\|_{\ell^{s_2}}$$

$$P_k(f \otimes g) = \sum_{i=-2}^2 \left[\sum_{j \leq k+i-2} 2^{s_1(k-j)} \right] P_j f P_{k+i} g$$

$\lesssim 1$

$2^{s_1 j}$ $2^{s_2 k}$

\Rightarrow The rest follows as before.

(iv) & (v) \Leftarrow paraproduct decomp & (i) - (iii).

□

⑧

Consider $\partial_t u = F(u) \zeta$. $F \in C_b^3$

Supp $u \in \mathcal{C}^s$, $0 < s < 1$.

$$\Rightarrow \frac{|F(u(t_1)) - F(u(t_2))|}{|t_1 - t_2|^s} \leq \|u\|_{\mathcal{C}^s} \cdot \|F'\|_{C_b^1}$$

$$\frac{|u(t_1) - u(t_2)|}{|t_1 - t_2|^s} \leq \|u\|_{\mathcal{C}^s}$$

i.e. $F \circ u \in \mathcal{C}^s$

$$F' \circ u, F'' \circ u \in \mathcal{C}^s$$

Write $u = X + Y$, where

$$\partial_t X = F(X + Y) \odot \zeta$$

$$\partial_t Y = \underbrace{F(X + Y) \odot \zeta}_{\text{good} \sim 0} + \boxed{F(X + Y) \odot \zeta}$$

Paralinearization.

GIP Lemma 2.b.

- Lemma 2 : $0 < s_2 \leq s_1 < 1$.

(i) If $F \in C_b^{1+s_2/s_1}$, then we have

$$F(f) = F'(f) \odot f + R(f)$$

$$R = R_F$$

for any $f \in C^{s_1}$,

where R satisfies

$$\|R(f)\|_{C^{s_1+s_2}} \lesssim \|F\|_{C_b^{1+s_2/s_1}} \left(1 + \|f\|_{C^{s_1}}^{1+s_2/s_1} \right).$$

(ii) If $F \in C_b^{2+s_2/s_1}$, then R is locally Lipschitz:

$$\|R(f) - R(g)\|_{C^{s_1+s_2}} \lesssim \|F\|_{C_b^{2+s_2/s_1}} \left(1 + \|f\|_{C^{s_1}}^{1+s_2/s_1} + \|g\|_{C^{s_1}}^{1+s_2/s_1} \right) \|f - g\|_{C^{s_1}}$$

Probematic resonant product

$$X+Y \sim \frac{1}{2} -$$

(10)

$$F(X+Y) \otimes \Xi$$

$$F(X+Y) \sim \frac{1}{2} -, \quad \Xi \sim -\frac{1}{2} -$$

$F \in C_b^2$

parallel
= $[F'(X+Y) \otimes (X+Y)] \otimes \Xi + \underbrace{R(X+Y) \otimes \Xi}_{1-}$.

$$= F'(X+Y) \overline{[(X+Y) \otimes \Xi]}$$

$$+ \text{com}_1(F'(X+Y), X+Y, \Xi) + R(X+Y) \otimes \Xi.$$

$\begin{matrix} \frac{1}{2}- & \frac{1}{2}- & -\frac{1}{2}- \\ \sim \frac{1}{2}- & & \end{matrix}$

$\underbrace{X \otimes \Xi}_{\text{BAD.}} + \underbrace{Y \otimes \Xi}_{\text{good}} \sim \frac{1}{2} - \text{ if } Y \sim 1-$

$$\cdot X \ominus Z$$

$$= \underline{\underline{f}}(F(x+y) \underline{\underline{\ominus}} Z) \ominus Z$$

$$= [F(x+y) \underset{\text{uu}}{\ominus} \underline{f(Z)}] \underset{\text{uu}}{\ominus} Z +$$

$$X = f(F(x+y) \ominus Z)$$

II

$$C_f(F(x+y), \overset{<1}{Z}) \ominus Z$$
$$(\frac{1}{2}-) + (\frac{1}{2}-) + 1 = 1-$$
$$\sim \frac{1}{2}-$$

0-

0-

$$= F(x+y) (\overset{1/2-}{f(Z)} \ominus \overset{0-}{Z})$$

$$+ \text{com}_1 (F(x+y), \overset{1/2- < 1}{f(Z)}, \overset{1/2-}{Z}, \overset{-1/2-}{Z}) + C_f(F(x+y), Z) \ominus Z$$

$$(\frac{1}{2}-) + (\frac{1}{2}-) + (-\frac{1}{2}-) = \frac{1}{2}-$$

$$\Rightarrow \partial_t X = F(X+Y) \odot \xi \sim -\frac{1}{2} \text{--} \quad (12)$$

$F \in C_b^2$

$$\begin{aligned} \partial_t Y &= F(X+Y) \odot \xi + \underline{R(X+Y) \odot \xi} \\ &\quad + \underline{\text{com}_1(F'(X+Y), X+Y, \xi)} \\ &\quad + \underline{F'(X+Y)} [Y \odot \xi] \\ &\quad + \underline{F'(X+Y)} [c_g(F(X+Y), \xi) \odot \xi] \\ &\quad + \underline{F'(X+Y)} \text{com}_1(F(X+Y), f(\xi), \xi) \\ &\quad + \underline{F'(X+Y)} \cdot \underline{F(X+Y)} (f(\xi) \odot \xi). \end{aligned} \quad \left. \right\} \sim 0 \text{--}$$

$$\Rightarrow X \sim \frac{1}{2} \text{--}, Y \sim 1 \text{--}$$

Note: In getting a difference estimate, we need to assume $F \in C_b^3$.

Pf of Lemma 2 (paralinearization)

$$(i) \quad R(f) = F(f) - F'(f) \otimes f$$

$$S_k(f) = \sum_{j < k} P_j f.$$

$$= \sum_{j=0}^{\infty} P_j F(f) - S_{j-2}(F'(f)) P_j f$$

freq $\sim 2^j$

$$=: \sum_{j=0}^{\infty} A_j$$

① $j \leq 3$: $\|A_j\|_{L^\infty} \lesssim \|F\|_{C_b^1} (1 + \|f\|_{L^\infty}).$

② $j > 3$: $P_j f = K_j^{\text{per}} * f, \quad \int K_j^{\text{per}} dx = \varphi_j(0) = 0.$

$$\cdot S_{j-2} f = K_{<j-2}^{\text{per}} * f, \quad \text{where} \quad K_{<j-2}^{\text{per}} = \sum_{k < j-2} K_k^{\text{per}}$$

$$\Rightarrow \int K_{<j-2}^{\text{per}} dx = 1. \quad \text{for } j > 3.$$

(14)

$$A_j(x) = \iint_{\mathbb{T}^d \times \mathbb{T}^d} K_j^{\text{per}}(x-y) K_{<j-2}^{\text{per}}(x-z) \\ \times [F(f)(y) - F'(f)(z) f(y)] dy dz$$

- View f (and $F(f)$) as a periodic func on \mathbb{R}^d
- $\begin{cases} K_j^{\text{per}}(x) = \sum_{n \in \mathbb{Z}^d} K_j(x - 2\pi n) \\ K_{<j-2}(x) = \sum_{n \in \mathbb{Z}^d} K_{<j-2}(x - 2\pi n) \end{cases}$

$$= \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_j(x-y) K_{<j-2}(x-z) [F(f)(y) - F'(f)(z) f(y)] dy dz$$

$$= \iint K_j(x-y) K_{<j-2}(x-z) \left[F(f)(y) - F(f)(z) - F'(f)(z) (f(y) - f(z)) \right] dy dz$$

$$\left(\Leftarrow \int K_j(x-y) \left(\int K_{<j-2}(x-z) F(f)(z) dz \right) dy = 0, \text{ etc.} \right)$$

(15)

$$\cdot F(f(y)) - F(f(z))$$

$$FTC = \int_0^1 F'(f(z) + \tau(f(y) - f(z))) dz \times (\underline{f(y) - f(z)}) .$$

$$\Rightarrow F(f(y)) - F(f(z)) - \underline{F'(f)(z)} (\underline{f(y) - f(z)})$$

$$= \int_0^1 \underbrace{F'(f(z) + \tau(f(y) - f(z)))}_{\leq \tau |f(y) - f(z)|} - \underbrace{F'(f(z))}_{F'(f(z))} dz \times \underline{(f(y) - f(z))}$$

$$\lesssim \|F\|_{C_b^{1+s_2/s_1}} \underbrace{|f(y) - f(z)|^{1+s_2/s_1}}_{\lesssim \|f\|_{C^{s_1}}^{1+s_2/s_1} \min_{\tilde{z}} |y - \tilde{z}|^{s_1+s_2}, \quad \tilde{z} \in z + (2\pi\mathbb{Z})^d} \leq |y - z|^{s_1+s_2}$$

$$|A_j(x)| \lesssim \|F\|_{C_b^{1+s_2/s_1}} \iint K_j(x-y) K_{<j-2}(x-z) |y-z|^{s_1+s_2} dy dz \times \|f\|_{C^{s_1}}^{s_1+1+s_2/s_1} \quad (16)$$

$$\begin{cases} K_j(x-y) = 2^{dj} K(2^j(x-y)) & K = \mathcal{F}^{-1}(\Phi_1(\frac{\cdot}{2})). \\ K_{<j-2}(x-z) = 2^{d(j-3)} K_{<1}(2^{j-3}(x-z)) \end{cases}$$

$$= \|F\|_{C_b^{1+s_2/s_1}} \iint \underbrace{2^{dj} K(2^j(x-y))}_{\text{W}} \underbrace{2^{d(j-3)} K_{<1}(2^{j-3}(x-z))}_{\text{W}} \times 2^{(s_1+s_2)j} |y-z|^{s_1+s_2} dy dz \times \|f\|_{C^{s_1}}^{1+s_2/s_1}$$

$$\times 2^{-s_1+s_2} j$$

$$\lesssim \|F\|_{C_b^{1+s_2/s_1}} \|f\|_{C^{s_1}}^{1+s_2/s_1} \cdot \underline{2^{-s_1+s_2} j}.$$

\Rightarrow The desired bd on $\|R(f)\|_{C^{s_1+s_2}}$.

(ii) Difference estimate on $R(f) - R(g)$

① $j \leq 3$: easy

② $j > 3$: $A_j(f)(x) - A_j(g)(x) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_j(x-y) K_{j-2}(x-z) \cdots dy dz$

* $\begin{aligned} & \int_0^1 F'(f(z) + \tau(f(y) - f(z))) - F'(f(z)) d\tau \times (f(y) - f(z)) \\ & - \int_0^1 F'(g(z) + \tau(g(y) - g(z))) - F'(g(z)) d\tau \times (g(y) - g(z)) \end{aligned}$

= I + II + III, where

* $\boxed{\text{I}} = \int_0^1 \underbrace{F'(f(z) + \tau(f(y) - f(z))) - F'(f(z))}_{\leq \|F\| C_b^{1+s_2/s_1} |f(y) - f(z)|^{s_2/s_1}} d\tau \times \underbrace{(f(y) - f(z)) - (g(y) - g(z))}_{\leq \|f-g\|_{C^{s_1}} |y-z|^{s_1}}$

* $\boxed{\text{II}} \lesssim \|F\|_{C_b^{1+s_2/s_1}} \|f\|_{C^{s_1}}^{s_2/s_1} \|f-g\|_{C^{s_1}} |y-z|^{s_1+s_2}$

$$\cdot \text{II} = \int_0^1 F'(f(z) + \tau(f(y) - f(z))) - F'(g(z) + \tau(g(y) - g(z))) d\tau \quad (18)$$

$$x (g(y) - g(z))$$

$$= \int_0^1 \int_0^1 F''(f(z) + \tau(f(y) - f(z)) + \theta(f(z) - g(z) + \tau((f(y) - f(z)) - (g(y) - g(z)))) d\theta d\tau$$

$$x ((f(z) - g(z)) + \tau((f(y) - f(z)) - (g(y) - g(z))) \times (g(y) - g(z))$$

$$=: \text{II}_1 \qquad \qquad \qquad =: \text{II}_2$$

$$\cdot \text{III} = - \int_0^1 F'(f(z)) - F'(g(z)) d\tau \times (g(y) - g(z))$$

$$= - \int_0^1 \int_0^1 F''(f(z) - \theta(f(z) - g(z))) d\theta d\tau$$

$$x (f(z) - g(z)) \times (g(y) - g(z))$$

$$\boxed{\text{II}_1 + \text{III}} = \int_0^1 \int_0^1 F''(\underline{f(z)} + \underline{\tau(f(y) - f(z))} \\ + \underline{\theta(f(z) - g(z))} + \underline{\tau((f(y) - f(z)) - (g(y) - g(z)))}) \\ - F''(\underline{f(z)} + \underline{\theta(f(z) - g(z))}) d\theta dz \\ \times (\underline{f(z) - g(z)} \times \underline{(g(y) - g(z))})$$

$\leq \|f - g\|_{L^\infty}$ $\leq \|g\|_{C^{s_1}} |y - z|^{s_1}$

$$\lesssim \|F\|_{C_b^{2+s_2/s_1}} \left(|f(y) - f(z)| + |g(y) - g(z)| \right)^{s_2/s_1}$$

$$\times \|g\|_{C^{s_1}} \|f - g\|_{L^\infty} |y - z|^{s_1}$$

$$\lesssim \|F\|_{C_b^{2+s_2/s_1}} \left(1 + \|f\|_{C^{s_1}} + \|g\|_{C^{s_1}} \right)^{1+s_2/s_1} \|f - g\|_{C^{s_1}} |y - z|^{s_1}.$$

*2

$$\cdot |\mathbb{I}_2| \leq \|F\|_{C_b^2} ((f(y) - f(z)) - (g(y) - g(z))) \times (g(y) - g(z)) \quad (20)$$

$$\leq \|F\|_{C_b^2} \|g\|_{e^{s_1}} \underbrace{\|f-g\|_{e^{s_2}}}_{\leq \|f-g\|_{e^{s_1}}} |y-z|^{s_1+s_2}$$

From \circledast_1 , \circledast_2 , and \circledast_3 , we obtain

$$|A_j(f)(x) - A_j(g)(x)| \lesssim \|F\|_{C_b^{2+s_2/s_1}} \left(1 + \|f\|_{e^{s_1}} + \|g\|_{e^{s_2}} \right)^{1+s_2/s_1} \|f-g\|_{e^{s_1}}$$

$$\times \iint_{\mathbb{R}^d \times \mathbb{R}^d} K_j(x-y) K_{<j-2}(x-z) |y-z|^{s_1+s_2} dy dz$$

$\lesssim 2^{-(s_1+s_2)\hat{j}}$ as on p. 16.



Rmk: It's not essential that F and its derivatives are bdd.

For example, if \exists non-decreasing func G s.t.

say $s_2 = s_1$

$$|F(f)|, |F'(f)|, \dots, |F''(f)| \leq G(\|f\|_{\mathcal{C}^{s_1}}).$$

(EX: $F(f) =$ polynomial (or analytic) in f .

