Advanced Ph.D. course

Stochastic PDEs with multiplicative noises

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Contents:

STOCHASTIC PDES WITH MULTIPLICATIVE NOISE - LECTURE NOTES

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1. Introduction and vocabulary

1.1. **A bit of vocabulary...** In this course, we study the problem of well-posedness for *stochastic partial differential equations (PDEs) with multiplicative noise*. Let us first give some examples of such equations:

- Stochastic nonlinear Schrödinger equation (SNLS):
$$i\partial_t u - \Delta u = N(u) + \sigma(u)\Phi\xi, \tag{1.1}$$
- Stochastic nonlinear wave equation (SNLW):
$$\partial_t^2 u - \Delta u = N(u) + \sigma(u)\Phi\xi, \tag{1.2}$$
- Stochastic nonlinear heat equation (SNLH):
$$\partial_t u - \Delta u = N(u) + \sigma(u)\Phi\xi, \tag{1.3}$$

Let us come back a bit on these equations, and clarify some vocabulary. First, note that, in each of these equations, the left-hand side is the part that qualifies the equation as Schrödinger, wave or heat. More particularly, the left-hand side also make us qualify (SNLS) and (SNLW) as *dispersive* equations, while (SNLH) is qualified as *dissipative* or *parabolic* equation. Also, $N(u)$ denotes the *nonlinearity* and $\sigma(u)\Phi\xi$ denotes the *stochastic noise*. Let us now precise the terms of the noise:

- ξ denotes a *space-time white noise*, which essentially denotes a space-time process such that, for any (t,x), $\xi(t,x)$ is a mean 0 Gaussian process with
$$E[\xi(t,x)\overline{\xi(s,y)}] = \delta(t-s)\delta(x-y),$$
- Φ is a smoothing operator in x, namely Φ is a bounded operator in $L^2(\mathcal{M})$ with $\mathcal{M} = \mathbb{R}^d$ or $\mathcal{M} = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$,
- σ is a function in u. If $\sigma(u) \equiv 1$, we say that we have an *additive* noise. Otherwise, *e.g.* $\sigma(u) = u$ or $\sigma(u) = u^k$, we say that we have a *multiplicative* noise.

Let us now introduce the notion of *mild solution* to an equation. To do so, we focus on the associated problem for (SNLS):
$$\begin{cases} i\partial_t u = \Delta u + N(u) + \sigma(u)\Phi\xi \\ u|_{t=0} = u_0. \end{cases} \tag{1.4}$$
Suppose, for now, that we only have the *linear* problem, namely:
$$\begin{cases} i\partial_t u = \Delta u, \\ u|_{t=0} = u_0. \end{cases} \tag{1.5}$$
Then, applying Fourier transform for the space variable $x \in \mathcal{M}$, our problem becomes

$$\begin{cases} i\partial_t \widehat{u} = -|\xi|^2 \widehat{u}, \\ \widehat{u}|_{t=0} = \widehat{u}_0, \end{cases}$$

which has for solution $\widehat{u}(t,\xi) = \widehat{u}_0(\xi) e^{it|\xi|^2}$. Therefore, inverse Fourier transform allows us to say that the solution for (1.5) is $u(t) = S(t)u_0$ where

$$S(t)f = e^{it\Delta} f = \mathcal{F}^{-1}[e^{it|\xi|^2} \widehat{f}]. \tag{1.6}$$

From this, we say that u is a *(mild) solution* to (SNLS) if u satisfies the following *mild* or *Duhamel formulation*

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')N(u)(t')\mathrm{d}t' - i\int_0^t S(t-t')\sigma(u)(t')\Phi \mathrm{d}W(t'). \tag{1.7}$$

In the previous equation, we say that

- $S(t)u_0$ is the linear solution,
- $\int_0^t S(t-t')N(u)(t')\mathrm{d}t'$ is the *nonlinear part of the equation*,
- $\int_0^t S(t-t')\sigma(u)(t')\Phi \mathrm{d}W(t')$ is the effect of *stochastic forcing*. We call it *stochastic convolution* and denote it $\Psi(t)$.

Note that the notion of mild solution for the other equations follows from a similar argument and is easily adaptable.

1.2. **Construction of the stochastic convolution.** Let us now explain the construction and the meaning of the stochastic convolution Ψ. First, we define the *L^2-cylindrical Wiener process* $W(t,x)$. This stochastic process can be essentially understood throughout this course as a Brownian motion. Indeed, on $\mathbb{R}^d$, we define it by

$$W(t,x) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n(x),$$

where e_n is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $(\beta_n)_{n\in\mathbb{N}}$ is a family of independent real-valued Brownian motions (see below for the definition). Besides, on $\mathbb{T}^d$, we define $W(t)$ by

$$W(t,x) = \sum_{n\in\mathbb{Z}^d} \beta_n(t) e^{2\pi i n\cdot x},$$

where $(\beta_n)_{n\in\mathbb{Z}^d}$ are again independent real-valued Brownian motions.

Remark 1.1. Note that, for Schrödinger's equation, we may choose (β_n) to be independent complex-valued Brownian motions assuming furthermore that, if we write $\beta_n = \operatorname{Re}\beta_n + i\operatorname{Im}\beta_n$, then $\operatorname{Re}\beta_n$ and $\operatorname{Im}\beta_n$ are independent real-valued Brownian motions.

Suppose from now on, without loss of generality, that $\mathcal{M} = \mathbb{R}^d$, since the computations are essentially the same on the torus. From the previous definition of W, we can write $dW(t') = \sum_{n\in\mathbb{N}} e_n(x) d\beta_n(t')$ and, since $e_n \in L^2(\mathcal{M})$ for any n, we can apply Φ on e_n. Thus, we define the stochastic convolution $\Psi(t)$ as

$$\Psi(t) = \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\big[\sigma(u)(t')\Phi(e_n)\big] d\beta_n(t') \tag{1.8}$$

However, then again we stumble onto a new problem since the integral we get is now a stochastic integral that we need to define. To do so, we will use the so-called *Wiener* and *Ito integrals*. Besides, we will also give a rigorous definition of a Brownian motion.

2. Construction of stochastic integrals

2.1. **Brownian motion and Wiener integral.** In this subsection, we define the Wiener integral, but first, we need to rigorously introduce the Brownian motion:

Definition 2.1. A *Brownian motion* $\{B(t)\}_{t\in\mathbb{R}_+}$ is a stochastic process satisfying

(1) $B(0) = 0$ almost surely,
(2) $B(t)-B(s) \sim \mathcal{N}(0, t-s)$ for any $t > s$, where $\mathcal{N}(0, t-s)$ denotes a gaussian random variable, with mean 0 and variance $(t-s)$,
(3) $B(t_1) - B(s_1)$ and $B(t_2) - B(s_2)$ are independent, provided that $t_1 \geq s_1 \geq t_2 \geq s_2$.

This definition gives quite useful properties on the Brownian motion:

Proposition 2.2. (1) A Brownian motion is almost surely continuous.
(2) Let B a Brownian motion. For any natural integer k and any $t > s$, we have

$$E[|B(t) - B(s)|^{2k}] = \frac{(2k)!}{2^k k!}(t-s)^k.$$

In general, we also have for any $p \geq 1$

$$E[|B(t) - B(s)|^p] \sim_p |t-s|^{\frac{p}{2}}.$$

Besides, we also have the following theorem:

Theorem 2.3 (Kolmogorov's continuity critterion)**.** *Let $\{X_t\}_{t\in\mathbb{R}_+}$ a stochastic process with values in a metric space. Suppose it satisfies*

$$E[d(X_s, X_t)^p] \leq C_0|t-s|^{1+\alpha}$$

for some $p > 1$ and $\alpha > 0$. Then,

$$P\left(\sup_{s\neq t} \frac{d(X_s, X_t)}{|s-t|^{\frac{\alpha}{p}-\varepsilon}} \geq \lambda\right) \leq \frac{c_1}{\lambda^p}$$

for any $0 < \varepsilon < \frac{\alpha}{p}$ and $\lambda > 0$. Namely, X_t is almost surely $\left(\frac{\alpha}{p} - \varepsilon\right)$-Hölder continuous and, in particular, continuous.

As an example, let us apply Kolomogorov's continuity criterion to a Brownian motion B. Using property 2 of a Brownian motion, we know that for any $p > 1$

$$E[|B(t) - B(s)|^p] \sim_p |t-s|^{\frac{p}{2}} =: |t-s|^{1+\alpha}.$$

Therefore, Theorem 2.3 tells us that B is $\left(\frac{\alpha}{p} - \varepsilon\right)$-Hölder continuous. The question that remains is how good can be that $\frac{\alpha}{p}$? Observe that

$$\frac{\alpha}{p} = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} \longrightarrow_{p\to\infty} \frac{1}{2},$$

so that, taking p large enough, B is $\left(\frac{1}{2} - \varepsilon\right)$-Hölder continuous.

Remark 2.4. The idea of being $\left(\frac{1}{2}-\varepsilon\right)$-Hölder continuous is that our Brownian motion is γ-Hölder continuous with $\gamma<\frac{1}{2}$ being as close as we want to $\frac{1}{2}$ without ever reaching it. Therefore, we denote this idea $\frac{1}{2}-$. We extend this notation to $x-$ for any real number x to say that we are strictly less than x, but as close to it as we want. We also have the $x+$ counterpart to say that we are strictly more than x but as close to it as we want. Namely, $x-=x-\varepsilon$ and $x+=x+\varepsilon$ for $\varepsilon>0$ small.

From our previous considerations, we then say that a Brownian motion has regularity $\frac{1}{2}-$, and we denote it as

$$B\sim\frac{1}{2}-\,.$$

Now, we call white noise the derivative of a Brownian motion, and since taking a derivative means losing one degree of regularity (this is understandable by using the definition of differentiation), we can see that a white noise dB has regularity $-\frac{1}{2}-$, namely

$$dB\sim-\frac{1}{2}-\,.$$

Remark 2.5. Note that when we define a space-time white noise $\xi=dW$, we use the formula $W(t)=\sum_{n\in\mathbb{N}}\beta_n(t)e_n$ with β_n some Brownian motions. This means that ξ is not unique, it's just *one* space-time white noise among other possible ones. Note also that, from this formula, we can retrieve the Brownian motion β_n with the formula

$$\beta_n(t)=\langle\mathbb{1}_{[0,t]}e_n,\xi\rangle_{L^2_{t,x}}.$$

Indeed, we have by orthonormality of the $(e_m)_{m\in NB}$:

$$\langle\mathbb{1}_{[0,t]}e_n,\xi\rangle_{L^2_{t,x}}=\sum_{m\in\mathbb{N}}\langle\mathbb{1}_{[0,t]}e_n,e_md\beta_m\rangle_{L^2_{t,x}}=\langle\mathbb{1}_{[0,t]},d\beta_n\rangle_{L^2_t}=\beta_n(t).$$

Let us now move onto the construction of the *Wiener integral*, that roughly allows us to integrate a deterministic function against a random measure satisfying some conditions. We want to construct a linear operator $I\colon L^2((a,b))\to L^2(\Omega)$ such that, for any deterministic function $f\in L^2((a,b))$, we can write

$$I(f)=\int_a^b f(t)dB(t)$$

where B is a Brownian motion, and with the following properties:

(1) $E[I(f)]=0$,
(2) $E[(I(f))^2]=\|f\|^2_{L^2((a,b))}=\int_a^b|f(t)|^2dt$.

Namely, we want this operator to be an isometry.

To construct this operator, we start with step functions $f(t)=\sum_{j=1}^n a_{j-1}\mathbb{1}_{[t_{j-1},t_j]}$. Using left-endpoints Riemann sum, we define $I(f)$ by

$$I(f)=\sum_{j=1}^n a_{j-1}\big(B(t_j)-B(t_{j-1})\big)$$

with B a Brownian motion. Then, we get from the definition of B:

$$E[I(f)] = \sum_{j=1}^{n} a_{j-1} E\big[B(t_j) - B(t_{j-1})\big] = 0,$$

and by independence

$$\begin{aligned}
E[(I(f))^2] &= E\Big[\sum_{j=1}^{n}\sum_{k=1}^{n} a_{j-1}a_{k-1}\big(B(t_j) - B(t_{j-1})\big)\big(B(t_k) - B(t_{k-1})\big)\Big] \\
&= \sum_{j\neq k} a_{j-1}a_{k-1}E\big[B(t_j) - B(t_{j-1})\big]E\big[B(t_k) - B(t_{k-1})\big] + \sum_{j=k} a_{j-1}^2 E\Big[\big(B(t_j) - B(t_{j-1})\big)^2\Big] \\
&= 0 + \sum_{j=k} a_{j-1}^2\big(t_j - t_{j-1}\big) \\
&= \int_a^b |f(t)|^2 dt
\end{aligned}$$

so the properties are satisfied. Now, we can define $I(f)$ for any $f \in L^2(a,b)$. To do so, we approximate f by step functions f_n, and then we define $I(f) = \lim_{n\to\infty} I(f_n)$ in $L^2(\Omega)$.

Remark 2.6. (1) If $f \in C^1$, we can define $I(f)$ as a *Paley-Wiener-Zygmund integral*:

$$I(f) = \int_a^b f dB = -\int_a^b f'(t)dB(t) + f(b)B(b) - f(a)B(a).$$

(2) If $f \in C^{\frac{1}{2}+}$, we can define $I(f)$ as a *Young integral*, which is a generalization of the Riemann-Stieltjes integral.

(3) If B is complex valued, the former definition yields

$$E[(I(f))^2] = 2\|f\|_{L^2(a,b)}^2.$$

2.2. **Ito integral.** Let us now explain the construction of the Ito integral, that roughly allows us, under certain conditions, to integrate a random function against a random measure. To that end, we need to use *filtrations*.

Definition 2.7 (Filtration)**.** Let $(\Omega, \mathcal{F}, P)$ a probability space. A filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a sequence of σ-fields such that, for any $t_1 \leq t_2$, we have

$$\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}.$$

From this definition, we say that a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is *adapted*, or *non-anticipating*, if, for any $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

Furthermore, we say that a stochastic process X is *progressively measurable* if the function

$$X\colon (t,\omega) \in [0,T] \times \Omega \to X(t,\omega)$$

is $\mathcal{B}_{[0,T]} \otimes \mathcal{F}_t$-measurable.