Then, Lemma 2 still holds by replacing

$$\|F\|_{\mathcal{C}_b^3} \text{ by } G(\|f\|_{\mathcal{C}^{s_1}}), \text{ etc.}$$

↔ This is good enough to solve $\partial_t u = F(u)\xi$

Lec 3 : July 10 , 2023 (MON)

①

Back to $\partial_t u = u \tilde{g}$ (linear case)

$$\widehat{If}(m) = \mathbb{1}_{n=0} \frac{\widehat{f}(n)}{in}$$

Goal: Solve the equation on (a subinterval of) $[0, 1]$.

View $[0, 1]$ as $[0, 1] \subset \mathbb{T} = [-\pi, \pi]$.

Need to introduce several objects.

① $X(t) =$ smooth cutoff function supported on $[-\frac{1}{2}, \frac{3}{2}]$
such that $X \equiv 1$ on $[0, 1]$.

② integration operator $\mathcal{J}(f)(t) = \int_0^t f(\tau) d\tau$

$$\Rightarrow \boxed{\mathcal{J}(f)(t) = If(t) - If(0) + t P_0(f)}$$

$$P_0(f) = \widehat{f}(0) = \int_{-\pi}^{\pi} f(t) dt \quad \text{for } f \in \mathbb{T}$$

Set $\mathcal{J}_X = X \cdot \mathcal{J}$

$$\begin{cases} \partial_t u = u \tilde{s} \\ u|_{t=0} = u_0 \end{cases} \Leftrightarrow u(t) = u_0 + \int_0^t u(t') \tilde{s}(t') dt' \quad (2)$$

$$\Leftrightarrow \underset{\text{on } [0, T]}{u(t) = u_0 + \int_X (u \tilde{s})(t)} \leftarrow \text{makes sense on } \Pi.$$

- Given $T > 0$, let $\chi_T(t) = \chi\left(\frac{t}{T}\right) \quad \chi_T \equiv 1 \text{ on } [0, T]$

and we consider

$$u = u_0 + \int_X (\underline{\chi_T} \cdot u \tilde{s}).$$

Note: not good to put χ_T outside \int_X since
we work in C^s with $s > 0$ (and $L^\infty \supset C^s$).

(3)

Lemma 3 (integral op)

$$(i) \quad \| I f \|_{\mathcal{C}^{s+1}} \lesssim \| f \|_{\mathcal{C}^s}, \quad \forall s \in \mathbb{R}$$

(ii) Let $s > -1$.

$$\| \underline{\mathcal{F}_X(f)} \|_{\mathcal{C}^{s+1}} \lesssim \| f \|_{\mathcal{C}^s}$$

$\begin{cases} X(t) I(f)(0) \\ \text{in } \mathcal{C}^\infty \quad \text{want this to make sense in } \mathcal{C}^{s+1} \subset L^\infty \\ \Rightarrow s > -1 \end{cases}$

Lemma 4 (time localization)

(i) $-1 < s_1 \leq s_2 \leq 0$. Then, given any $\varepsilon > 0$, $\exists C > 0$ s.t.

$$\| X_T f \|_{\mathcal{C}^{s_1}} \leq C \frac{T^{s_2 - s_1 - \varepsilon}}{} \| f \|_{\mathcal{C}^{s_2}}$$

for $0 < T \leq 1$.

(4)

(iii) $0 < s < 1$. $\forall \varepsilon > 0 \exists C > 0$ s.t.

$$\| J_X(X_T \cdot f) \|_{e^s} \leq C T^{\frac{s}{2}} \| f \|_{e^{s-1+\varepsilon}}$$

- $u = u_0 + J_X(X_T \cdot u \tilde{\zeta})$

Write $u = X + Y$, where $\frac{u \otimes \tilde{\zeta}}{\text{worst}} + u \otimes \tilde{\zeta}$

$$X = J_X(X_T \cdot (X+Y) \otimes \tilde{\zeta})$$

$$Y = u_0 + J_X(X_T \cdot (X+Y) \otimes \tilde{\zeta})$$

- Assume $\tilde{\zeta} \sim -\frac{1}{2} - \varepsilon$.

$\Rightarrow X \sim \frac{1}{2} - 2\varepsilon$ (in order to get T^θ).

$Y \sim \frac{1}{2} + a\varepsilon$ for some $a > 0$.

(5)

As in Lec 4, we need to rewrite $X \ominus \tilde{z}$

$$X \ominus \tilde{z} = \oint_X ((X+Y) \ominus \tilde{z}) \underset{\substack{P_{\neq 0} \\ \text{NO } X_T}}{\ominus} \tilde{z}$$

$$\left\{ \begin{array}{l} X = \oint_X (X_T \cdot (X+Y) \ominus \tilde{z}) \\ = \oint_X ((X+Y) \ominus \tilde{z}) \end{array} \right. \text{on } [0, T].$$

- $$\begin{aligned} & \oint_X ((X+Y) \ominus \tilde{z})(t) \\ &= \underline{X(t) \cdot I((X+Y) \ominus \tilde{z})(t)} - \underline{X(0) \cdot I((X+Y) \ominus \tilde{z})(0)} \\ &+ \cancel{X(t) \cdot t \cdot P_0((X+Y) \ominus \tilde{z})} \end{aligned}$$

$$(X+Y) \ominus P_{\neq 0} \tilde{z} = P_{\neq 0} ((X+Y) \ominus \tilde{z})$$

2nd term: $X \ominus \tilde{z}$ makes sense since X is smooth

(need $X \sim \frac{1}{2} + 2\varepsilon$)

(6)

$$\cdot X \cdot I((X+Y) \odot P_{\pm} \vec{z})$$

$$= \underbrace{X \cdot [(X+Y) \odot I(\vec{z})]} + X \cdot C_I(X+Y, \vec{z})$$

$\frac{1}{2}-2\varepsilon$

$-\frac{1}{2}-\varepsilon$

$1-3\varepsilon$

Lemma 5: $C_I(f, g) = I(f \odot g) - f \odot I(g)$

$s_1 < 1$ Then,

$$\|C_I(f, g)\|_{e^{s_1+s_2+1}} \lesssim \|f\|_{e^{s_1}} \|g\|_{e^{s_2}}.$$

$$\cdot [X \cdot ((X+Y) \odot I(\vec{z}))] \ominus \vec{z}$$

$$= \underbrace{[X \ominus ((X+Y) \odot I(\vec{z}))]} \ominus \vec{z}$$

$$+ \underbrace{[X \overset{\sim \infty}{\ominus} ((X+Y) \odot I(\vec{z}))]} \ominus \vec{z}$$

$\sim \infty$

↑ makes sense

(7)

$$[X \otimes ((X+Y) \otimes I(\tilde{z}))] \otimes \tilde{z}$$

$$\begin{aligned}
 &= X [((X+Y) \otimes I(\tilde{z})) \otimes \tilde{z}] + com_1(X, (X+Y) \otimes I(\tilde{z}), \tilde{z}) \\
 &= X \cdot [(X+Y)(I(\tilde{z}) \otimes \tilde{z})] + X \cdot com_1(X+Y, I(\tilde{z}), \tilde{z}) \\
 &\quad + com_1(X, (X+Y) \otimes I(\tilde{z}), \tilde{z})
 \end{aligned}$$

Labels for terms:
 Top term: $\frac{1-\varepsilon}{2} \quad \frac{1}{2}-\varepsilon \quad -\frac{1}{2}-\varepsilon$
 Middle term: $\frac{1}{2}-2\varepsilon \quad \frac{1}{2}-\varepsilon \quad -\frac{1}{2}-\varepsilon$
 Bottom term: $\frac{1}{2}-4\varepsilon$

Lemma b: $com_1(f, g, h) = (f \otimes g) \otimes h - f(g \otimes h)$

$$\cdot s_1 < 1, \quad s_2 + s_3 < 0, \quad s_1 + s_2 + s_3 > 0$$

$$\|com_1(f, g, h)\|_{e^{s_1+s_2+s_3}} \lesssim \|f\|_{e^{s_1}} \|g\|_{e^{s_2}} \|h\|_{e^{s_3}}$$

⑧

$$\cdot X = \oint_X (X_T \cdot (X+Y) \otimes \bar{z})$$

$$\cdot Y = u_0 + \oint_X (X_T \cdot (X+Y) \odot \bar{z})$$

$$+ \oint_X (X_T \cdot Y \otimes \bar{z})$$

$$+ \oint_X (X_T \cdot \underbrace{I((X+Y) \otimes \bar{z})(0)}_{\text{const.}} \cdot X \otimes \bar{z})$$

$$+ \oint_X (X_T \cdot (X \cdot C_I(X+Y, \bar{z})) \otimes \bar{z})$$

$$+ \oint_X (X_T \cdot [X \otimes ((X+Y) \otimes I(\bar{z}))] \otimes \bar{z})$$

$$+ \oint_X (X_T \cdot \text{com}_1(X, (X+Y) \otimes I(\bar{z}), \bar{z}))$$

$$+ \oint_X (X_T \cdot X \cdot \text{com}_1(X+Y, I(\bar{z}), \bar{z}))$$

$$+ \oint_X (X_T \cdot X \cdot (X+Y) (\underbrace{I(\bar{z}) \otimes \bar{z}}_{-\varepsilon}))$$

$\Rightarrow Y \sim 1 - 2\varepsilon$ at best

- Enhanced data set

$$\Xi = (\mu_0, \Xi_1, \Xi_2) = (\mu_0, \zeta, I(\zeta) \ominus \zeta) \quad \begin{matrix} -\frac{1}{2}-\varepsilon \\ -\varepsilon \end{matrix} \text{ for white noise } \zeta \quad \text{Lec 1: via stoch. analy.}$$

$$\in \mathbb{R} \times \mathbb{C}^{\frac{1}{2}-\varepsilon} \times \mathbb{C}^{-\varepsilon}$$

- We now view Ξ_1 and Ξ_2 as given deterministic distributions in $\mathbb{C}^{\frac{1}{2}-\varepsilon}$ and $\mathbb{C}^{-\varepsilon}$, resp., and solve the system.

- $\Gamma_1(X, Y, \Xi_1) = \text{RHS of the } X\text{-eqn.}$

- $\Gamma_2(X, Y, \mu_0, \Xi_1, \Xi_2) = \text{RHS of the } Y\text{-eqn.}$

WTS: $\Gamma = (\Gamma_1, \Gamma_2)$ is a contraction on $\mathbb{C}^{\frac{1}{2}-2\varepsilon} \times \mathbb{C}^{\frac{1}{2}+\alpha\varepsilon}$

$$\|\Gamma_1(X, Y, \Xi_1)\|_{\mathbb{C}^{\frac{1}{2}-2\varepsilon}} \stackrel{\text{Lem 4}}{\lesssim} T^\theta \| (X+Y) \odot \Xi_1 \|_{\mathbb{C}^{-\frac{1}{2}-\varepsilon}}$$

$$\stackrel{\text{prod esti}}{\lesssim} T^\theta \|X+Y\|_{\mathbb{C}^{\frac{1}{2}-2\varepsilon}} \|\Xi_1\|_{\mathbb{C}^{-\frac{1}{2}-\varepsilon}}$$

(10)

$$\begin{aligned} & \Gamma_2(X, Y, u_0, E_1, E_2) \\ &= u_0 + \oint_X (X_T \cdot (A_1 + \dots + A_8)). \end{aligned}$$

By Lem 4 (time localization),

$$\begin{aligned} & \| \oint_X (X_T \cdot (A_1 + \dots + A_8)) \|_{e^{\frac{t}{2} + \alpha \varepsilon}} \\ & \lesssim T^\theta \| A_1 + \dots + A_8 \|_{\overline{e}^{-\frac{1}{2} + (\alpha+1)\varepsilon}}. \end{aligned}$$

- $\| A_1 \|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} \lesssim \| A_1 \|_{e^{-3\varepsilon}} \lesssim \| X + Y \|_{e^{\frac{1}{2}-2\varepsilon}} \| E_1 \|_{e^{\frac{1}{2}-\varepsilon}}$
- $\| A_2 \|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} \lesssim \| A_2 \|_{e^{(\alpha-1)\varepsilon}} \lesssim \| Y \|_{e^{\frac{1}{2}+\alpha\varepsilon}} \| E_1 \|_{e^{-\frac{1}{2}-\varepsilon}}$

need $\alpha > 1$

$$\cdot \|A_3\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} \leq \|\mathcal{I}((x+y) \otimes \mathbb{E}_1)\|_{e^\varepsilon} \|X \otimes \mathbb{E}_1\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} \quad (11)$$

$$\lesssim \underbrace{\|(x+y) \otimes \mathbb{E}_1\|_{e^{-1+\varepsilon}}}_{\left(\lesssim \|x+y\|_{e^{\frac{1}{2}-2\varepsilon}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}} \right)} \|X\|_{e^{100}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}}$$

$$\left(\lesssim \|x+y\|_{e^{\frac{1}{2}-2\varepsilon}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}} \right)$$

$$\lesssim \|X\|_{e^{100}} \|x+y\|_{e^{\frac{1}{2}-2\varepsilon}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}}^2.$$

$$\cdot \|A_4\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} = \|(X \cdot \underbrace{C_I(x+y, \mathbb{E}_1)}_{1-3\varepsilon}) \otimes \mathbb{E}_1\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}}$$

$$\lesssim \|X \cdot C_I(x+y, \mathbb{E}_1)\|_{e^{1-3\varepsilon}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}}$$

$$\lesssim \|X\|_{e^{100}} \|x+y\|_{e^{\frac{1}{2}-2\varepsilon}} \|\mathbb{E}_1\|_{e^{-\frac{1}{2}-\varepsilon}}^2$$

$$\|A_5\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} = \|\underbrace{[X \odot ((X+Y) \odot I(B_1))] \ominus B_1}_{\frac{1}{2}-\varepsilon}\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} \quad (12)$$

$$\lesssim \|X \odot ((X+Y) \odot I(B_1))\|_{e^{100}} \|B_1\|_{e^{-\frac{1}{2}-\varepsilon}}$$

$$\lesssim \|X\|_{e^{100}} \|X+Y\|_{e^{\frac{1}{2}-2\varepsilon}} \|I(B_1)\|_{e^{\frac{1}{2}-\varepsilon}} \|B_1\|_{e^{-\frac{1}{2}-\varepsilon}}$$

$$\lesssim \|X\|_{e^{100}} \|X+Y\|_{e^{\frac{1}{2}-2\varepsilon}} \|B_1\|_{e^{\frac{-1-\varepsilon}{2}}}^2$$

$$\cdot \|A_6\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} = \|\underbrace{\text{com}_1(X, (X+Y) \odot I(B_1), B_1)}_{1-3\varepsilon}\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}}$$

$$\lesssim \|X\|_{e^{100}} \|X+Y\|_{e^{\frac{1}{2}-2\varepsilon}} \|B_1\|_{e^{\frac{-1-\varepsilon}{2}}}^2$$

$$\cdot \|A_7\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}} = \|X \cdot \underbrace{\text{com}_1(X+Y, I(B_1), B_1)}_{\frac{1}{2}-4\varepsilon}\|_{e^{-\frac{1}{2} + (\alpha+1)\varepsilon}}$$

$$\lesssim \|X\|_{e^{100}} \|X+Y\|_{e^{\frac{1}{2}-2\varepsilon}} \|B_1\|_{e^{\frac{-1-\varepsilon}{2}}}^2 \quad \downarrow \frac{1}{2}-4\varepsilon.$$

(13)

$$\begin{aligned}
 & \|A_8\|_{e^{-\frac{1}{2}+(a+1)\varepsilon}} = \|\underbrace{X \cdot (X+Y)}_{e^{\frac{1}{2}-2\varepsilon}} \cdot \underbrace{B_2}_{e^{-\varepsilon}}\|_{e^{-\frac{1}{2}+(a+1)\varepsilon}} \\
 & \lesssim \|X \cdot (X+Y)\|_{e^{\frac{1}{2}-2\varepsilon}} \|B_2\|_{e^{-\varepsilon}} \\
 & \lesssim \|X\|_{e^{100}} \|X+Y\|_{e^{\frac{1}{2}-2\varepsilon}} \|B_2\|_{e^{-\varepsilon}}.
 \end{aligned}$$

Hence, we obtained

$$\begin{aligned}
 & \|\Gamma(X, Y, u_0, B_1, B_2)\|_{e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+a\varepsilon}} \\
 & \leq |u_0| + C T^\theta \left(1 + \|B_1\|_{e^{\frac{1}{2}-\varepsilon}}^2 + \|B_2\|_{e^{-\varepsilon}} \right) \\
 & \quad \times \|(X, Y)\|_{e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+a\varepsilon}}
 \end{aligned}$$

$a > 1$
 and
 $\frac{1}{2} + a\varepsilon \leq 1 - 2\varepsilon$

Let $R = 2|u_0|$ and $B_R \subset e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+a\varepsilon}$
 \sqsubset closed ball of radius R .

By choosing $T = T(\|\Xi_1\|_{e^{-\frac{1}{2}-\varepsilon}}, \|\Xi_2\|_{e^{-\varepsilon}}) > 0$ (14)

suff. small, we have

↑ indep of R (and α_0)

$$\|\Gamma(x, Y, u_0, \Xi_1, \Xi_2)\|_{e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+\alpha\varepsilon}} \text{ in the linear case}$$

$$\leq |u_0| + \underbrace{\frac{1}{2}R}_{= \frac{1}{2}R} = R \quad \forall (x, Y) \in B_R$$

$$\Rightarrow \Gamma : B_R \hookrightarrow$$

$$\text{Also, } \|\Gamma(x_1, Y_1) - \Gamma(x_2, Y_2)\|_{e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+\alpha\varepsilon}}$$

$$\leq C' T^\theta \left(1 + \|\Xi_1\|_{e^{-\frac{1}{2}-\varepsilon}}^2 + \|\Xi_2\|_{e^{-\varepsilon}} \right) \leq \frac{1}{2}$$

$$x \|\Gamma(x_1, Y_1) - \Gamma(x_2, Y_2)\|_{e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+\alpha\varepsilon}}$$

$\Rightarrow \Gamma$ is a contraction on $B_R \subset e^{\frac{1}{2}-2\varepsilon} \times e^{\frac{1}{2}+\alpha\varepsilon}$. (15)

\rightarrow local well-posedness of the system.

deterministic. soln maps depends conti on $\Xi = (u_0, E_1, E_2)$

\Rightarrow pathwise local well-posedness $\begin{cases} \partial_t u = u \\ u|_{t=0} = u_0 \end{cases}$

① A similar argument with paralinearization yields

pathwise LWP of $\begin{cases} \partial_t u = F(u) \\ u|_{t=0} = u_0 \end{cases} \quad F \in C_b^3$

② For the LWP argument, we only need to choose

$T > 0$ depending on $\|\Xi_1\| e^{-\frac{1}{2}-\varepsilon}$ and $\|\Xi_2\| e^{-\varepsilon}$.

In particular, independent of u_0 (and R)

\Rightarrow global well-posedness (in the linear case) ⑯

$$\cdot (X(0), Y(0)) = (0, u_0) \xrightarrow{LWP} (X(\tau), Y(\tau))$$

$$\cdot (X^1(\tau), Y^1(\tau)) = (0, X(\tau) + Y(\tau)) \xrightarrow{LWP} (X^1(2\tau), Y^1(2\tau))$$

$$\begin{aligned} \cdot (X^j(\tau), Y^j(\tau)) &= (0, X^{j-1}(j\tau) + Y^{j-1}(j\tau)) \\ &\mapsto (X^j((j+1)\tau), Y^j((j+1)\tau)), \dots \end{aligned}$$

Lec 4 : July 11, 2023 (TUE)

Last time

Lemma 3: (i) $\|If\|_{e^{s+1}} \lesssim \|f\|_{e^s}$, $\forall s \in \mathbb{R}$

(ii) $s > -1$. $\|J_X(f)\|_{e^{s+1}} \lesssim \|f\|_{e^s}$.

Pf: (i). It suffices to prove

$$(*) \quad \|\Delta_j(I^f)\|_{L^\infty} \lesssim 2^{-j} \|\Delta_j f\|_{L^\infty}$$

Idea: $\frac{\langle n \rangle}{\langle m \rangle} \downarrow_{n \neq 0} \Leftarrow$ Mihlin multiplier
 but Mihlin multiplier thm holds
 on L^p , $1 < \underline{p} < \infty$

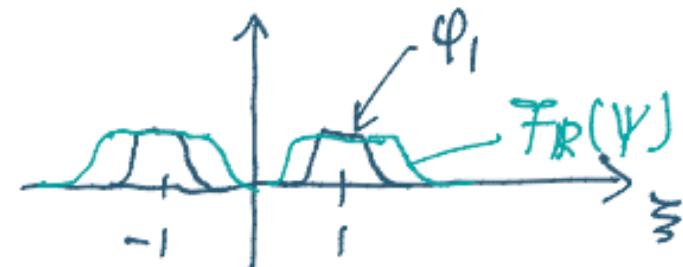
$$\begin{aligned}
 \|If\|_{C^{s+1-\varepsilon}} &\stackrel{\text{Sobolev}}{=} \|If\|_{W^{s+1-\frac{\varepsilon}{2}, p}} \quad p = p(\varepsilon) \gg \\
 &\stackrel{\text{Minkin}}{\lesssim} \|f\|_{W^{s-\frac{\varepsilon}{2}, p}} \\
 &\lesssim \|f\|_{W^{s-\frac{\varepsilon}{2}, \infty}} \\
 &\lesssim \|f\|_{C^s}
 \end{aligned} \tag{2}$$

- Let $\psi \in \mathcal{F}(\mathbb{R})$ s.t. $\mathcal{F}_{\mathbb{R}}(\psi)(0) = 0$ and

$$\mathcal{F}_{\mathbb{R}}(\psi)(z) \cdot \varphi_1(z) = \psi(z), \quad \forall z \in \mathbb{R}$$

- For $j \in \mathbb{N}$, set

$$\mathcal{F}_{\mathbb{R}}(\psi_j)(z) = \mathcal{F}_{\mathbb{R}}(\psi)(2^{-j+1}z)$$



(3)

Then, Ψ_j = periodization of ψ_j

$$\text{func on } \mathbb{T} = \sum_{n \in \mathbb{Z}} \Psi_j(x + 2\pi n).$$

Poisson

$$\widehat{\Psi}_j(n) = \mathcal{F}_{\mathbb{R}}(\Psi_j)(n)$$

$$\Rightarrow \widehat{\Psi}_f(n) \cdot \Psi_j(n) = \psi_j(n), \quad \forall j \in \mathbb{N}, n \in \mathbb{Z}.$$

Let $n \neq 0$.

$$\begin{aligned} \mathcal{F}_{\mathbb{T}}(\Delta_j(I_f))(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{\Psi}_j(n) \widehat{f}(n) d\theta \\ &= -i 2^{-j+1} \frac{2^{j-1}}{n} \mathcal{F}_{\mathbb{R}}(\psi_j)\left(\frac{n}{2^{j-1}}\right) \widehat{\psi}_j(n) \widehat{f}(n) \end{aligned}$$

Define Ψ on \mathbb{R}

$$\text{by } \mathcal{F}_{\mathbb{R}}(\Psi)(z) = \sum_{j=1}^{\infty} \mathcal{F}_{\mathbb{R}}(\psi_j)(z) \Rightarrow \Psi \in \mathcal{J}(\mathbb{R})$$

Recall $\mathcal{F}_{\mathbb{R}}(\psi_j)(0) = 0$.

Set $\tilde{\Psi}_j$ by $\rightarrow \tilde{\Psi}_j(x) = 2^{j-1} \tilde{\Psi}(2^{j-1}x)$ ④

$$\begin{aligned}\mathcal{F}_{\mathbb{R}}(\tilde{\Psi}_j)(\xi) &= \mathcal{F}_{\mathbb{R}}(\tilde{\Psi})\left(\frac{\xi}{2^{j-1}}\right) \\ &= \frac{2^{j-1}}{\xi} \mathcal{F}_{\mathbb{R}}(\tilde{\Psi})\left(\frac{\xi}{2^{j-1}}\right)\end{aligned}$$

and $\tilde{\Psi}_j$ = periodization of $\tilde{\Psi}_j = \sum_{n \in \mathbb{Z}} \tilde{\Psi}_j(x + 2\pi n)$

$$\Rightarrow \hat{\tilde{\Psi}}_j(n) = \mathcal{F}_{\mathbb{R}}(\tilde{\Psi}_j)[n].$$

\Rightarrow so, $\boxed{\Delta_j(I f) = -i 2^{-j+1} \tilde{\Psi}_j * (\Delta_I f)}$

$$\begin{aligned}\Rightarrow \|\Delta_j(I f)\|_{L^\infty(\mathbb{T})} &\stackrel{\text{Young}}{\lesssim} 2^{-j} \|\tilde{\Psi}_j\|_{L^1(\mathbb{T})} \|\Delta_I f\|_{L^\infty(\mathbb{T})} \\ &\leq \|\tilde{\Psi}_j\|_{L^1(\mathbb{R})} \\ &\text{ch.v. } \|\tilde{\Psi}\|_{L^1(\mathbb{R})} < \infty.\end{aligned}$$

(5)

$\Rightarrow (*) \Rightarrow$ Lemma 3 (i).

$$\text{(ii)} \quad g_X(f)(t) = X(t) If(t) - X(t) If(0) + tX(t) \cdot P_0(f)$$

$$\cdot \|X If\|_{e^{s+1}} \stackrel{\text{prod.}}{\lesssim} \|If\|_{e^{s+1}} \stackrel{(i)}{\lesssim} \|f\|_{e^s}.$$

$$\cdot \|X(t) If(0)\|_{e^{s+1}} \leq \|X\|_{e^{s+1}} \|If\|_{L^\infty}$$

$$\stackrel{(i)}{\lesssim} \|If\|_{e^{s+1}} \quad \underline{\Leftarrow s > -1}$$

$$\stackrel{(i)}{\lesssim} \|f\|_{e^s}.$$

$$\cdot \underbrace{\|tX(t) P_0(f)\|_{e^{s+1}}}_{C^\infty} = \|tX(t)\|_{e^{s+1}} |P_0(f)|$$

$$\lesssim \|f\|_{e^s}.$$



Rmk: In the proof of Lemma 3(i), if we replace

- the L^∞ -norm by the L^p -norm, $1 \leq p \leq \infty$.

- \mathcal{I} by $|\nabla|^s$

$$\Rightarrow \Delta_j(M^s f) = -i 2^{s(j-1)} (\tilde{\Phi}_j * \Delta_i f)$$

$$\stackrel{\text{Young}}{\Rightarrow} \|\nabla^s \Delta_i f\|_{L^p} \lesssim 2^{sj} \|\Delta_i f\|_{L^p} \quad (\text{Bernstein's ineq.})$$

$$+ s \in \mathbb{R}$$

Lemma 4: (i) $-1 < s_1 \leq s_2 \leq 0$. Then, $\forall \varepsilon > 0 \exists C > 0$ s.t.

see P. 17

$$\|\chi_T f\|_{e^{s_1}} \leq C \underbrace{T^{s_2 - s_1 - \varepsilon}}_{\text{for } 0 < T \leq 1} \|f\|_{e^{s_2}}$$

for $0 < T \leq 1$.

(ii) $0 < s < 1$. $\forall \varepsilon > 0 \exists C > 0$ s.t.

$$\|\varphi_\chi(\chi_T \cdot f)\|_{e^s} \leq C \underbrace{T^{\frac{\varepsilon}{2}}}_{\text{for } 0 < T \leq 1} \|f\|_{e^{s-1+\varepsilon}}$$

(7)

Pf: (ii). $-1 < s+1 < 0$

$$\| \mathcal{F}_X(\chi_T \cdot f) \|_{\ell^s} \stackrel{\text{Lem 3 (ii)}}{\lesssim} \| \chi_T f \|_{\ell^{s-1}}$$

$$\stackrel{\text{Lem 4 (i)}}{\lesssim} T^{2/2} \| f \|_{\ell^{s-1+\varepsilon}}$$

(i) Suppose that we have

(**) $\boxed{\| \chi_T f \|_{W^{s_1, p}} \lesssim T^{s_2 - s_1} \| f \|_{W^{s_2, p}}}$

for $-1 < s_1 \leq s_2 \leq 0$ andany suff. large (but finite) $p \gg 1$.

Then, $\| \chi_T f \|_{\ell^{s_1}} \lesssim \| \chi_T f \|_{W^{s_1, \infty}} \stackrel{\text{Sobolev}}{\lesssim} \| \chi_T f \|_{W^{s_1 + \varepsilon, p}} \quad \varepsilon p > 1.$

$$\stackrel{(**)}{\lesssim} T^{\frac{s_2 - s_1 - 2\varepsilon}{2}} \| \chi_T f \|_{W^{s_2 - \varepsilon, \infty}} \lesssim T^{\frac{s_2 - s_1 - 2\varepsilon}{2}} \| f \|_{\ell^{s_2}}$$

⑧

- By duality, (***) \Leftrightarrow

$$\|\chi_T f\|_{W^{\sigma_1, g}} \lesssim T^{\sigma_2 - \sigma_1} \|f\|_{W^{\sigma_2, g}}$$

for $0 \leq \sigma_1 \leq \sigma_2 < 1$ and

$g > 1$ suff. close to 1.

- Case 1: $\sigma_1 = 0$. We separately estimate the contributions from (a) $\langle n \rangle \gtrsim T^{-1}$ and (b) $\langle n \rangle \ll T^{-1}$.

$\underline{T}_T =$ smooth freq projection onto $\{n \in \mathbb{Z} : \langle n \rangle \gtrsim T^{-1}\}$
 $\underline{=}$ (multiplier $\underline{1 - \phi(T\langle n \rangle)}$).