From these definitions, observe that a progressively measurable process is always adapted. However, an adapted process is not always progressively measurable. But if a process is adapted and has left (or right) continuity, then it is progressively adapted. For instance, an adapted and *càdlàg*[1] process is progressively measurable.

[1] *càdlàg* stands for "continuité à droite et limite à gauche" and means "right continuity, with a limit from the left".

Now, the idea for constructing the Ito integral is the followiing: assume that a Brownian motion B satisfies:

- $B(t)$ is $\mathcal{F}_t$-measurable for any $t \geq 0$,
- $B(t) - B(s)$ is independent of $\{\mathcal{F}_s\}_{s<t}$ for any $t \geq 0$.

Set $L^2_{ad}\big([a,b] \times \Omega\big)$ the set of all functions $f(t,\omega)$ such that f is adapted to $\{\mathcal{F}_t\}$ and

$$\int_a^b E[f^2(t)]dt < \infty.$$

Then, we define the Itô integral on $L^2_{ad}\big([a,b] \times \Omega\big)$ so that we can denote it

$$I(f) = \int_a^b f(t)dB(t)$$

and it satisfies the following properties:

(1) $I(f)$ is centered
$$E\big[I(f)\big] = 0,$$
(2) and we have the Ito isometry:
$$E\big[(I(f))^2\big] = \int_a^b E\big[f^2(t)\big]dt.$$

Step 1: To do so, we use a similar idea as for Wiener integral and start with step stochastic processes:

$$f(t,\omega) = \sum_{j=1}^n a_{j-1}(\omega)\mathbb{1}_{[t_{j-1},t_j)}(t)$$

with (a_j) such that

(1) a_j is $\mathcal{F}_{t_j}$-measurable, so that it does not "peek in the future",
(2) $\sum a_j^2 < \infty$.

Then, we define the Ito integral as the left endpoint sum:

$$I(f)(\omega) = \sum_{j=1}^n a_{j-1}(\omega)\big(B(t_j) - B(t_{j-1})\big)$$

and we will show our two conditions are satisfied.

But first, we need to recall a few properties about *conditional expectation.*

Definition 2.8. Let $X \in L^1(\Omega, \mathcal{F})$ and $\mathcal{G} \subset \mathcal{F}$ a sub σ-field. The conditional expectation of X given $\mathcal{G}$ is defined by the unique random variable Y such that

(1) Y is $\mathcal{G}$-measurable,
(2) for any $A \in \mathcal{G}$, $\int_A XdP = \int_A YdP$.

We denote this random variable Y by $E[X|\mathcal{G}]$.

Remark 2.9. In the previous definition, Y is given by the Radon-Nikodym theorem in the following sense:

Since $X \in L^1(\Omega, \mathcal{F})$, the application

$$\mu \colon A \in \mathcal{G} \to \int_A XdP$$

is absolutely continuous with respect to $P|_{\mathcal{G}}$. Therefore, Radon-Nikodym theorem gives the existence of a $\mathcal{G}$-measurable function Y such that

$$d\mu = Y dP|_{\mathcal{G}} \iff \forall A \in \mathcal{G}, \quad \mu(A) = \int_A Y dP|_{\mathcal{G}}.$$

Note also that if $X \in L^2(\Omega, \mathcal{F})$, then $E[X|\mathcal{G}] = P_{L^2(\Omega,\mathcal{G})}(X)$.

Proposition 2.10. (1) $E\big[E[X|\mathcal{G}]\big] = E[X]$[2],

(2) If X is $\mathcal{G}$-measurable, $E[X|\mathcal{G}] = X$,

(3) If X and $\mathcal{G}$ are independent, in the sense that $\{X \in V\}$ and $A \in \mathcal{G}$ are independent for any $V \in B_{\mathbb{R}}$ and $A \in \mathcal{G}$, then

$$E[X|\mathcal{G}] = E[X],$$

(4) if Y is $\mathcal{G}$-measurable and $E[XY] < \infty$, then

$$E[XY|\mathcal{G}] = Y E[X|\mathcal{G}].$$

Using these properties, we can prove the two conditions needed for the Ito integral:

Proof of the two properties of Ito integral. We prove them separately.

Part 1: the integral is centered. Indeed, using properties (1), (4) and (3) of the conditional expectation, we get, for any j

$$\begin{aligned} E\big[a_{j-1}\big(B(t_j) - B(t_{j-1})\big)\big] &= E\Big[E\big[a_{j-1}\big(B(t_j) - B(t_{j-1})\big)\big|\mathcal{F}_{t_{j-1}}\big]\Big] \\ &= E\Big[a_{j-1}E\big[\big(B(t_j) - B(t_{j-1})\big)\big|\mathcal{F}_{t_{j-1}}\big]\Big] \\ &= E\Big[a_{j-1}E\big[B(t_j) - B(t_{j-1})\big]\Big] = 0 \end{aligned}$$

by definition of the Brownian motion B.

Part 2: the Ito isometry. Let i, j be two integers. Suppose first that $i \neq j$ and, without loss of generality, that $i < j$. Then, we have from propertie (1), (4) and (3) of the conditional expectation:

$$\begin{aligned} &E\Big[a_{i-1}a_{j-1}\big(B(t_i) - B(t_{i-1})\big)\big(B(t_j) - B(t_{j-1})\big)\Big] \\ &\qquad = E\Big[a_{i-1}a_{j-1}\big(B(t_i) - B(t_{i-1})\big)E\big[B(t_j) - B(t_{j-1})|F_{t_{j-1}}\big]\Big] = 0 \end{aligned}$$

Now, if we assume that $i = j$, the same properties give

$$\begin{aligned} E\big[a_{j-1}^2\big(B(t_j) - B(t_{j-1})\big)^2\big] &= E\Big[a_{j-1}^2 E\big[\big(B(t_j) - B(t_{j-1})\big)^2|\mathcal{F}_{t_{j-1}}\big]\Big] \\ &= E\big[a_{j-1}^2\big](t_j - t_{j-1}) \end{aligned}$$

and the result follows by summation over j. $\square$

[2] This property is often used in computing an expectation by conditioning.

Step 2: Now, we can define Ito integral for any function $f \in L^2_{ad}\big([a,b]\times\Omega\big)$. Indeed, for any function f in that space, there exists a sequence $\{f_n\}$ of step stochastic processes converging to f in $L^2_{ad}\big([a,b]\times\Omega\big)$. Then, we define the Ito integral by

$$I(f) = \int_a^b f(t)dB := \lim_{n\to\infty} I(f_n).$$

Then, we have a few properties that come around naturally:

Proposition 2.11. (1) The operator I is linear.

(2) $E[I(f)] = 0$ for any $f \in L^2_{ad}\big([a,b]\times\Omega\big)$.

(3) There is the so-called *Ito isometry*: for any $f \in L^2_{ad}\big([a,b]\times\Omega\big)$, we have

$$E\big[(I(f))^2\big] = \int_a^b E\big[(f(t))^2\big]dt.$$

(4) For any $f, g \in L^2_{ad}\big([a,b]\times\Omega\big)$, we have

$$E\left[\int_a^b f(t)dB \int_a^b g(t)dB\right] = \int_a^b E\big[f(t)g(t)\big]dt.$$

To sum up, the operator $I\colon L^2_{ad}\big([a,b]\times\Omega\big) \to L^2(\Omega)$ is an isometry.

Using Ito integral, we can also define an equivalent to "Taylor expansion of order 2" for a stochastic process using the following theorem:

Theorem 2.12 (Ito's lemma)**.** *Let X a stochastic process. Assume X satisfies the following stochastic differential equation*

$$dX = fdt + gdB$$

where f and g are deterministic functions. Then, considering $F(X)$ for F a C^2 function, we have

$$\begin{aligned} dF &= \partial_x F dX + \frac{1}{2}\partial_x^2 F (dX)^2 \\ &= \partial_x F (fdt + gdB) + \frac{1}{2}\partial_x^2 F g^2 dt \end{aligned}$$

For instance, we can apply Ito's lemma on the case where

$$F(x) = \frac{x^2}{2} \quad \text{and} \quad X = B.$$

We get then

$$\frac{B^2}{2}(t) = \int_0^t BdB + \frac{1}{2}t \iff \int_0^t BdB = \frac{B^2}{2}(t) - \frac{1}{2}t.$$

The term $-\frac{1}{2}t$ on the righthand side of the previous equation is called *Ito correction.* We see then that the behaviour of a stochastic integral can be somewhat different from the behaviour of deterministic integral.

Remark 2.13. The Ito correction that appears in Ito integral can be removed if we use a different construction, such as *Stratonovich integral.* In this case, we define the integral with the midpoint sum, and it yields, for instance,

$$\int_0^t B \circ dB = \frac{B^2(t)}{2}.$$

SPDE'S WITH MULTIPLICATIVE NOISE

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1. Function Spaces

1.1. **Non-Homogeneous Sobolev Spaces.** In the study of elliptic and parabolic PDE's the Sobolev space $W^{k,p}$ is defined to be the set of measurable functions who, along with their first k weak derivatives, lie in L^p, where k and p are positive integers. Denote by $\mathcal{S}$ the *Schwartz space*, that is, the space of rapidly decaying C^∞ functions on $\mathbb{R}^d$. Over this space of functions, the definition of the Sobolev space can be extended to fractional exponents $s \in (0,1)$. More precisely, this is the space for which the norm,

$$||f||_{W^{s,p}} = ||f||_L^p + \left(\int \int \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} \right)^{1/p}, \tag{1.1}$$

is bounded for $f \in \mathcal{S}(\mathbb{R}^d)$. Next, recall the Fourier transform characterisation of the space $H^k = W^{k,2}$ wherein, for complex valued functions it is proven that for any function $g \in L^2(\mathbb{R}^d)$ that $g \in H^k(\mathbb{R}^d)$ if and only if $(1+|y|^s)\,\hat{f} \in L^2(\mathbb{R}^d)$. The Schwartz space has the nice property that it makes the Fourier transform into an automorphism (an isomorphism from a space to itself) which allows us to extend the definition of the *fractional Sobolev space* to all $s \in \mathbb{R}$.

Definition 1.1 (Fractional Sobolev Space: $H^s(\mathbb{R}^d)$)**.** *For $s \in \mathbb{R}$ define the the fractional Sobelov space to be*

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d) \;:\; \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \left| \hat{f}(\xi) \right|^2 d\xi < \infty \right\} \tag{1.2}$$

where f are Schwarz functions and $\langle \cdot \rangle = \left(1 + |\cdot|^2\right)^{\frac{1}{2}}$, what are known as Japanese Brackets. In addition the space is equipped with the following norm,

$$||f||_{H^s} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \left| \hat{f}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}. \tag{1.3}$$

The above definition corresponds to the case where $p = 2$, but by using the *Bessel Potential* we can go one step further and define the fractional Sobolev space for all $1 < p < \infty$.

Definition 1.2 (Bessel Potential Space $W^{s,p}$, or L^p_s)**.** *For $s \in \mathbb{R}$ and $1 < p < \infty$, define the norm of the Bessel potential space by,*

$$||f||_{W^{s,p}} = ||\langle \nabla \rangle^s f||_{L^p}, \tag{1.4}$$

where $\widehat{\langle \nabla \rangle^s f}(\xi) = \langle \xi \rangle^s \hat{f}(\xi)$ denotes the Bessel potential of order $-s$. Then the Bessel potential space is the set of Scwhartz functions f for which $||\langle \nabla \rangle^s f||^2_{L^p} < \infty$.

Remark 1.3. *The spaces $W^{s,p}$ and H^s are standard in the study of dispersive PDE's but* (1.3) *and* (1.4) *are generally different from* (1.1)*, which is used in the study of elliptic and parabolic PDE's and is obtained by real interpolation. The spaces defined in* (1.4) *and* (1.1) *are the same when s is an integer and $1 < p < \infty$* [8][6]*, in which case the norms correspond to that in the classical definition of the Sobolev space introduced for instance in* [5].

1.2. **Homogeneous Sobolev Spaces.** Next we introduce the Homogeneous analogues to the above spaces. The homogeneous version of $H^s(\mathbb{R}^d)$ is denoted by $\dot{H}^s(\mathbb{R}^d)$ and defined as the completion of $\mathcal{S}(\mathbb{R}^d)$ under the norm

$$||f||_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{1.5}$$

Likewise, this extends to all $1 < p < \infty$ whence we denote the homogeneous version of $W^{s,p}(\mathbb{R}^d)$ by $\dot{W}^{s,p}(\mathbb{R}^d)$ and define it as the closure of $\mathcal{S}(\mathbb{R}^d)$ under the norm

$$||f||_{\dot{W}^{s,p}} = |||\nabla|^s f||_{L^p}\,, \tag{1.6}$$

where $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ denotes the *Reisz potential* of order $-s$. This space is thus known as the Reisz potential space.

Remark 1.4. *On the Taurus $\mathbb{T}^d$ the Fourier transform is simply the fourier coefficient and the frequency ξ would be a member of $\mathbb{Z}^d$. However, in $\mathbb{R}^d$ the weight of ξ is zero when $\xi = 0$. Hence,* (1.6) *is not a norm but a semi-norm over the Schwartz space. For instance, if $\hat{f}$ is a distribution supported at $\xi = 0$, then*

$$||f||_{\dot{W}^{s,p}} = 0, \quad \forall s > 0.$$

The only distribution supported at $\xi = 0$ is the Dirac delta and its derivatives, whose inverse Fourier transform is the set of all polynomials, since

$$\mathcal{F}^{-1}(\delta) = 1, \mathcal{F}^{-1}(\delta') = -2i\pi x, \ldots.$$

In order to make (1.6) *a norm we must work in the space of tempered distributions, the dual of the Schwartz space, denoted $\mathcal{S}'$. Additionally, to remove zeros, we have to quotient out the polynomials, hence identifying functions that differ by a polynomial (see definition 1.5 below).*

Definition 1.5 (Reisz Potential Space $\dot{W}^{s,p}$, or $\dot{L}^p_s$)**.** *Let $\mathcal{S}'$ be the set of tempered distributions, the set of continuous linear functionals on $\mathcal{S}$, and let $\mathbf{P}$ be the set of all polynomials. Then for $s \in \mathbb{R}$ and $1 < p < \infty$, for $f \in \mathcal{S}'\backslash\mathbf{P}$ define the norm of the Reisz potential space by,*

$$||f||_{\dot{W}^{s,p}} = |||\nabla|^s f||_{L^p}\,, \tag{1.7}$$

where $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ denotes the Reisz potential of order $-s$. Here $\mathcal{S}'\backslash\mathbf{P}$ means we quotient out the polynomials. Then the Reisz potential space is the set of tempered distributions f quotient polynomials for which $|||\nabla|^s f||^2_{L^p} < \infty$.