• By Mihlin multiplier theorem (with $(1 - \Phi(T\langle m \rangle)) (T\langle m \rangle)^{-\sigma_2}$) ⑨

$$\| X_T \pi_T(f) \|_{W^{0,q}} \leq \| \pi_T(f) \|_{L^q} \lesssim T^{\sigma_2} \| f \|_{W^{\sigma_2,q}}$$

L^q

$$(\langle m \rangle \gtrsim T^{-1} \Rightarrow 1 \lesssim T^{\sigma_2} \langle m \rangle^{\sigma_2} \quad \forall \alpha \in (\mathbb{Z}_{\geq 0})^d)$$

Mihlin multiplier theorem: $m(\zeta)$ satisfying $|2^\alpha m(\zeta)| \leq \frac{C_\alpha}{|\zeta|^{1+\alpha}}$

$$\Rightarrow \| T_m f \|_{L^q(\mathbb{R}^d)} \lesssim \| f \|_{L^q(\mathbb{R}^d)} \quad 1 < q < \infty$$

Transference.

$$\stackrel{\Rightarrow}{\text{Gra vol. 1 ch. 4}} \| T_m f \|_{L^q(\mathbb{T}^d)} \lesssim \| f \|_{L^q(\mathbb{T}^d)}$$

$$\widehat{T_m f}(\vec{\zeta}) = m(\vec{\zeta}) \widehat{f}(\vec{\zeta})$$

$$\cdot \quad \|(\text{Id} - \Pi_T) f\|_{L^\infty} \stackrel{H-Y}{\lesssim} \sum_{\substack{n \in \mathbb{Z} \\ |n| \ll T^{-1}}} |\hat{f}(n)| \langle n \rangle^{\sigma_2} \cdot \langle n \rangle^{-\sigma_2}$$

q close to 1
 $\Rightarrow q < 2 < q'$

Hölder

$$\lesssim \|1_{|n| \ll T^{-1}} \langle n \rangle^{\sigma_2}\|_{l^q} \|\langle n \rangle^{\sigma_2} \hat{f}(n)\|_{l^{q'}}$$

$$\stackrel{H-Y}{\lesssim} \underline{T^{\sigma_2 - \frac{1}{q}}} \|f\|_{W^{\sigma_2, q}}$$

$$\left(\sum_{|n| \ll T^{-1}} \langle n \rangle^{-\sigma_2 q} \right) \lesssim T^{-1(-\sigma_2 q + 1)} \quad \text{for } \underline{\sigma_2 q < 1}.$$

$$\Rightarrow \|\chi_T (\text{Id} - \Pi_T) f\|_{L^q}$$

$$\leq \|\chi_T\|_{L^q} \|(\text{Id} - \Pi_T) f\|_{L^\infty}$$

$$\lesssim T^{\frac{1}{q}} \cdot T^{\sigma_2 - \frac{1}{q}} \|f\|_{W^{\sigma_2, q}} \Rightarrow \underline{\text{Case 1}}.$$

$\sigma_2 < 1$ so can choose
 $q > 1$ close to 1
 s.t. $\sigma_2 q < 1$

(10)

$$m(\xi) = \underbrace{(1 - \phi(T\langle \xi \rangle))}_{=1 \text{ on } \langle \xi \rangle \gtrsim T^{-1}} \underbrace{(T\langle \xi \rangle)^{-\sigma_2}}_{\leq 1} \quad \sigma_2 \geq 0. \quad (11)$$

- $|m(\xi)| \lesssim 1$
- 2nd factor $T^{-\sigma_2} \langle \xi \rangle^{-\sigma_2}$ $\partial^k \langle \xi \rangle^{-\sigma_2} \sim \langle \xi \rangle^{-\sigma_2-k}$
 $\Rightarrow |\partial^k (T^{-\sigma_2} \langle \xi \rangle^{-\sigma_2})| \lesssim \frac{1}{\langle \xi \rangle^k}$.

- 1st factor $\partial^k (1 - \phi(T\langle \xi \rangle)) = \partial^k \phi(T\langle \xi \rangle)$
 Note that $\text{supp}(\partial^k \phi(T\langle \xi \rangle)) = \{\langle \xi \rangle \sim T^{-1}\}$.

- $\partial \phi(T\langle \xi \rangle) = \phi'(T\langle \xi \rangle) \cdot \frac{T}{\langle \xi \rangle} \cdot \xi$
 $\Rightarrow |\partial \phi(T\langle \xi \rangle)| \lesssim T \sim \frac{1}{\langle \xi \rangle}$

$$\Rightarrow |\partial^k \phi(T\langle \xi \rangle)| \lesssim \frac{1}{\langle \xi \rangle^k}$$

$$\Rightarrow |\partial^k m(\xi)| \lesssim \frac{1}{\langle \xi \rangle^k}$$

- Case 2: $0 < \sigma_1 = \sigma_2 < 1$.

$\Pi_T =$ smooth freq proj onto $\{\langle n \rangle \gtrsim T^{-1}\}$.

- By Mihlin & transference principle,

$$\|(\text{Id} - \Pi_T)(\chi_T f)\|_{W^{\sigma_2, q}} \lesssim T^{-\sigma_2} \|\gamma_T f\|_{W^{0, q}}$$

case 1

$$\begin{aligned} m(n) &= \phi(T\langle n \rangle) (T\langle n \rangle)^{\sigma_2} \\ &\quad \text{on } \{\langle n \rangle \ll T^{-1}\}. \end{aligned}$$

- Write

$$\begin{aligned} \Pi_T(\chi_T f) &= \underbrace{\Pi_T(\chi_T \Pi_T f)}_{=: \text{I}} + \underbrace{\Pi_T(\chi_T (\text{Id} - \Pi_T) f)}_{=: \text{II}} \end{aligned}$$

(13)

$$\| \mathbb{I} \|_{W^{\sigma_2, q}} \lesssim_{\text{Mihlin}} T^{2-\sigma_2} \| \partial_t^2 (\chi_T (\text{Id} - \Pi_T) f) \|_{W^{0, q}}$$

really $1 - \delta_t^2$

$$m(n) = (1 - \phi(\tau \langle n \rangle)) (\tau \langle n \rangle)^{\sigma_2} (Tn)^{-2}$$

$\sim (\tau \langle n \rangle)^{\sigma_2 - 2}$
 $\tau \langle n \rangle > 1$

$$\textcircled{1} \quad T^{2-\sigma_2} \| \partial_t^2 \chi_T (\text{Id} - \Pi_T) f \|_{W^{0, q}}$$

$$\lesssim T^{\sigma_2} \| \chi_{2T} (\text{Id} - \Pi_T) f \|_{W^{0, q}}$$

$$\stackrel{\text{case 1}}{\lesssim} T^{-\sigma_2} \cdot T^{\sigma_2} \| (\text{Id} - \Pi_T) f \|_{W^{\sigma_2, q}}$$

$$\stackrel{\text{Mihlin}}{\lesssim} \| f \|_{W^{\sigma_2, q}}$$

$$m(n) = \phi(\tau \langle n \rangle)$$

(14)

$$\textcircled{2} T^{2-\sigma_2} \left\| \underbrace{\partial_t \chi_T}_{L^\infty} \cdot \partial_t ((Id - \Pi_T) f) \right\|_{W^{0,q}}$$

$$\lesssim T^{1-\sigma_2} \left\| \partial_t ((Id - \Pi_T) f) \right\|_{W^{0,q}} \quad \sim \cancel{T^{(m)}}^{1-\sigma_2} > 0$$

Mihlin

$$\lesssim \|f\|_{W^{\sigma_2,q}}$$

$$m(m) = \phi(T\langle n \rangle) T^{1-\sigma_2} n \langle m \rangle^{-\sigma_2}$$

$$\langle m \rangle \ll T^{-1}.$$

$$\textcircled{3} T^{2-\sigma_2} \left\| \cancel{\chi_T} \cdot \partial_t^2 ((Id - \Pi_T) f) \right\|_{W^{0,q}}$$

$$\stackrel{\text{Mihlin}}{\lesssim} \|f\|_{W^{\sigma_2,q}} \quad m(n) = \phi(T\langle n \rangle) T^{2-\sigma_2} n^2 \langle n \rangle^{-\sigma_2}$$

$$\Rightarrow \|\mathbb{I}\|_{W^{\sigma_2,q}} \lesssim \|f\|_{W^{\sigma_2,q}}$$

Next, we consider $\Pi_T (\underline{\chi_T} \cdot \Pi_T f)$.

$$(\ast\ast\ast) \quad \text{Claim: } \|\Pi_T (e^{inot} \Pi_T f)\|_{W^{\sigma_2, q}} \lesssim \langle T n_0 \rangle^{\sigma_2} \|f\|_{W^{\sigma_2, q}}$$

for any $0 \leq \sigma_2 \leq 2$.

When $\sigma_2 = 0$, $(\ast\ast\ast)$ follows from Mihlin twice.

When $\sigma_2 = 2$,

$$\|\Pi_T (e^{inot} \Pi_T f)\|_{W^{2, q}} \stackrel{\text{Mihlin.}}{\lesssim} \|f\|_{W^{0, q}} + \|\partial_t^2 (e^{inot} \Pi_T f)\|_{W^{0, q}}$$

$$\|\partial_t^2 e^{inot} \cdot \Pi_T f\|_{W^{0, q}} \lesssim n_0^2 \|\Pi_T f\|_{W^{0, q}}$$

$$\stackrel{\text{Mihlin}}{\lesssim} \langle T n_0 \rangle^2 \|f\|_{W^{2, q}} \quad \langle n \rangle \gtrsim T^{-1}$$

$$\|\partial_t e^{inot} \cdot \partial_t \Pi_T f\|_{W^{0, q}} \lesssim \langle T n_0 \rangle \|f\|_{W^{2, q}}$$

$$\|\cancel{e^{inot} \cdot \partial_t^2 \Pi_T f}\|_{W^{0, q}} \lesssim \|f\|_{W^{2, q}}$$

$\sigma_2 = 0 \wedge \sigma_2 = 2 \implies$ interpolation (***) for $0 \leq \sigma_2 \leq 2.$ (1b)

$$\| \Pi_T (\chi_T \Pi_T f) \|_{W^{\sigma_2, q}} \stackrel{\text{claim}}{\leq} \sum_{n_0 \in \mathbb{Z}} T |F_R(\chi)(Tn_0)| \langle Tn_0 \rangle^{\sigma_2} \|f\|_{W^{\sigma_2, q}}$$

Write in Fourier Series

$$= \sum_{n_0 \in \mathbb{Z}} T F_R(\chi)(Tn_0) e^{int}$$

$\uparrow \sim \int_{\mathbb{R}} \langle z \rangle^{\sigma_2} |F_R(\chi)(z)| dz < \infty$

Riemann Sum approx.

So, we prove

$$\| \chi_T f \|_{W^{\sigma_1, q}} \lesssim T^{\sigma_2 - \sigma_1} \|f\|_{W^{\sigma_2, q}}$$

$$\left. \begin{array}{l} \textcircled{1} \quad \sigma_1 = 0 \leq \sigma_2 < 1 \\ \textcircled{2} \quad 0 \leq \sigma_1 = \sigma_2 < 1 \end{array} \right\} \begin{array}{l} \text{interpolation} \\ \implies \text{for any } 0 \leq \sigma_1 \leq \sigma_2 < 1 \end{array}$$



Appendix : Lemma 4': $-1 < s_1 \leq s_2 \leq 0$. Then,

$$\|\chi_T f\|_{C^{s_1}} \lesssim \underbrace{T^{s_2 - s_1}}_{\text{NO } \varepsilon\text{-loss}} \|f\|_{C^{s_2}}$$

for $0 < T \leq 1$

$$(\text{LHS}) \sim \sup_j \|\Delta_j (\chi_T \cdot f)\|_{W^{s_1, \infty}}$$

duality

$$= \sup_{\|G\|_{W^{-s_1, 1}} = 1} \int_{\mathbb{T}} G \cdot \Delta_j (\chi_T f) dt$$

$$\hookrightarrow = \sum_k \int (\chi_T \cdot \Delta_j G) \cdot \Delta_k f dt \quad \Delta_j = \text{self adjoint}$$

$$\stackrel{\circlearrowleft}{=} \int \sum_k \tilde{\Delta}_k (\chi_T \cdot \Delta_j G) \cdot \Delta_k f dt$$

$$\left(\sum_k \tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1} \Rightarrow \tilde{\Delta}_k \Delta_k = \Delta_k \right)$$

$$\begin{aligned}
 &= \int \sum_k \tilde{\Delta}_k \langle \nabla \rangle^{-s_2} (\chi_T \cdot \Delta_j G) \cdot \langle \nabla \rangle^{s_2} \Delta_k f \, dt \\
 &\leq \underbrace{\|\chi_T \cdot \Delta_j G\|_{W^{-s_2, 1}} \|\sup_k \langle \nabla \rangle^{s_2} \Delta_k f\|_{L^\infty}}_{= \sup_k \|\Delta_k f\|_{W^{s_2, \infty}} \sim \|f\|_{\mathcal{C}^{s_2}}} \\
 &\quad = \sup_k \|\Delta_k f\|_{W^{s_2, \infty}} \sim \|f\|_{\mathcal{C}^{s_2}}
 \end{aligned} \tag{18}$$

• Hence, it suffices to show (with $\sigma_1 = -s_2$, $\sigma_2 = -s_1$)

$$\|\chi_T \cdot \Delta_j G\|_{W^{\sigma_1, 1}} \lesssim T^{\sigma_2 - \sigma_1} \underbrace{\|\Delta_j G\|_{W^{\sigma_2, 1}}}_{\lesssim \|G\|_{W^{-s_1, 1}} = 1}$$

for $0 \leq \sigma_1 \leq \sigma_2 < 1$

• Write $\chi_T \Delta_j G = \chi_T \otimes \Delta_j G + \chi_T \ominus \Delta_j G + \chi_T \oslash \Delta_j G$.