Definition 1.5 is only included for completeness. In practice, particularly in the study of PDE's and SPDE's, it doesn't matter since we work in L^p and often L^2 which means that the functions have decay properties in the limit corresponding to the Sobolev order p. Hence instead of considering tempered distributions as in the formal definition we can work in Schwartz space and apply boundary conditions that the function be supported on a finite domain, this is equivalent to the definitions in (1.5) and (1.6). This pre-empts the need for the considerations in remark 1.4, which are nonetheless necessary in analysis.

2. Some Results on Sobolev Spaces

First we introduce the fractional extension of the *Gagliardo–Nirenberg–Sobolev inequality* which leads us to the *Sobolev embedding theorem.* Finally we state the algebra property.

Theorem 2.1 (Sobolev Inequality)**.** *Let* $1 < p < q < \infty$ *such that* $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$ *then for* $f \in \mathcal{S}(\mathbb{R}^d)$

$$||f||_{L^q(\mathbb{R}^d)} \lesssim ||f||_{\dot{W}^{s,p}(\mathbb{R}^s)}, \tag{2.1}$$

while for $f \in \mathcal{S}(\mathbb{T}^d)$*,*

$$\left|\left|f - \bar{f}\right|\right|_{L^q(\mathbb{T}^d)} \lesssim ||f||_{\dot{W}^{s,p}(\mathbb{T}^d)} \tag{2.2}$$

where $\bar{f}$ *is the average value of* f *on it's support.*

Theorem 2.2 (Sobolev Embedding)**.** *If* $1 < p < \infty$ *and* $sp > d$ *then*

$$||f||_{L^\infty} \lesssim ||f||_{W^{s,p}}. \tag{2.3}$$

Theorem 2.3 (Algebra Property)**.** *If* $1 < p < \infty$ *and* $sp > d$ *then*

$$||fg||_{W^{s,p}(\mathbb{R}^n)} \lesssim ||f||_{W^{s,p}(\mathbb{R}^n)} ||g.||_{W^{s,p}(\mathbb{R}^n)}. \tag{2.4}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$*.*

A proof of the embedding theorem can be found in [4], section 4.4. The proof of the Algebra property of $W^{s,p}$ can be found in [12]. For H^s, which does not require Littlewood-Paley theory a proof can be found on page 18 of [9]. Yet another proof, without para-products and credited to Marcinkiewicz and Zygmund can be found in [6], Theorem 5.5.1.

3. Stochastic Convolution Revisited

Consider the *linear stochastic Schrödinger equation*,

$$i\partial_t u - \nabla u = \sigma(u)\Phi\xi. \tag{3.1}$$

The solution to this equation is given by the stochastic convolution,

$$\begin{aligned}\Psi(t) &= \int_0^t S\left(t - t'\right)\left(\sigma(u)(t')\Phi \; dW(t')\right) \\ &= \sum_{n\in\mathbb{N}} \int_0^t S\left(t - t'\right)\left(\sigma(u)(t')\Phi(e_n)\right) \; d\beta_n(t')\end{aligned} \tag{3.2}$$

where Φ is the *'smoothing operator'*. In the study of Schröodinger's equation we work in H^s because it is a Hilbert space, a vector space equipped with a metric which is closed under the norm induced by the inner product. Hilbert spaces are used in the mathematically rigorous formulation of quantum mechanics.

Definition 3.1 (Hilbert-Schmidt Norm). *Given any operator $T : X \to Y$, with X and Y Hilbert spaces we say that T is Hilbert-Schmidt if, for any orthonormal basis e_n of X, it holds that*

$$||T||_{HS(X,Y)} = \left(\sum_n ||Te_n||_Y^2 \right)^{\frac{1}{2}} < \infty. \tag{3.3}$$

$||T||_{HS(X,Y)}$ is known as the Hilbert-Schmidt norm.

3.1. **Additive Case.** Although this is a course on SPDE's with multiplicative noise, first we will focus on the easier case, where the multiplicative factor is just be one i.e. where $\sigma(u) \equiv 1$ and we have what is known as the additive case. Hence the stochastic convolution in this case becomes

$$\Psi(t) = \int_0^t S(t-t')\Phi \; dW(t'). \tag{3.4}$$

We want to find out the regularity of this equation, which is the space that it's sample paths will belong to almost surely.

Proposition 3.2. *On $\mathbb{R}^d$ for $\Phi \in HS\left(L^2, H^s\right)$, and $s \in \mathbb{R}$ we have that*

(1) *$\Psi \in C_t H_x^s$, almost surely.*
(2) *Let $1 \leq q < \infty$ and $d \geq 3$ then for finite $2 \leq r \leq \frac{2d}{d-2}$ we have*

$$\Psi \in L_T^q W_x^{s,r} = L^q \left([0,T]; W^{s,r}(\mathbb{R}^d)\right) \tag{3.5}$$

almost surely for any $T > 0$.

Part (1) says that Ψ is continuous in time and H^s in x if the smoothing operator is Hilbert-Schmidt. That is sometimes enough to prove well posedness but for rougher initial data we need (2) in which we have the addition of integrability up to r over x, as well as differentiability up to s because

$$||f||_{W^{s,r}} = ||\langle\nabla\rangle^s f||_{L_x^r}.$$

Note that the upper bound on r above is precisely the index required for the (non-homogeneous) Sobolev embedding, i.e. it is the largest r for which it holds that $||\cdot||_{L^r} \leq ||\cdot||_{H^1}$. It is also the maximum r for which the *Strichartz estimate* hold, and as you will see, this is used in the proof.

Notation 3.1. *When capital letters are used in the subscript then we are referring to the interval from zero to the value indicated by the subscript. For example, the T in L_T^q refers to the interval $[0,T]$.*

Remark 3.3. *On $\mathbb{T}^d$ part (1) of proposition 3.2 holds true and we also have $\Psi \in C_t W_x^{s-\varepsilon,\infty}(\mathbb{T}^d)$, $\forall \varepsilon > 0$. This holds because the taurus is a bounded domain and in that case it follows from Hölder's inequality that L^∞ is the strongest norm in terms of the integrability of a function. This implies that the L^∞ norm controls all other norms, a statement that also true in $\mathbb{R}^d$ for low dimensions $d = 1, 2$ but not for $d > 2$ because in higher dimensions the decay of the functions are not sufficient to determine integrability.*

Proof of Proposition 3.2 (1). For part one, we begin by proving that $\Psi \in C_t$ almost surely. First we have,

$$\langle \nabla \rangle^s \Psi(t) = \sum_{n \in \mathbb{N}} \int_0^t S(t-t') \langle \nabla \rangle^s \Phi(e_n) d\beta_n(t') \tag{3.6}$$

which can be rewritten in the L^2 norm with s derivatives as,

$$\mathbb{E}\left[||\Psi(t)||^2_{H^s} \right] = \mathbb{E}\left[||\langle \nabla \rangle^s \Psi(t)||^2_{L^2} \right] = \int \int \cdots dx dP. \tag{3.7}$$