(19)

$$\textcircled{1} \quad \chi_T \ominus \Delta_j G = \sum_{|m-j| \leq 2} \Delta_m \chi_T \Delta_j G$$

$$\Rightarrow \|\chi_T \ominus \Delta_j G\|_{W^{\sigma_1, 1}} \lesssim \sum_k 2^{\sigma_1 k} \|\Delta_k (\chi_T \ominus \Delta_j G)\|_{L^1}$$

$$\leq \sum_k 2^{\sigma_1 k} \mathbf{1}_{k \leq j+10} \underbrace{2^{-\sigma_2 j}}_{|m-j| \leq 2} \sum_{|m-j| \leq 2} \|\Delta_m \chi_T\|_{L^\infty} \cdot \underbrace{2^{\sigma_2 j}}_{\textcolor{red}{=}} \|\Delta_j G\|_{L^1}$$

$$\left(\lesssim \sum_{k \leq j+10} 2^{\sigma_1(k-j)} \cdot 2^{(\sigma_1 - \sigma_2)j} \lesssim 2^{(\sigma_1 - \sigma_2)m} \right)$$

$$\lesssim \sup_m 2^{\overbrace{(\sigma_1 - \sigma_2)m}^{\leq 0}} \|\Delta_m \chi_T\|_{L^\infty} \|\Delta_j G\|_{W^{\sigma_2, 1}}.$$

$$\stackrel{\text{WTS}}{\lesssim} T^{\sigma_2 - \sigma_1}$$

\Leftarrow ④ & ⑤ on the next page.

a) If $2^{-m} \lesssim T$, then

$$2^{(\sigma_1 - \sigma_2)m} \|\Delta_m \chi_T\|_{L^\infty} \lesssim T^{\sigma_2 - \sigma_1} \|\chi_T\|_{L^\infty} = T^{\sigma_2 - \sigma_1}$$

b) If $2^{-m} \gg T$, then

$$\begin{aligned} & 2^{(\sigma_1 - \sigma_2)m} \Delta_m \chi_T(t) = 2^{(\sigma_1 - \sigma_2)m} \int_{\mathbb{R}} \chi\left(\frac{t-t_1}{T}\right) dt_1 \\ & \lesssim 2^{\underbrace{(-\sigma_2 + \sigma_1)m}_{>0}} \int_{\mathbb{R}} \chi\left(\frac{t_1}{T}\right) dt_1 \\ & \ll T^{-1+\sigma_2-\sigma_1} \cdot T \\ & = T^{\sigma_2 - \sigma_1} \end{aligned}$$

$\sum_{n \in \mathbb{Z}} K_m(t - t_1 + 2\pi n)$
 $\int_{\mathbb{R}} K_m^{\text{per}}(t - t_1) \chi\left(\frac{t_1}{T}\right) dt_1$
 $\leq 2^m \cdot \sum_n K(2^m(t - t_1 + 2\pi n))$
 $\lesssim 2^m \cdot \frac{1}{\langle 2^m(t - t_1) + 2^m \cdot 2\pi n \rangle^{100}}$
 $\lesssim 2^m$

$$\Rightarrow \text{Hence, } \|\chi_T \otimes \Delta_j G\|_{W^{\sigma_1, 1}} \lesssim T^{\sigma_2 - \sigma_1} \|\Delta_j G\|_{W^{\sigma_2, 1}}$$

(21)

$$\begin{aligned}
 & \|X_T \odot \Delta_j G\|_{W^{\sigma_1, 1}} \leq \sum_k 2^{\sigma_1 k} \|\Delta_k (X_T \odot \Delta_j G)\|_1 \\
 & \quad S_{j-2}(f) = \sum_{k < j-2} \Delta_k f \\
 & \lesssim \sum_k \underbrace{1_{|k-j| \leq 5}}_{\sim 2^{(\sigma_1 - \sigma_2)j}} 2^{\sigma_1 k} 2^{-\sigma_2 j} \|S_{j-2}(X_T)\|_{L^\infty} 2^{\sigma_2 j} \|\Delta_j G\|_1 \\
 & \quad \sim \|\Delta_j G\|_{W^{\sigma_2, 1}}
 \end{aligned}$$

$$\text{WTS: } 2^{(\sigma_1 - \sigma_2)j} \|S_{j-2}(X_T)\|_{L^\infty} \lesssim T^{\sigma_2 - \sigma_1}$$

\Leftarrow just repeat the argument ④ & ⑤ on p(20)

$$\left\{
 \begin{array}{l}
 \text{In ⑤, replace } K_m^{\text{per}} \text{ by } K_{<j-2}^{\text{per}}(t) = \sum_{n \in \mathbb{Z}} K_{<j-2}(t + 2\pi n) \\
 \Rightarrow |K_{<j-2}^{\text{per}}(t)| \lesssim 2^{j-2} \sum_{n \in \mathbb{Z}} \frac{1}{\langle 2^{j-2}t + 2^{j-2} \cdot 2\pi n \rangle^{100}} \\
 \sim 2^j
 \end{array}
 \right.$$

- It remains to show

freq supp $\sim 2^m$

(22)

$$\|\chi_T \odot \Delta_j G\|_{W^{\sigma_1,1}} = \left\| \sum_{m > j+2} \Delta_m \chi_T \Delta_j G \right\|_{W^{\sigma_1,1}} \lesssim T^{\sigma_2 - \sigma_1} \|\Delta_j G\|_{W^{\sigma_2,1}}$$

$$\cdot (\text{LHS}) \lesssim \sum_{m > j+2} 2^{\sigma_1 m} \|\Delta_m \chi_T \cdot \Delta_j G\|_{L^1} \quad \sigma_2 + \varepsilon = \frac{1}{1} - \frac{1 - \sigma_2 - \varepsilon}{1}$$

$$\textcircled{1} \text{ Bernstein's ineq, } \|\Delta_j G\|_{L^{\frac{1}{1-\sigma_2-\varepsilon}}} \lesssim \|\Delta_j G\|_{W^{\sigma_2+\varepsilon,1}} \sim 2^{\varepsilon j} \|\Delta_j G\|_{W^{\sigma_2,1}}$$

$$\textcircled{2} \quad \underbrace{\sum_{m > j+2} 2^{-\varepsilon m}}_{\sim 2^{-\varepsilon j}} 2^{\varepsilon m} 2^{\sigma_1 m} \|\Delta_m \chi_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}} \lesssim \|\nabla^{\sigma_1 + \varepsilon} \chi_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}}$$

$$\left. \begin{array}{l} \|\chi_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}} \sim T^{\sigma_2 + \varepsilon} \\ \|\partial_t^2 \chi_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}} \sim \sigma_2 + \varepsilon - 2 \end{array} \right\} \text{interpolate} \Rightarrow \|\nabla^{\sigma_1 + \varepsilon} \chi_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}} \sim T^{\sigma_2 + \varepsilon - (\sigma_1 + \varepsilon)} \sim T^{\sigma_2 - \sigma_1}$$

(23)

Putting together, we have

$$\begin{aligned}
 (\text{LHS}) &\lesssim \sum_{m>j+2} 2^{\sigma_1} \|\Delta_m X_T\|_{L^{\frac{1}{\sigma_2+\varepsilon}}} \cdot \|\Delta_j G\|_{L^{\frac{1}{1-\sigma_2-\varepsilon}}} \\
 &\lesssim \left(T^{\sigma_2-\sigma_1} \cancel{2^{-\varepsilon j}} \right) \cdot \cancel{\left(2^{\varepsilon j} \|\Delta_j G\|_{W^{\sigma_2,1}} \right)} \\
 &\lesssim T^{\sigma_2-\sigma_1} \|\Delta_j G\|_{W^{\sigma_2,1}}
 \end{aligned}$$

□

Part 2: 1-d SNLW with multiplicative space-time white noise

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = F(u) \xi & \text{on } \mathbb{T} \\ (u, \partial_t u) \Big|_{t=0} = (u_0, u_1). \end{cases}$$

↑ space-time white noise.

- SNLW in the null coordinates

$$x_1 = \frac{t+x}{\sqrt{2}}, \quad x_2 = \frac{t-x}{\sqrt{2}}$$

$$\Leftrightarrow \partial_{x_1} \partial_{x_2} v = \frac{1}{2} F(v) \tilde{\xi}$$

↑ two-parameter white noise

- We instead consider

(SNLW_{null}) $\partial_{x_1} \partial_{x_2} v = F(v) \tilde{\xi}$ on $D = [0, 1]^2$

$$\begin{aligned} \text{BC: } v(x_1, 0) &= v_1(x_1), & 0 \leq x_1 \leq 1 \\ v(0, x_2) &= v_2(x_2), & 0 \leq x_2 \leq 1 \end{aligned}$$

compatibility condition

$$v_1(0) = v_2(0)$$

(2)

- Goal: Prove pathwise well-posedness of (SNLW_{null}).

$$\cdot \underline{F(v) = v}: \quad \partial_{x_1} \partial_{x_2} v = v \tilde{s}$$

integrate in x_2

$$\Rightarrow \partial_{x_1} v(x_1, x_2) - \partial_{x_1} v(x_1, 0) = \int_0^{x_2} (v \tilde{s})(x_1, y_2) dy_2$$

integrate in x_1

$$\Rightarrow v(x_1, x_2) = \underbrace{v(x_1, 0)}_{\text{''}} + \underbrace{v(0, x_2)}_{\text{''}} - v(0, 0) \\ + \int_0^{x_1} \int_0^{x_2} (v \tilde{s})(y_1, y_2) dy_2 dy_1$$

$$\cdot \mathbb{T}^2 \cong [-\pi, \pi)^2 \supset D = [0, 1]^2.$$

$\chi(\vec{x}) = \chi(x_1, x_2) =$ smooth cutoff function s.t.

$$\cdot \chi \equiv 1 \text{ on } D$$

$$\cdot \chi \equiv 0 \text{ on } [-\frac{1}{2}, \frac{3}{2}]^C$$

\Rightarrow We consider

$$v(\vec{x}) = \chi(\vec{x}) \bar{v}_0(\vec{x}) + \chi(\vec{x}) \int_0^{x_1} \int_0^{x_2} (v \zeta)(\vec{y}) d\vec{y},$$

where $\bar{v}_0(\vec{x}) = v_1(x_1) + v_2(x_2) - v_1(0)$

formally closed in the space of periodic functions.

- $\zeta \sim (-\frac{1}{2}^-, -\frac{1}{2}^-) \stackrel{\text{expect}}{\Rightarrow} v \sim (\frac{1}{2}^-, \frac{1}{2}^-)$

\Rightarrow The product $v \zeta$ suffers deficiency of regularities
in two independent directions

A new idea is needed:

bi-parameter paracontrolled distributions

- Bi-parameter Besov spaces $B_{p,q}^{s_1,s_2}$ unisotropic (4)

For $f = f(x_1, x_2)$ on $\mathbb{T}_{x_1}^{d_1} \times \mathbb{T}_{x_2}^{d_2}$,

$$\|f\|_{B_{p,q}^{s_1,s_2}} = \left\| 2^{\vec{s} \cdot \vec{j}} \| \Delta_{j_1}^{(1)} \Delta_{j_2}^{(2)} f \|_{L_{x_1,x_2}^p(\mathbb{T}^{d_1+d_2})} \right\|_{\ell_{j_1,j_2}^q(\mathbb{Z}_{\geq 0}^2)}$$

$$\vec{s} = (s_1, s_2), \quad \vec{j} = (j_1, j_2)$$

- bi-parameter Hölder-Besov space $C^{s_1,s_2}(\mathbb{T}^{d_1+d_2}) = B_{\infty,\infty}^{s_1,s_2}(\mathbb{T}^{d_1+d_2})$
- bi-parameter Lipschitz space $\Lambda^{s_1,s_2}(\mathbb{T}^{d_1+d_2}), \quad 0 < s_1, s_2 < 1$
(Hölder-Zygmund)

$$\begin{aligned} \|f\|_{\Lambda^{s_1,s_2}} &= \|f\|_{L^\infty} + \left\| \sup_{h_1 \neq 0} \frac{|\delta_{h_1}^{(1)} f|}{|h_1|^{s_1}} \right\|_{L^\infty} + \left\| \sup_{h_2 \neq 0} \frac{|\delta_{h_2}^{(2)} f|}{|h_2|^{s_2}} \right\|_{L^\infty} \\ &\quad + \left\| \sup_{h_1, h_2 \neq 0} \frac{|\delta_{h_1}^{(1)} \delta_{h_2}^{(2)} f|}{|h_1|^{s_1} |h_2|^{s_2}} \right\|_{L^\infty}. \end{aligned}$$

$$\text{where } \delta_{h_1}^{(1)} f(\vec{x}) = f(x_1 + h_1, x_2) - f(x_1, x_2)$$

(5)

$$\delta_{h_2}^{(2)} f(\vec{x}) = f(x_1, x_2 + h_2) - f(x_1, x_2)$$

$$\Rightarrow \text{Then, } \underline{\mathcal{C}^{s_1, s_2}} = \underline{\Lambda^{s_1, s_2}}, \quad 0 < s_1, s_2 < 1$$

- \Leftrightarrow need this to prove
- bi-parameter paralinearization
 - $F(u) \in \mathcal{C}^{s_1, s_2}$ for $u \in \mathcal{C}^{s_1, s_2}$, $F \in C_b^2$

• bi-parameter paraproducts: f, g on $\mathbb{T}^{d_1} \times \mathbb{T}^{d_2}$.

$$\begin{aligned} fg &= f \circ \otimes g = f \odot \otimes g + f \odot \ominus g + f \odot \oplus g \\ &= \underline{f \otimes \otimes g} + \underline{f \ominus \otimes g} + \underline{f \oplus \otimes g} \\ &\quad + \underline{f \otimes \ominus g} + \underline{f \ominus \ominus g} + \underline{f \oplus \ominus g} \\ &\quad + \underline{f \otimes \oplus g} + \underline{f \ominus \oplus g} + \underline{f \oplus \oplus g} \end{aligned}$$

⑥

$$\cdot f \circledast \circledcirc g = \sum_{\substack{j_1 < k_1 - 2 \\ j_2 < k_2 - 2}} \Delta_{j_1}^{(1)} \Delta_{j_2}^{(2)} f \cdot \Delta_{k_1}^{(1)} \Delta_{k_2}^{(2)} g , \text{ etc.}$$

- The terms involving only paraproducts (such as $f \circledast \circledcirc g$) are always well defined as distributions
- The terms involving resonant products (such as $f \circledast \circledcirc g$) are well defined (in general) only when the sum of the regularities of the corresponding variable is positive.

Aside: "bi-parameter paraproduct"
 by Muscalu - Pipher - Tao - Thiele '04
 ⇒ use it to prove the bi-parameter Coifman-Meyer thm

(7)

Lemma 7: (product estimates)

$$s_1, s_2, t_1, t_2 \in \underline{\mathbb{R} \setminus \{0\}}, \quad 1 \leq p, p_1, p_2, q \leq \infty$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

$$(i) \|f \odot g\|_{B_{p,q}^{s_1 \wedge 0 + t_1, s_2 \wedge 0 + t_2}} \lesssim \|f\|_{B_{p_1,q}^s} \|g\|_{B_{p_2,q}^t}$$

$$a \wedge b = \min(a, b)$$

$$(ii) \|f \odot g\|_{B_{p,q}^{s_1 \wedge 0 + t_1, s_2 + t_2 \wedge 0}} \lesssim \|f\|_{B_{p_1,q}^s} \|g\|_{B_{p_2,q}^t}$$

$$(iii) \underline{s_2 + t_2 > 0}$$

$$\|f \odot g\|_{B_{p,q}^{s_1 \wedge 0 + t_1, s_2 + t_2}} \lesssim \|f\|_{B_{p_1,q}^s} \|g\|_{B_{p_2,q}^t}$$

$$(iv) \underline{s_1 + t_1, s_2 + t_2 > 0}$$

$$\|f \odot g\|_{B_{p,q}^{s_1 + t_1, s_2 + t_2}} \lesssim \|f\|_{B_{p_1,q}^s} \|g\|_{B_{p_2,q}^t}$$

$$\Rightarrow s_1 + t_1, s_2 + t_2 > 0$$

$$\|fg\|_{B_{p,q}^{s_1+t_1, s_2+t_2}} \lesssim \|f\|_{B_{p_1,q}^s} \|g\|_{B_{p_2,q}^t}$$

- The proof is analogous to the one-parameter case.

- Paracontrolled distribution: f is paracontrolled by g if

$$f = f' \odot g + R$$

\uparrow smoother.

- high freq behavior of $f \approx$ high freq behavior of g .

- Define π_1 and π_2 by

$$\Delta \tilde{f} = \Delta_{j_1}^{(1)} \Delta_{j_2}^{(2)}$$

$$\pi_1 f = \sum_{j_1 \geq j_2} \Delta \tilde{f} \Leftrightarrow x_1 - \text{freq} \geq x_2 - \text{freq}.$$

$$\pi_2 = \text{Id} - \pi_1$$

(9)

Lemma 8: $\sigma \geq 0$.

$$\|\pi_1 f\|_{B^{s_1-\sigma, s_2+\sigma}_{p,q}} \lesssim \|f\|_{B^{s_1, s_2}_{p,q}}$$

$$\|\pi_2 f\|_{B^{s_1+\sigma, s_2-\sigma}_{p,q}} \lesssim \|f\|_{B^{s_1, s_2}_{p,q}}$$

for any $s_1, s_2 \in \mathbb{R}$, $1 \leq p, q \leq \infty$

- Let $g \in C^{s,s} \Rightarrow \pi_1 g \sim (s_1 - \sigma, s_2 + \sigma)$
↑ Bad in the x_1 -direction.
- We say that f is bi-paracontrolled by g if

$$f = f_1 \underset{\substack{\text{worst behavior} \\ \text{in } x_1}}{\circlearrowleft} \pi_1 g + f_2 \underset{\substack{\text{worst behavior} \\ \text{in } x_2}}{\circlearrowleft} \pi_2 g + R$$

↑ smoother

- Back to (SNLWnull):

$$\underline{v}(\vec{x}) = V_0(\vec{x}) + \chi(\vec{x}) \int_0^{x_1} \int_0^{x_2} (v \tilde{\zeta})(\vec{y}) d\vec{y}.$$

$$= V_0(\vec{x}) + \underline{\int_X (v \tilde{\zeta})(\vec{x})}$$

white

$$\left(\begin{array}{l} \cdot \quad \underline{\int^{(1)} f}(\vec{x}) = \int_0^{x_1} f(y_1, x_2) dy_1, \quad \tilde{\zeta} = \pi_1 \tilde{\zeta} + \pi_2 \tilde{\zeta}, \\ \cdot \quad \underline{\int^{(2)} f}(\vec{x}) = \int_0^{x_2} f(x_1, y_2) dy_2 \\ \cdot \quad \chi_1(x_1), \chi_2(x_2) \leftarrow \text{smooth cutoff in } x_j \\ \quad \quad \chi = \chi_1 \cdot \chi_2 \\ \cdot \quad \underline{\int_X} = \underline{\int_{X_1}} \circ \underline{\int_{X_2}} = \chi_1 \underline{\int^{(1)}} \circ (\chi_2 \underline{\int^{(2)}}) = \chi \cdot \underline{\int} \end{array} \right)$$

- $v = V_0 + f_X(v \underline{\xi}) \rightarrow \underline{\xi} = \pi_1 \underline{\xi} + \pi_2 \underline{\xi}$

- Bi-paracontrolled ansatz: $v = X + Y$.

$$X = f_X(\underline{v} \odot \circlearrowleft \pi_1 \underline{\xi}) + f_X(\underline{v} \circlearrowright \odot \pi_2 \underline{\xi})$$

$$Y = V_0 + f_X(\underline{v} \circlearrowleft \odot \pi_1 \underline{\xi}) + f_X(\underline{v} \odot \circlearrowright \pi_2 \underline{\xi}).$$

where $\circlearrowright = \odot + \circlearrowleft$.

- Bad news: $\underline{v} \circlearrowleft \odot \pi_1 \underline{\xi}$ contains $Y \circlearrowleft \odot \pi_1 \underline{\xi}$.

$$\cdot \underline{\xi} \sim (-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon)$$

$$\Rightarrow X \sim (\frac{1}{2} - 2\varepsilon, \frac{1}{2} - 2\varepsilon), Y \sim (\frac{1}{2} + a\varepsilon, \frac{1}{2} + a\varepsilon)$$

- Want $Y \circlearrowleft \odot \pi_1 \underline{\xi} \sim (-\frac{1}{2} + (a+1)\varepsilon, -\frac{1}{2} + (a+1)\varepsilon)$
 $(\frac{1}{2} + a\varepsilon, \frac{1}{2} + a\varepsilon) \quad (-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon)$

\circlearrowleft makes sense b/c $(\frac{1}{2} + a\varepsilon) + (-\frac{1}{2} - \varepsilon) > 0$. for $a > 1$.

to get T^θ .

(12)

$\Rightarrow Y \ominus \odot \underline{\pi_1} \underline{\gamma}$ is a well-defined distribution

↑
but $\sim \underline{((\alpha-1)\varepsilon, -\frac{1}{2}-\varepsilon)}$ ← good for the
 X -eqn.

View this as
 $(-\frac{1}{2}-(\alpha+3)\varepsilon, -\frac{1}{2}+(\alpha+1)\varepsilon)$
BAD good

↑
Not good
need $-\frac{1}{2} + (\alpha+1)\varepsilon$.

$$x_1: \left(\frac{1}{2} + \alpha\varepsilon\right) + \left(-\frac{1}{2} - (\alpha+3)\varepsilon\right) = -3\varepsilon < 0.$$

$\Rightarrow Y \ominus \odot \pi_1 \gamma$ makes sense but does NOT have enough regularity to be in the Y -eqn.

→ Modified paracontrolled ansatz: There are ill-defined products.

$$X = \oint_X (\nu \odot \circ \pi_1 \bar{\zeta}) + \oint_X (\nu \circ \odot \pi_2 \bar{\zeta})$$

$$+ \oint_X (Y \ominus \odot \pi_1 \bar{\zeta}) + \oint_X (Y \odot \ominus \pi_2 \bar{\zeta}).$$

Use this
for substitution.

$$Y = V_0 + \oint_X (\nu \triangleright \circ \pi_1 \bar{\zeta}) + \oint_X (\nu \circ \triangleright \pi_2 \bar{\zeta})$$

$$+ \oint_X (Y \ominus \geq \pi_1 \bar{\zeta}) + \oint_X (Y \geq \ominus \pi_2 \bar{\zeta})$$

$$+ \oint_X (X \ominus \circ \pi_1 \bar{\zeta}) + \oint_X (X \circ \ominus \pi_2 \bar{\zeta})$$

$$\left(\frac{1}{2} - 2\varepsilon, \frac{1}{2} - 2\varepsilon \right) \quad \underline{\left(\frac{1}{2} - \varepsilon - \sigma, \frac{1}{2} - \varepsilon + \sigma \right)}$$

$$\left(\frac{1}{2} - 2\varepsilon \right) + \left(-\frac{1}{2} - \varepsilon - \sigma \right) = -3\varepsilon - \sigma < 0 \Rightarrow \text{X}$$

$$\text{Let } X_T(x_1, x_2) = X\left(\frac{x_1}{T}, \frac{x_2}{T}\right).$$

We consider

$$X = f_X(X_T \cdot V \ominus \odot \pi_1 \xi) + f_X(X_T \cdot V \odot \ominus \pi_2 \xi)$$

$$+ f_X(X_T \cdot Y \ominus \odot \pi_1 \xi) + f_X(X_T \cdot Y \odot \ominus \pi_2 \xi)$$

$$Y = V_0 + f_X(X_T \cdot V \ominus \odot \pi_1 \xi) + f_X(X_T \cdot V \odot \ominus \pi_2 \xi)$$

$$+ f_X(X_T \cdot Y \ominus \odot \pi_1 \xi) + f_X(X_T \cdot Y \odot \ominus \pi_2 \xi)$$

$$+ f_X(X_T \cdot [X \ominus \odot \pi_1 \xi]) + f_X(X_T \cdot [X \odot \ominus \pi_2 \xi])$$

WANT: RHS (without $f_X(X_T \cdot)$) of the X -eqn $\sim (-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon)$

RHS ("") of the Y -eqn $\sim (\frac{1}{2} + (a+1)\varepsilon, \frac{1}{2} + (a+1)\varepsilon)$

• Analysis of the Y-eqn:

by symmetry

ill-defined terms: $X \ominus \odot \pi_1 \bar{z}$ and $X \odot \ominus \pi_2 \bar{z}$



Substitute the X-eqn. (without X_T) $\textcircled{<} + \textcircled{=} + \textcircled{>}$

$$\begin{aligned}
 \cdot \underline{X \ominus \odot \pi_1 \bar{z}} &= J_X(v \ominus \odot \pi_1 \bar{z}) \ominus \odot \pi_1 \bar{z} + J_X(v \odot \ominus \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z} \\
 &\quad + J_X(Y \ominus \odot \pi_1 \bar{z}) \ominus \odot \pi_1 \bar{z} + J_X(Y \ominus \odot \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z} \\
 &= J_X(v \ominus \odot \pi_1 \bar{z}) \ominus \odot \pi_1 \bar{z} + \underline{J_X(v \odot \ominus \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z}} \\
 &\quad + \underline{J_X(v \odot \ominus \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z}} + \underline{J_X(v \odot \ominus \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z}} \\
 &\quad + \cancel{J_X(Y \ominus \odot \pi_1 \bar{z}) \ominus \odot \pi_1 \bar{z}} + \cancel{J_X(Y \ominus \odot \pi_2 \bar{z}) \ominus \odot \pi_1 \bar{z}} \\
 &= I_1^Y(v) + I_2^Y(v) + I_3^Y(v) + I_4^Y(v) + I_5^Y(Y) + I_6^Y(Y)
 \end{aligned}$$

BAD

- $I_2^Y(v), I_3^Y(v), I_5^Y(Y) = \text{good terms} \sim (\frac{1}{2} + (q+1)\varepsilon, \frac{1}{2} + (q+1)\varepsilon)$ (16)

- On $I_1(Y)$: $I_1^Y(v) = \oint_X (v \odot \circlearrowleft \pi_i \tilde{s}) \odot \circlearrowright \pi_i \tilde{s}$ $\xrightarrow{\quad \textcolor{red}{\cancel{\circlearrowleft + \circlearrowright}} \geq 0 \quad}$

$\circlearrowright = \odot \circlearrowleft + \odot \circlearrowright$

- $\oint_X (v \odot \circlearrowleft \pi_i \tilde{s})(\vec{x}) = \oint_{X_2}^{(2)} (\chi_1 \cdot \oint_{X_1}^{(1)} (v \odot \circlearrowleft P_{\neq 0}^{(1)} \pi_i \tilde{s}))(\vec{x})$

$$\begin{aligned} \left(\oint^{(1)} f \right)(\vec{x}) &= \int_0^{x_1} f(y_1, x_2) dy_1 & \widehat{I_1(f)}(\vec{n}) &= \widehat{f}(\vec{n}) \mathbf{1}_{n_1 \neq 0} \\ &= I_1(f)(\vec{x}) - I_1(f)(0, x_2) + x_1 P_0^{(1)}(f)(x_2) \end{aligned}$$

$$= \oint_{X_2}^{(2)} (\chi_1 \cdot I_1(v \odot \circlearrowleft P_{\neq 0}^{(1)} \pi_i \tilde{s}))(\vec{x})$$

$$- \boxed{\chi_1(x_1) \cdot \oint_{X_2}^{(2)} (I_1(v \odot \circlearrowleft P_{\neq 0}^{(1)} \pi_i \tilde{s}))(\underline{0}, x_2)}$$

smooth

$$=: A_1 + A_2$$

good.

17

- A_2 is good and $A_2 \otimes \circ \pi_1 \bar{\zeta}$ makes sense.

$$\begin{aligned}
 \bullet A_1 &= \oint_{X_2}^{(2)} \left(X_1 \cdot \underline{I_1} (v \otimes \circ P_{\neq 0}^{(1)} \pi_1 \bar{\zeta}) \right) \\
 &= \oint_{X_2}^{(2)} \left(X_1 \cdot v \otimes \circ I_1(\pi_1 \bar{\zeta}) \right) + \underline{\oint_{X_2}^{(2)} \left(X_1 \cdot C_{I_1}(v, P_{\neq 0}^{(1)} \pi_1 \bar{\zeta}) \right)} \\
 &\quad \text{good}
 \end{aligned}$$

• $C_{I_1}(f, g) = I_1(f \otimes \circ g) - f \otimes \circ I_1(g)$

$$\begin{aligned}
 \bullet & \oint_{X_2}^{(2)} \left(X_1 \cdot v \otimes \circ I_1(\pi_1 \bar{\zeta}) \right) \otimes \circ \pi_1 \bar{\zeta} \\
 & \quad \uparrow \quad \text{only in the } x_1\text{-variable} \\
 & = (X_1 \otimes_1 \oint_{X_2}^{(2)} (v \otimes \circ I_1(\pi_1 \bar{\zeta}))) \otimes \circ \pi_1 \bar{\zeta} \\
 & + \underline{(X_1 \otimes_1 \oint_{X_2}^{(2)} (v \otimes \circ I_1(\pi_1 \bar{\zeta}))) \otimes \circ \pi_1 \bar{\zeta}} \\
 & \quad \uparrow \text{Smooth} \quad \text{good}
 \end{aligned}$$

$$\cdot \left(\chi_1 \underbrace{\otimes}_1 \mathcal{J}_{\chi_2}^{(2)} (\nu \otimes \odot I_1(\pi, \zeta)) \right) \underbrace{\ominus}_0 \pi, \zeta$$

$$= \chi_1 \left(\mathcal{J}_{\chi_2}^{(2)} (\nu \otimes \odot I_1(\pi, \zeta)) \ominus \odot \pi, \zeta \right)$$

$$- \underbrace{\text{com}_1^{(1)} (\chi_1, \mathcal{J}_{\chi_2}^{(2)} (\nu \otimes \odot I_1(\pi, \zeta)), \pi, \zeta)}_{\text{good}}$$

$$\cdot \boxed{\text{com}_1^{(1)} (f, g, h) = (f \otimes g) \ominus \odot h - f (g \ominus \odot h)}$$

↑
only in χ_1

$$\cdot \boxed{\text{com}_2^{(1)} (f, g, h) = \mathcal{J}_{\chi_2}^{(2)} (f \odot g) \ominus \odot h - \mathcal{T}^{(2)} (f, g, h)}$$

$\chi_3 = \chi_2$

$$\mathcal{T}^{(2)} (f, g, h)(x_1, x_2, x_3) = \chi_2(x_2) \int_0^{x_2} f(x_1, y_2) \left\{ g(x_1, y_2) \ominus, h(x_1, x_3) \right\} dy_2$$

(19)

Define a tri-parameter commutator.

$$\text{COM}_2^{(1)}(f, g, h)(x_1, x_2, x_3) = f_{x_2}^{(2)}(f \otimes \circ g)(x_1, x_2) \ominus, h(x_1, \underline{x_3}) \\ - T^{(2)}(f, g, h)(x_1, x_2, x_3)$$

$$\Rightarrow \text{COM}_2^{(1)}(f, g, h)(x_1, x_2) = \text{COM}_2^{(1)}(f, g, h)(x_1, x_2, x_3) \Big|_{x_3=x_2}$$

- $\underbrace{f_{x_2}^{(2)}(v \otimes \circ I_1(\pi, \zeta)) \ominus \circ \pi, \zeta}_{= T^{(2)}(v, I_1(\pi, \zeta), \pi, \zeta) \Big|_{x_3=x_2}} + \underbrace{\text{COM}_2^{(1)}(v, I_1(\pi, \zeta), \pi, \zeta)}_{\text{good.}}$