It follows that,

$$\begin{aligned}
\mathbb{E}\left[||\Psi(t)||^2_{H^s} \right] &= \int \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathbb{E}\left[\int_0^t \cdots d\beta_n(t_1) \overline{\int_0^t \cdots d\beta_m(t_2)} \right] dx \\
&= 2 \sum_{n \in \mathbb{N}} \int_0^t \left|\left| S(t-t') \Phi(e_n) \right|\right|^2_{H^s} dt' \\
&= 2t \left|\left| \Phi \right|\right|^2_{HS(L^2;H^2)},
\end{aligned}$$

where $S(t-t') = e^{-i(t-t')|\xi|^2}$ on the Fourier side and can be dropped as it is a unitary operator, see (4.4) on the next page. We now move on to continuity, which we want to prove for a fixed T. For this T we want to show that $\Psi \in C_t H^s_x$ almost surely. We cannot show that it holds everywhere since the set of T is uncountable, hence we can't say that there is a set of probability one for which this holds for all T. To proceed we use Kolmogorov's continuity criterion. First we rewrite in the L^2 norm as before, and note that for $h > 0$,

$$\mathbb{E}\left[\int_0^{t+h} \cdots d\beta_n(t_1) \overline{\int_0^t \cdots d\beta_m(t_2)} \right] \sim 2t$$

since the second integral only goes up to the shorter timescale t, which implies that

$$\begin{aligned}
\mathbb{E}\left[||\Psi(t+h) - \Psi(t)||^2_{H^s} \right] &= \langle \nabla \rangle^s (\Psi(t+h)\overline{\Psi(t+h)} - \Psi(t+h)\overline{\Psi(t)} \\
&\quad - \Psi(t)\overline{\Psi(t+h)} + \Psi(t)\overline{\Psi(t)}) \\
&\sim 2(t+h) \left|\left| \Phi \right|\right|^2_{HS} - 2t \left|\left| \Phi \right|\right|^2_{HS} - 2t \left|\left| \Phi \right|\right|^2_{HS} + 2t \left|\left| \Phi \right|\right|^2_{HS} \\
&= 2h \left|\left| \Phi \right|\right|^2_{HS}
\end{aligned}$$

for $h > 0$. Note that Ψ is Gaussian, hence an homogeneous Wiener chaos of order one, meaning that we can apply order hypercontractivity to get

$$\begin{aligned}
\mathbb{E}\left[||\Psi(t+h) - \Psi(t)||^p_{H^s} \right] &\leq C_p \left(\mathbb{E}\left[||\Psi(t+h) - \Psi(t)||^2_{H^s} \right] \right)^{p/2} \\
&\lesssim |h|^{p/2} \left|\left| \Phi \right|\right|^p_{HS(L^2;H^s)}, \quad \text{for } p >> 1.
\end{aligned}$$

Then by Kolmogorov's continuity criterion it follows that $\Psi \in C_t H^s_x$, almost surely. ☐

4. Properties of the Solution to Schrödinger's Equation

Firstly, recall the linear Schrödinger equation

$$\begin{cases} i\partial_t u = \nabla u \\ u|_{t=0} = u_0, \end{cases} \tag{4.1}$$

which has the solution

$$u(t) = S(t)u_0, \quad \text{where } S(t) = e^{-it\Delta}, \tag{4.2}$$

and

$$\widehat{S(t)f}(\xi) = e^{it|\xi|}f. \tag{4.3}$$

This last term introduces oscillation which, for large ξ diverges as time goes by. The solution (4.2) implies the conservation of homogeneous Sobolev norms, i.e. that $S(t)$ is unitary in $\dot{H}^s$ which can be written

$$||S(t)f||_{\dot{H}^s} = ||f||_{\dot{H}^s} \quad \forall s \in \mathbb{R}. \tag{4.4}$$

Further, the Fourier transform (4.3) can be used to prove the dispersive estimate for $\xi \in \mathbb{R}^d$, which can be written as:

$$||S(t)f||_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}} ||f||_{L^1_x}, \quad t \neq 0. \tag{4.5}$$

This is in a sense a smoothing off over long time scales, since large (unbounded) peaks will be eliminated by t in the denominator on the right hand side which bounds the infinity norm on the left.

Proposition 4.1 (Strichartz Estimate)**.** *Let $2 \leq q, r \leq \infty$ and $(q, r, d) \neq (2, \infty, 2)$. We say that the ordered pair (g, r) is admissible if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

For admissible pairs $(g, r), (\tilde{g}, \tilde{r})$ we have the following estimates:

(1) *The Homogeneous Case. For $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$||S(t)f||_{L^q_t L^r_x} \lesssim ||f||_{L^2_x} \tag{4.6}$$

Note that if $d = 1$ then $r \leq \infty$, if $d = 2$ then $r < \infty$ and if $d \leq 3$ then $r \leq \frac{2d}{d-2}$.

(2) *The Dual Case.*

$$\left|\left|\int_{\mathbb{R}} S(-t)F(t)dt\right|\right|_{L^2_x} \lesssim ||F||_{L^{q'}_t L^{r'}_x} \tag{4.7}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

(3) *Non-homogeneous/Retarded Case.*

$$\left|\left|\int_0^t S(t-t')F(t')dt'\right|\right|_{L^q_t L^r_x} \lesssim ||F||_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{4.8}$$

If $f \in L^2_x$ then $S(t)f \in L^q_t L^r_x$ and in particular, $S(t)f \in L^r_x$, almost surely for $t \in \mathbb{R}$. On $\mathbb{R}^d$ the proof uses the unitary and dispersive estimated along with the Sobeolov inequality (4.4), (4.5) and (2.1) respectively. Some Strichartz estimates extend to $\mathbb{T}^d$, details of which can be found in the work by Jean Bourgain [1] and collaboratively with Ciprian Demeter in [2], the latter being particularly influential. The proof of the non-endpoint case can be found in [11] and [13], while that for the endpoint estimates can be found in [7] for Schrödinger's equation in $d \geq 3$ and the wave equation in $d \geq 4$. There are other kinds of Strichartz estimates for the wave equation in specific cases, such as in [3] for critically decaying potentials. Finally for the heat equation there is the analogous result known as the *Schauder estimate*,

$$||P(t)f||_{W^{s,q}} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}} ||f||_{L^p_x}, \quad \text{for } t > 0. \tag{4.9}$$

We are now in a position to prove the rest of Proposition 3.2.

Proof of Proposition 3.2 (2). Recall again that $\Psi(t) = \int_0^t S(t-t')\Phi dW(t')$ and we assume that $\Phi \in HS(L^2; H^s)$. Additionally we have

$$\Phi W = \sum_{n\in\mathbb{N}} \beta_n(t)\phi(e_n), \quad \text{where } \{e_n\}_{n\in\mathbb{N}} = \text{O.N.B of } L^2(\mathbb{R}^d).$$

Like previously in (3.7) we can rewrite the H^s norm as the L^2 norm by taking s derivatives. Further, independence of the Gaussians, order hypercontractivity and Ito's isometry followed by the change of variables $t - t' = \tau$, we have that,

$$||\langle\nabla\rangle^s\Psi(t,x)||_{L^2(\Omega)} = \left|\left|\left(\int_0^t |S(t-t')\langle\nabla\rangle^s\Phi(e_n)(x)|^2 dt'\right)^{1/2}\right|\right|_{l_n^2} \tag{4.10}$$

$$= ||S(\tau)\langle\nabla\rangle^s\Phi(e_n)(x)||_{l_n^2 L_\tau^2([0,t])}. \tag{4.11}$$

From this, for all $p \geq 2$, we get

$$||\langle\nabla\rangle^s\Psi(t,x)||_{L^p(\Omega)} \lesssim p^{1/2}\,||\langle\nabla\rangle^s\Psi(t,x)||_{L^2(\Omega)} \tag{4.12}$$

$$\sim p^{1/2}\,||S(\tau)\langle\nabla\rangle^s\Psi(e_n)(x)||_{l_n^2 L_\tau^2([0,t])}. \tag{4.13}$$

Next, for $q < \infty$ and $r \leq \frac{2d}{d-2}$ (the admissibility condition), for finite $T > 0$ it follows from the Minkowski integral inequality that

$$\left|\left|||\Psi||_{L_T^q W_x^{s,r}}\right|\right|_{L^p(\Omega)} \leq \left|\left|||\langle\nabla\rangle^s\Psi(t,x)||_{L^p(\Omega)}\right|\right|_{L_T^q L_x^r} \tag{4.14}$$

$$\leq C_p \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{l_n^2 L_\tau^2([0,t])}\right|\right|_{L_T^q L_x^r} \tag{4.15}$$

for $p \geq q \vee r = \max\{q, r\}$. Now if we replace $([0,t])$ by $([0,T])$ and apply Minkowski's inequality again we get

$$C_p \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{l_n^2 L_\tau^2([0,t])}\right|\right|_{L_T^q L_x^r} \leq C_p T^{1/q} \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{L_\tau^2([0,t];L_x^r)}\right|\right|_{l_n^2}. \tag{4.16}$$

Given $2 \leq r \leq \infty$, let $(\tilde{q}, r)$ be admissible for some $\tilde{q} \geq 2$ then we can apply Hölder's inequality to obtain

$$C_p T^{1/q} \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{L_\tau^2([0,t];L_x^r)}\right|\right|_{l_n^2} \leq C_p T^{\theta} \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{L_\tau^{\tilde{q}};L_x^r}\right|\right|_{l_n^2}. \tag{4.17}$$

Then by applying Strichartz estimate we get

$$C_p T^{\theta} \left|\left|||S(\tau)\langle\nabla\rangle^s\Phi(e_n)||_{L_\tau^{\tilde{q}};L_x^r}\right|\right|_{l_n^2} \leq C_p T^{\theta}\,||||\Phi(e_n)||_{H^s}||_{l_n^2} \tag{4.18}$$

$$= ||\Phi||_{HS(L^2;H^s)} \tag{4.19}$$

$$< \infty$$

This shows, for finite T, that $\Psi \in L_T^q W_x^{s,r}$ almost surely. □

Remark 4.2. *To be more rigorous in the proof above one would consider a sequence of operators Φ_k converging to Φ but with stronger smoothing properties, then consider the corresponding sequence Ψ_k. The result of the subsequent calculations would be a Cauchy sequence converging to Ψ and having the correct properties. Details can be found in* [10], *lemma 2.1.*

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LECTURE NOTES ON STOCHASTIC PDES WITH MULTIPLICATIVE NOISES

MARTIN KOLODZIEJCZYK

Contents

1. Lecture 3 (Part II)

1.1. **Local Well-Posedness for SNLS with additive noise.** Let us first recall the stochastic non-linear Schröndiger equation (SNLS) with additive noise on $\mathbb{R}^d$ is defined as

$$\begin{cases} i\partial_t u - \Delta u = |u|^{2k}u + \phi\xi, \\ u|_{t=0} = u_0 \in H^s, \end{cases} \tag{1.1}$$

where ξ is a space-time white noise on $\mathbb{R}^d$ and $\phi \in HS\left(L^2; H^s\right)$ with $s \in \mathbb{R}$ and $k \in \mathbb{N}$. Let us also recall that $S(t)f := \mathcal{F}^{-1}(e^{-it|\cdot|^2}\hat{f}(\cdot))$, $t \in \mathbb{R}$, so that if we denote by Ψ the stochastic convolution defined as

$$\Psi(t) := \int_0^t S(t-t')\phi \, dW(t'), \ t \geq 0 \tag{1.2}$$

then $\Psi \in C_T H^s_x \cap L^q_T W^{s,r}_x$, $\mathbb{P}$-a.s. for any given finite $q \geq 1$ and $r \geq 2$ satisfying the following condition

$$\begin{cases} r < +\infty, \, d = 1, 2 \\ r \leq \frac{d}{d-2}, \, d \geq 3. \end{cases}$$

We refer interested readers to [1] or [2] for the proofs of those facts. To prove the local well-posedness of (1.1) we first need to state what we mean by a solution to (1.1).

Definition 1.1. *A H^s-valued stochastic process $u(t)$, $t \in [0,T]$, is said to be a mild solution to* (1.1) *if the following integral equality is satisfied $\mathbb{P}$-a.s.*

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')|u(t')|^{2k}u(t')dt' - i\Psi(t), \, t \in [0,T].$$

Our goal is to prove the local well-posedness (LWP) for SNLS when $s > \frac{d}{2}$(i.e. when H^s has an algebra structure). We aim to show that there exists a local mild solution to (1.1) and, moreover, we look to prove the stability under perturbations with respect to both the initial datum $u_0 \in H^s$ and the noise $\phi \in HS(L^2; H^s)$. To achieve this, we apply the Banach fixed point theorem to the map $\Gamma_{u_0,\phi} : H^s \times C_T H^s \to C_T H^s$ defined as

$$\Gamma_{u_0,\phi}(t) := S(t)u_0 - i\int_0^t S(t-t')|u(t')|^{2k}u(t')dt' - i\Psi(t), \, t \in [0,T]. \tag{1.3}$$

1.1.1. *Local existence and uniqueness for SNLS with additive noise.* To prove local existence and uniqueness for SNLS, we first need to recall some estimates involving the addends in the RHS of (1.3).

i) Let us first note that from the unitarity of $\{S(t)\}_{t\in\mathbb{R}}$ we have

$$\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s}, \, \forall t \in \mathbb{R},$$

so that we infer

$$\|S(t)u_0\|_{C_T H^s} = \|u_0\|_{H^s}.$$

ii) As for the non-linear term, from Minkowski's inequality for multiple integrals, the unitarity of $\{S(t)\}_{t\in\mathbb{R}}$ and the algebra property of H^s(which we know it holds since $s > \frac{d}{2}$) we obtain

$$\left\| \int_0^t S(t-t')|u(t')|^{2k}u(t')dt' \right\|_{H^s} \leq \int_0^t \| |u(t')|^{2k}u(t')\|_{H^s}\, dt'$$

$$\leq \int_0^t \|u(t')\|_{H^s}^{2k+1}\, dt' \leq T\|u\|_{C_T H^s}^{2k+1}, \text{ for all } t \in [0,T],\ \mathbb{P}\text{-a.s.}$$

iii) To conclude, we recall that the stochastic convolution Ψ satisfies the following

$$\left\| \|\Psi\|_{C_T H^s} \right\|_{L^2(\Omega)} \leq C(T)\, \|\phi\|_{HS(L^2;H^s)} \tag{1.4}$$

where $C(T)$ is a suitable constant depending on T. From this we infer the existence of a random variable $\omega \to C(T, \|\phi\|_{HS(L^2;H^s)}, \omega)$ such that

$$\|\Psi\|_{C_T H^s} \leq C(T; \|\phi\|_{HS(L^2;H^s)}, \omega) < \infty,\ \mathbb{P}\text{-a.s.}$$