$\hookrightarrow A_1(\pi, \zeta, \pi, \zeta)(x_1, x_2, x_3) = I_1(\pi, \zeta)(x_1, x_2) \ominus \pi, \zeta(x_1, x_3)$

tri-parameter quadratic stochastic object.

$$B^{(2)}(f, G)(x_1, x_2) = \left. \int_{X_2}^{(2)} (f(x_1, \cdot) G(x_1, \cdot, x_3)) \right|_{\substack{x_3=x_2}} \\ = X_2(x_2) \int_0^{x_2} f(x_1, y_2) G(x_1, y_2, x_3) dy_2 \Big|_{\substack{x_3=x_2}}$$

(20)

$$\Rightarrow J^{(2)}(\nu, I_1(\pi, \zeta), \pi, \zeta) \\ = B^{(2)}(\nu, A_1(\pi, \zeta, \pi, \zeta)).$$

$$\begin{aligned} \cdot I_1^Y(\nu) &= - \left(\underline{X_1}(x_1) \cdot \int_{X_2}^{(2)} (I_1(\nu \otimes \circ \pi, \zeta)) \underline{(0, x_2)} \right) \otimes \circ \pi, \zeta \\ &\quad + \int_{X_2}^{(2)} (X_1 \cdot \underline{C_{I_1}(\nu, \pi, \zeta)}) \otimes \circ \pi, \zeta \\ &\quad + \left(\underline{X_1} \otimes \underline{1} \int_{X_2}^{(2)} (\nu \otimes \circ I_1(\pi, \zeta)) \right) \otimes \circ \pi, \zeta \\ &\quad + \underline{\text{Com}_1^{(1)}}(X_1, \int_{X_2}^{(2)} (\nu \otimes \circ I_1(\pi, \zeta)), \pi, \zeta) \\ &\quad + X_1 \cdot \underline{\text{Com}_2^{(1)}}(\nu, I_1(\pi, \zeta), \pi, \zeta) \\ &\quad + X_1 \cdot B^{(2)}(\nu, A_1(\pi, \zeta, \pi, \zeta)). \end{aligned}$$

Lec 6 : July 17, 2023 (Mon)

Bad terms in the Y-eqn: $X \ominus \odot \pi_1 \xi$ and $X \odot \ominus \pi_2 \xi$

$$\begin{aligned} X \ominus \odot \pi_1 \xi &= I_1^Y(v) + \underline{I_2^Y(v)} + \underline{I_3^Y(v)} + I_4^Y(w) + \underline{I_5^Y(y)} + I_6^Y(Y) \\ X \odot \ominus \pi_2 \xi &= II_1^Y(v) + II_2^Y(v) + II_3^Y(v) + II_4^Y(w) + II_5^Y(y) + II_6^Y(Y) \end{aligned}$$

BAD terms = expand each into a sum of 6 terms

- 14 commutators
- 6 tri-parameter quadratic stoch. objects

$$A_j(\pi_k \xi, \pi_l \xi) = I_j(\pi_k \xi)(x_1, x_2) \ominus, \pi_l \xi(x_1, x_2) =$$

Good terms in the Y-eqn: WTS $\sim \left(-\frac{1}{2} + (a+1)\varepsilon, -\frac{1}{2} + (a+1)\varepsilon\right)$ ②

- $$\begin{aligned} \bullet \quad V \odot \odot \pi_1 \xi &\sim (-(\alpha+5)\varepsilon, -\frac{1}{2} + (\alpha+1)\varepsilon). \quad \checkmark \\ (\frac{1}{2}-2\varepsilon, \underline{\frac{1}{2}-2\varepsilon}), (\underline{-\frac{1}{2}-\varepsilon}, -\frac{1}{2}-\varepsilon) \\ \pi_1 \hookrightarrow & \underline{(-\frac{1}{2}-(\alpha+3)\varepsilon, -\frac{1}{2}+(\alpha+1)\varepsilon)} \\ \Rightarrow & (\frac{1}{2}-2\varepsilon) + (-\frac{1}{2}+\alpha+1)\varepsilon = (\alpha-1)\varepsilon > 0 \quad \text{if } \alpha > 1. \end{aligned}$$

- $$Y \oplus \mathbb{Z} \pi_1 \cong ((a-1)\varepsilon, (a-1)\varepsilon), \quad \forall$$

$$\left(\frac{1}{2} + a\varepsilon, \frac{1}{2} + a\varepsilon\right) \quad \left(-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon\right)$$

$$\bullet I_2^Y(v) = \left(\underbrace{\mathbb{J}_X(v \odot \pi_2 \xi)}_{(\frac{1}{2}-2\varepsilon, \frac{1}{2}-2\varepsilon)} \right) \oplus \pi_1 \xi. \xrightarrow{\pi_1} \\ (-\frac{1}{2}-(a+3)\varepsilon, -\frac{1}{2}+(a+1)\varepsilon) \\ \downarrow \\ \underbrace{(-\frac{1}{2}+(a+4)\varepsilon, -\frac{1}{2}-(a+6)\varepsilon)}_{(-\frac{1}{2}+(a+4)\varepsilon, -\frac{1}{2}-(a+6)\varepsilon)} \\ \downarrow \\ \mathbb{J}_X \Rightarrow (\frac{1}{2}+(a+4)\varepsilon, \frac{1}{2}-(a+6)\varepsilon). \longrightarrow I_2^Y(v) \sim \left(\varepsilon, -\frac{1}{2}+(a+1)\varepsilon \right)$$

$$\cdot I_3^Y(v) = f_X(v \otimes \pi_2 \bar{z}) \oplus \pi_1 \bar{z} \quad \checkmark$$

better than $\oplus \pi_1 \bar{z}$ in $I_2^Y(v)$

$$\cdot I_5^Y(Y) = f_X(Y \oplus \underbrace{\pi_1 \bar{z}}_{(\frac{1}{2} + a\varepsilon, \frac{1}{2} + a\varepsilon)} \oplus \pi_1 \bar{z})$$

$$(\frac{1}{2} - (a+3)\varepsilon, -\frac{1}{2} + (a+1)\varepsilon)$$

$$(1 + (a-1)\varepsilon, \frac{1}{2} - \varepsilon) \rightarrow I_5^Y(Y) \sim (\frac{1}{2} - 4\varepsilon, -\frac{1}{2} + (a+1)\varepsilon)$$

On the X-eqn:

$$X = f_X(X^T \cdot v \otimes \pi_1 \bar{z}) + f_X(X^T \cdot v \otimes \underbrace{\pi_2 \bar{z}}_{BAD})$$

$$+ f_X(X^T \cdot Y \oplus \pi_1 \bar{z}) + f_X(X^T \cdot Y \otimes \pi_2 \bar{z}).$$

WTS

$$\sim (-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon).$$

$$(\frac{1}{2} + a\varepsilon, \frac{1}{2} + a\varepsilon) \underbrace{(-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon)}_{((a-1)\varepsilon, -\frac{1}{2} - \varepsilon)}$$

(4)

- BAD terms: $v \odot \ominus \pi_1 \xi$ and $v \odot \oplus \pi_2 \xi$

$$v \odot \ominus \pi_1 \xi = \underbrace{v \odot \ominus \pi_1 \xi}_{\left(\frac{1}{2}-2\varepsilon, \frac{1}{2}-2\varepsilon\right)} + \overline{\underbrace{v \odot \ominus \pi_1 \xi}_{\left(-\frac{1}{2}-\varepsilon, -\frac{1}{2}-\varepsilon\right)}} + \underbrace{v \odot \oplus \pi_1 \xi}_{\left(-\frac{1}{2}-\varepsilon, -3\varepsilon\right)}$$

$\hookrightarrow v \odot \ominus \pi_1 \xi \sim \underbrace{\left(-\frac{1}{2}-5\varepsilon, \varepsilon\right)}_{\left(-\frac{1}{2}-5\varepsilon, -\frac{1}{2}+3\varepsilon\right)}$ \uparrow NOT good $< -\frac{1}{2}-\varepsilon$

Need this regularity to make sense of \ominus_2

$$v \odot \ominus \pi_1 \xi = \underbrace{x \odot \ominus \pi_1 \xi}_{\begin{array}{c} \uparrow \\ \text{Substitute the } X\text{-eqn.} \end{array}} + \underbrace{y \odot \ominus \pi_1 \xi}_{\left(-\frac{1}{2}-\varepsilon, (\alpha-1)\varepsilon\right)}$$

$$= \underbrace{\left(\frac{1}{2}+\alpha\varepsilon, \frac{1}{2}+\alpha\varepsilon\right)}_{\text{NOT good}}$$

$$X \odot \ominus \pi_1 \zeta = J_X (V \odot \ominus \pi_1 \zeta) \odot \ominus \pi_1 \zeta = A_1 \quad (5)$$

$$+ J_X (Y \ominus \odot \pi_1 \zeta) \odot \ominus \pi_1 \zeta = A_2$$

$$+ J_X (Y \odot \ominus \pi_2 \zeta) \odot \ominus \pi_1 \zeta = A_3$$

$$+ J_X (V \odot \odot \pi_2 \zeta) \odot \ominus \pi_1 \zeta \quad \text{BAD}$$

//

$$\odot = \odot - \odot - \ominus$$

$$J_X (V \odot \odot \pi_2 \zeta) \underline{\odot} \underline{\ominus} \pi_1 \zeta$$

$$- J_X (V \odot \odot \pi_2 \zeta) \odot \ominus \pi_1 \zeta = A_4$$

$$- J_X (V \odot \odot \pi_2 \zeta) \underline{\ominus} \underline{\ominus} \pi_1 \zeta = I_4^Y (V)$$

Need to expand them into sums of 6 terms (each)

(6)

$$\cdot A_1 = \oint_{\Gamma} (\nu \odot \odot \pi_1 \zeta) \odot \odot \cancel{\pi_1 \zeta} = A_{11} + A_{12} + A_{13},$$

$\stackrel{=}{\cancel{}} = \odot + \odot + \odot$

$$\cdot A_{12} = \oint_{\Gamma} (\nu \odot \odot \pi_1 \zeta) \odot \odot \pi_1 \zeta \sim \left(-\frac{1}{2} - \varepsilon, \frac{1}{2} \right)$$

$\downarrow \pi_1$
 $(-\frac{1}{2} - 5\varepsilon, -\frac{1}{2} + 3\varepsilon)$
 $\underbrace{_{(\frac{1}{2} - 5\varepsilon, 1 + \varepsilon)}}$

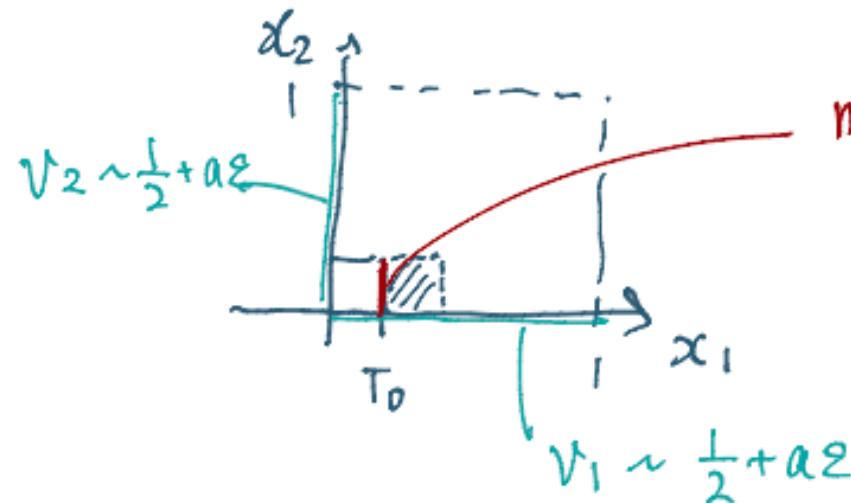
A_{13} is similar.

can't make use of this π_1

$$\cdot A_{11} = \underbrace{\oint_{\Gamma} (\nu \odot \odot \pi_1 \zeta)}_{(\frac{1}{2} - 4\varepsilon, \frac{1}{2} + 2\varepsilon)} \odot \odot \cancel{\pi_1 \zeta} \stackrel{=}{\cancel{}} (-\frac{1}{2} - \varepsilon, -\frac{1}{2} - \varepsilon) \sim \left(-\frac{1}{2} - \varepsilon, \varepsilon \right).$$

A_2, A_3, A_4 can be estimated in an analogous manner.

- Once we obtain the reformulated system for X and Y , LWP follows from a standard contraction argument. ⑦
- Construction of the stochastic objects is similar to that in Part 1.
- When $F(V) = V$, GWP holds but is more complicated than that in Part 1.



new BC is given by

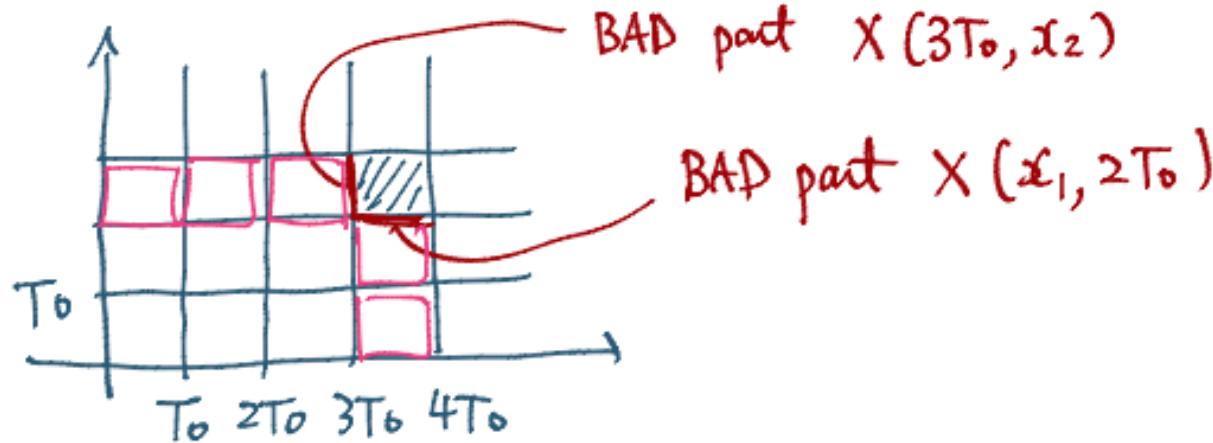
$$V(T_0, x_2) = X(T_0, x_2) + Y(T_0, x_2)$$

$\uparrow \in C_{x_2}^{\frac{1}{2}-\alpha_2}$ ← NOT good

must be included in the X -eqn.

\Rightarrow In substituting the X -eqn, we must use the structure of $X(T_0, x_2)$ from the previous step.

⑧



- Back to the commutator estimates in Part 1.

Lemma 5 from Lec 3: $C_I(f, g) = \underbrace{I(f \otimes g)}_{\leftarrow} - \underbrace{f \otimes Ig}_{\uparrow e^{s_2+1}}$ for $s_1 > 0$.

$$\|C_I(f, g)\|_{e^{s_1+s_2+1}} \lesssim \|f\|_{e^{s_1}} \|g\|_{e^{s_2}}$$

(9)

• Lemma 9:

$$[\Delta_k I, f]g = \underline{\Delta_k I(fg)} - \underline{f} \Delta_k I g.$$

$$\Rightarrow \| [\Delta_k I, f]g \|_{L^\infty} \lesssim \underline{2^{-2k}} \| \underline{\partial_x f} \|_{L^\infty} \| g \|_{L^\infty}.$$

• Pf of Lemma 5, assuming Lemma 9:

$$\Delta_j C_I(f, g) = \Delta_j (I(f \otimes g) - f \otimes I g)$$

$$= \sum_{k=0}^{\infty} \underline{\Delta_j} (\underbrace{\Delta_k (I(f \otimes g))}_{\sim 2^k} - \underbrace{f \otimes \Delta_k I g}_{\sim 2^k})$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \underline{\Delta_j} \left(\underline{\Delta_k I(S_{l-3}f)} \underline{\Delta_l g} \right. \\ \left. - \underline{S_{l-3}f} \underline{\Delta_k I(\Delta_l g)} \right)$$

$$= [\Delta_k I, S_{l-3}(f)](\Delta_l g)$$

$$S_\ell(f) = \sum_{k=0}^{\ell} \Delta_k f$$

(10)

$$= \sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ |k-j| \leq 10}}^{\infty} \Delta_j [\Delta_k I, S_{l-3} f] (\Delta_l g).$$

\uparrow
 $|j-l| \leq 15.$

$$\Rightarrow \|\Delta_j C_I(f, g)\|_{L^\infty} \leq \sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ |k-j| \leq 10}}^{\infty} \|[\Delta_k I, S_{l-3} f] (\Delta_l g)\|_{L^\infty}$$

$\stackrel{\text{Bernstein}}{\lesssim} \frac{2^l \|S_{l-3} f\|_{L^\infty}}{|j-l|} \|\Delta_l g\|_{L^\infty}$

$$\lesssim \frac{2^{-2j}}{m m} \sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ |k-j| \leq 10}}^{\infty} \frac{\|2x S_{l-3} f\|_{L^\infty}}{|j-l|} \|\Delta_l g\|_{L^\infty}$$

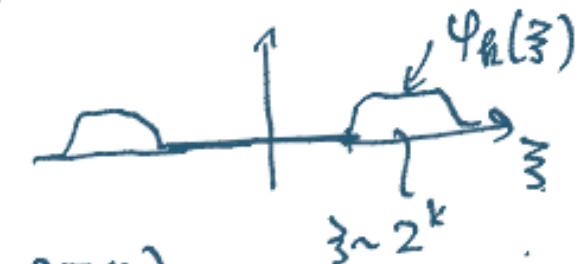
$\stackrel{\text{Bernstein}}{\lesssim} \frac{2^{-s_1 l} \cdot 2^{s_1 l}}{2^{-s_2 l} \cdot 2^{s_2 l}} \|\Delta_l g\|_{L^\infty}$

$$\lesssim 2^{-(s_1 + s_2 + 1)j} \|f\|_{e^{s_1}} \|g\|_{e^{s_2}}.$$

uniformly in $j \in \mathbb{Z}_{\geq 0}$ 

Pf of Lemma 9: $[\underline{\Delta_k I}, f](g) = \underline{\Delta_k I}(fg) - f\underline{\Delta_k I}g.$ (11)

① $k \geq 1$: $\varphi_k(\zeta) =$ multiplier for $\Delta_k.$



Set $\Phi_k(x) = \sum_{n \in \mathbb{Z}} I_R(F_R^{-1}(\varphi_k))(x - 2\pi n)$
function on \mathbb{R}

where $F_R(I_R f)(\zeta) = \frac{1}{i\sqrt{3}} F_R f(\zeta).$

$$\Rightarrow \widehat{\Phi}_k(n) = \begin{cases} \frac{1}{i\pi} \varphi_k(n), & n \neq 0, \\ 0, & n = 0 \end{cases}$$

$$[\Delta_k \mathcal{I}, f](g)(x)$$

$$= \sum_{n \in \mathbb{Z}} \int_0^{2\pi} \mathcal{I}_{\mathbb{R}}(\mathcal{F}_{\mathbb{R}}^{\gamma}(\varphi_k))(x-y-2\pi n) \\ \times (f(y) - f(x)) g(y) dy$$

View them as periodic function on \mathbb{R} .

$$= \int_{\mathbb{R}} \mathcal{I}_{\mathbb{R}}(\mathcal{F}_{\mathbb{R}}^{\gamma}(\varphi_k))(x-y) \\ \times (f(y) - f(x)) g(y) dy.$$

$$= \frac{1}{2^k} \int_0^1 \int_{\mathbb{R}} \boxed{\mathcal{I}_{\mathbb{R}}(\mathcal{F}_{\mathbb{R}}^{\gamma}(\varphi_k))(x-y)} \\ \times (y-x) \cdot 2^k \partial_x f(x + \tau(y-x)) g(y) dy$$

\uparrow \uparrow

$$\frac{1}{2^k} = \frac{2^k}{2^k} \cdot \frac{1}{2^k} \quad \frac{1}{2^k} \cdot 2^k G(2^k(x-y))$$

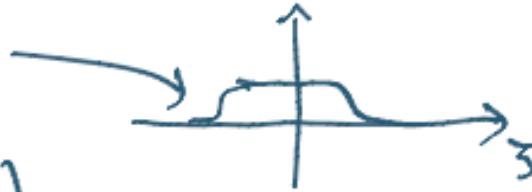
$$G(z) = -z \mathcal{I}_{\mathbb{R}}(\mathcal{F}_{\mathbb{R}}^{\gamma}(\varphi))(z)$$

$$\varphi(\tfrac{z}{2}) = \varphi_1(2z)$$

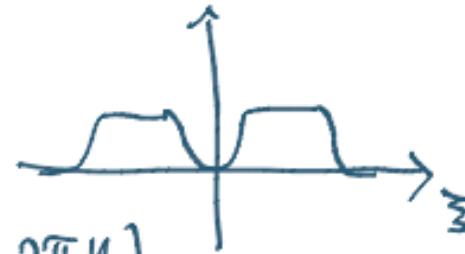
$$\lesssim \frac{1}{2^{2k}} \|\partial_x f\|_{L^\infty} \|g\|_{L^\infty}.$$

①③

$$\textcircled{2} \quad \underline{k=0}: \quad \varphi_0(\xi) = \phi(|\xi|)$$



Set $\tilde{\varphi}_0(\xi) = \varphi_0(\xi)(1 - \varphi_0(10\xi))$
 $\Rightarrow \tilde{\varphi}_0(0) = 0.$



Set $\Phi_0(x) = \sum_{n \in \mathbb{Z}} I_R(F_R^{-1}(\tilde{\varphi}_0))(x - 2\pi n)$

and repeat Step ①

□

Lec 7 : July 24, 2023 (MON)

①

From Lec 3, it remains to prove

Lemma 6: $\text{com}_r(f, g, h) = (f \otimes g) \ominus h - f(g \ominus h)$

$0 < s_1 < 1, \quad s_2 + s_3 < 0, \quad s_1 + s_2 + s_3 > 0$

does not make sense

$$\| \text{com}_r(f, g, h) \|_{\ell^{s_1+s_2+s_3}} \lesssim \| f \|_{\ell^{s_1}} \| g \|_{\ell^{s_2}} \| h \|_{\ell^{s_3}}$$

Lemma 10: $[\Delta_k, \otimes](f, g) = \Delta_k(f \otimes g) - f \cdot \Delta_k g.$

$0 < s_1 < 1$. Then, we have

$$\| [\Delta_k, \otimes](f, g) \|_{\ell^\infty} \lesssim 2^{-(s_1+s_2)k} \| f \|_{\ell^{s_1}} \| g \|_{\ell^{s_2}}.$$

(2)

By assuming Lemma 10, we first prove Lemma b.

Pf of Lemma b:

$$(f \odot g) \ominus h = \sum_{k,l=0}^{\infty} \Delta_k (f \odot g) \Delta_l h$$

$|k-l| \leq 2$

$$= \sum_{k,l,m=0}^{\infty} \Delta_k (\Delta_m f \odot g) \Delta_l h$$

$|k-l| \leq 2$

$m \leq k+10$

$$= \sum_{k,l,m=0}^{\infty} (\Delta_m f) (\Delta_k g) (\Delta_l h)$$

$|k-l| \leq 2$

$m \leq k+10$

$$+ \sum_{k,l,m=0}^{\infty} [\Delta_k, \odot] (\Delta_m f, g) \Delta_l h$$

$|k-l| \leq 2$

$m \leq k+10$

from Lem 10.

$$\sum_{m \leq k+10} \Delta_m f = S_{k+10}(f)$$

$$\Rightarrow \text{com}_1(f, g, h) = \sum_{\substack{k, l, m=0 \\ |k-l| \leq 2 \\ m > k+10}}^{\infty} (\Delta_m f)(\Delta_k g)(\Delta_l h) \sim 2^m \quad (\text{Fourier supp})$$

$$+ \sum_{\substack{k, l=0 \\ |k-l| \leq 2}}^{\infty} [\Delta_k, \Theta](S_{k+10}(f), g) \Delta_l h$$

$$=: A_1 + A_2.$$

On A_1 :

$$\|\Delta_j A_1\|_{L^\infty} \leq \sum_{\substack{k, l, m=0 \\ |k-l| \leq 2 \\ m > k+10}}^{\infty} \mathbb{1}_{j \leq m+C_0} \|\Delta_m f\|_{L^\infty} \|\Delta_k g\|_{L^\infty} \|\Delta_l h\|_{L^\infty}$$

$$\leq \|f\|_{e^{s_1}} \|g\|_{e^{s_2}} \|h\|_{e^{s_3}}$$

$\frac{-s_1 m}{2} \begin{bmatrix} -s_2 k & -s_3 l \\ 2 & 2 \end{bmatrix} \sim 2^{-(s_2 + s_3)k}$

$$\sum_{k,m=0}^{\infty} \frac{1}{j \leq m + c_0} 2^{-S_1 m} 2^{-\underbrace{(S_2 + S_3) k}_{< 0}}$$

$$\sum_{m=0}^{\infty} \frac{1}{j \leq m + c_0} \geq \frac{-(S_1 + S_2 + S_3)m}{2} > 0$$

$$\sup_{\mathbb{R}^n} \text{inj} \Rightarrow \|A_1\|_{C^{s_1+s_2+s_3}} \lesssim \|f\|_{C^{s_1}} \|g\|_{C^{s_2}} \|h\|_{C^{s_3}}.$$

$$\begin{aligned} \text{On } A_2: \quad A_2 &= \sum_{\substack{k, l=0 \\ |k-l| \leq 2}}^{\infty} [\Delta_k, \otimes] (S_{k+10}(f), g) \Delta_l h \\ &= \underbrace{\Delta_k (S_{k+10}(f)) \otimes g}_{\sim 2^k} - \underbrace{S_{k+10}(f) \Delta_k g}_{\lesssim 2^k} \end{aligned}$$

$$\|\Delta_j A_2\|_{L^\infty} \lesssim \sum_{\substack{k,l=0 \\ |k-l| \leq 2}}^{\infty} \mathbf{1}_{j \leq k+C_0} 2^{-(S_1+S_2)k} \times 2^{-S_3 l} \times 2^{S_3 l} \|\Delta_l h\|_{L^\infty} \leq \|S_{R+10}(f)\|_{\ell^{S_1}} \|g\|_{\ell^{S_2}}$$

$$\sum_{k=0}^{\infty} \mathbf{1}_{j \leq k+C_0} 2^{-(S_1+S_2+S_3)k} > 0 \lesssim 2^{-(S_1+S_2+S_3)j}$$

\Rightarrow Take sup in j .



Pf of Lemma 10: WTS: $\|[\Delta_k, \Theta](f, g)\|_{L^\infty} \lesssim 2^{-(S_1+S_2)k} \|f\|_{C^{S_1}} \|g\|_{C^{S_2}}$. ⑥

$$[\Delta_k, \Theta](f, g) = \Delta_k(f \Theta g) - f(\Delta_k g)$$

$$= (\Delta_k(f \Theta g) - f \underset{\text{red}}{\Theta} \Delta_k g) + f \Theta \Delta_k g$$

$$=: B_1 + B_2.$$

On B_2 :

$$B_2 = \sum_{m=k-2}^{\infty} \Delta_m f \cdot \Delta_k g \times 2^{-S_1 m} 2^{-S_2 k}$$

$$\Rightarrow \|B_2\|_{L^\infty} \leq \sum_{m=k-2}^{\infty} 2^{-S_1 m} 2^{-S_2 k} \|f\|_{C^{S_1}} \|g\|_{C^{S_2}}$$

$$\lesssim 2^{-(S_1+S_2)k} \quad (\text{b/c } \underline{S_1 > 0}).$$

• On B_1 :

$$\underline{\text{Lemma 9'}}: \|\llbracket \Delta_k f \rrbracket(g)\|_{L^\infty} \lesssim 2^{-k} \|\partial_x f\|_{L^\infty} \|g\|_{L^\infty}$$

\Leftarrow a slight modification of the proof of Lemma 9
(just drop I and I_{RR}).

$$\begin{aligned} B_1 &= \sum_{\substack{m, l=0 \\ m < l-2}}^{\infty} \underbrace{\Delta_k (\Delta_m f \Delta_l g)}_{2^l \text{ b/c } m < l-2} - \Delta_m f \underbrace{\Delta_k \Delta_l g}_{=} \\ &= [\Delta_k, \Delta_m f] (\Delta_l g) \end{aligned}$$

$$\begin{aligned} \|B_1\|_{L^\infty} &\stackrel{\text{Lem 9'}}{\lesssim} \sum_{\substack{m, l=0 \\ m < l-2 \\ |k-l| \leq 5}}^{\infty} \underbrace{2^{-k}}_{\substack{\sim 2^m \|\Delta_m f\|_{L^\infty} \\ \text{Bernstein}}} \|\partial_x \Delta_m f\|_{L^\infty} \|\Delta_l g\|_{L^\infty} \\ &\quad \times \underbrace{2^{-s_1 m}}_{\substack{\sim 2^{s_1 m} \\ \text{Bernstein}}} \underbrace{2^{-s_2 l}}_{\substack{\sim 2^{s_2 l} \\ \text{Bernstein}}} \times \underbrace{2^{s_1 m}}_{\substack{\sim 2^{s_1 m} \\ \text{Bernstein}}} \times \underbrace{2^{s_2 l}}_{\substack{\sim 2^{s_2 l} \\ \text{Bernstein}}} \end{aligned}$$

$$\Rightarrow \|B_1\|_{L^\infty} \lesssim \sum_{m, l=0}^{\infty} 2^{-k} 2^{(1-s_1)m} 2^{-s_2 k} \|f\|_{\ell^{s_1}} \|g\|_{\ell^{s_2}} \quad (8)$$

$m < l - 2$
 $|k - l| \leq 5$

$m < k - 7$
 $b/c \quad s_1 < 1$

$\lesssim 2^{-k} 2^{(1-s_1)k} 2^{-s_2 k} = 2^{-(s_1 + s_2)k}$

□