Once this is in place, from $i)$, $ii)$ and $iii)$ we obtain for $0 \leq T \leq 1$ and for any given $u_0 \in H^s$, $\phi \in HS(L^2; H^s)$ and $u \in C_T H^s$ the following

$$\|\Gamma_{u_0,\phi}(u)\|_{C_T H^s} \leq \|u_0\|_{H^s} + \|\Psi\|_{C([0,1];H^s)} + C_1 T \|u\|_{C_T H^s}^{2k}. \tag{1.5}$$

Similarly, from the algebra property of H^s we obtain for $u, v \in C_T H^s$ and any given $u_0 \in H^s$, $\phi \in HS(L^2; H^s)$ the following

$$\|\Gamma_{u_0,\phi}(u) - \Gamma_{u_0,\phi}(v)\|_{C_T H^s} \leq \int_0^t \left\| |u(t')|^{2k}u(t') - |v(t')|^{2k}v(t') \right\|_{C_T H^s} dt'$$

$$\leq C_2 T \left(\|u\|_{C_T H^s}^{2k} + \|v\|_{C_T H^s}^{2k} \right) \|u - v\|_{C_T H^s}, \tag{1.6}$$

where the second inequality follows form the fact that the term $|u|^{2k}u - |v|^{2k}v$ can be rearranged in the following way $|u|^{2k}u - |v|^{2k}v = P_{2k}(u, \bar{u}, v, \bar{v})(u - v) + Q_{2k}(u, \bar{u}, v, \bar{v})(u - v)$ where P_{2k} and Q_{2k} are polynomials of degree $2k$. If we now set $R = R_\omega := 2\left(\|u_0\|_{H^s} + \|\Psi\|_{C([0,1];H^s)}\right)$ and if $u, v \in \bar{B}_R \subset C_T H^s$ then it follows that

$$\|\Gamma_{u_0,\phi}(u)\|_{C_T H^s} \leq \frac{1}{2}R + C_1 T R^{2k+1},$$

$$\|\Gamma_{u_0,\phi}(u) - \Gamma_{u_0,\phi}(v)\|_{C_T H^s} \leq 2C_2 T R^{2k} \|u - v\|_{C_T H^s}.$$

Therefore, by choosing $T = T_\omega = T(R_\omega)$ small enough so that $C_1 T R^{2k+1} \leq \frac{1}{2}R$ and $2C_2 T R^{2k} \leq \frac{1}{2}$ we obtain that for any given $u_0 \in H^s$ and $\phi \in HS(L^2; H^s)$ the fixed point map $\Gamma_{u_0,\phi} : \bar{B}_R \to \bar{B}_R$ is a contraction. Henceforth, from the Banach fixed point theorem we infer the existence of a unique $u = u(\omega) \in \bar{B}_R$ such that $u = \Gamma_{u_0,\phi}(u)$ on $[0, T_\omega]$.

1.1.2. *Stability under perturbations for SNLS with additive noise.* Let $u_{0,1}, u_{0,2} \in H^s$, $\phi_1, \phi_2 \in HS(L^2; H^s)$ and let be their stochastic convolutions Ψ_1, Ψ_2 defined as $\Psi_j(t) = \int_0^t S(t-t')\phi_j \, dW(t')$, $t \in [0,T]$, $j = 1,2$, respectively. Let $u_j(t)$, $t \in [0,T]$, $j = 1,2$ be the corresponding solutions to (SNLS):

$$\begin{cases} i\partial_t u_j - \Delta u_j = |u_j|^{2k} u_j + \phi_j \xi, \\ u_j|_{t=0} = u_{0,j}. \end{cases}$$

By eventually choosing $T = T_\omega$ even smaller we have that $u_1 = \Gamma_{u_{0,1},\phi_1}(u_1)$ and $u_2 = \Gamma_{u_{0,2},\phi_2}(u_2)$ for all $t \in [0, T_\omega]$, $\mathbb{P}$-a.s. Hence, we infer

$$\begin{aligned}
&\|u_1 - u_2\|_{C_T H^s} = \left\|\Gamma_{u_{0,1},\phi_1}(u_1) - \Gamma_{u_{0,2},\phi_2}(u_2)\right\|_{C_T H^s} \\
&\leq \|u_{0,1} - u_{0,2}\|_{H^s} + \|\Psi_1 - \Psi_2\|_{C_T H^s} \\
&+ \left\| \int_0^{\cdot} S(\cdot - t') \left(|u_1(t')|^{2k} u_1(t') - |u_2(t')|^{2k} u_2(t') \right) dt' \right\|_{C_T H^s} \\
&\leq \|u_{0,1} - u_{0,2}\|_{H^s} + \|\Psi_1 - \Psi_2\|_{C_T H^s} + \frac{1}{2} \|u_1 - u_2\|_{C_T H^s} . \qquad (1.7)
\end{aligned}$$

Where the above chain of inequalities holds for $T = T_\omega$ sufficiently small. Hence, by moving the third addend in (1.7) to the LHS we obtain

$$\|u_1 - u_2\|_{C_T H^s} \leq 2 \left(\|u_{0,1} - u_{0,2}\|_{H^s} + \|\Psi_1 - \Psi_2\|_{C_T H^s} \right). \qquad (1.8)$$

To conclude, it is enough to recall that from exponential Chebyshev's inequality we know that the stochastic convolution satisfies

$$\|\Psi_1 - \Psi_2\|_{C([0,1];H^s)} \leq K \|\phi_1 - \phi_2\|_{HS(L^2;H^s)}, \qquad (1.9)$$

outside a set of probability less than ce^{-cK^2}. Indeed, once we know this, it follows that there exists a non-negative random variable M such that $M < +\infty$, $\mathbb{P}$-a.s. and

$$\|\Psi_1 - \Psi_2\|_{C([0,1];H^s)} \leq M \|\phi_1 - \phi_2\|_{HS(L^2;H^s)}, \ \mathbb{P}\text{-a.s.} \qquad (1.10)$$

Hence, in (1.8) we obtain

$$\|u_1 - u_2\|_{C_T H^s} \leq 2 \left(\|u_{0,1} - u_{0,2}\|_{H^s} + M \|\phi_1 - \phi_2\|_{HS(L^2;H^s)} \right), \ \mathbb{P}\text{-a.s.}$$

which shows the stability under perturbation on the initial datum and the noise.

Remark 1.2.
- *We emphasize that the same proof works on $\mathbb{T}^d$ in place of $\mathbb{R}^d$.*
- *So far we have proven uniqueness of the solution only in $\bar{B}_R \subset C_T H^s$. For the sake of clarity, uniqueness in fact holds in the entire space $C_T H^s$(unconditional uniqueness). To achieve this, two different methods can be implemented.*

Method 1: *Let us first note that if $u \in C_T H^s$ then from the continuity in time of the solution we know that $\lim_{t\to 0^+} \|u(t)\|_{H^s} = \|u_0\|_{H^s} \leq \frac{1}{2}R$. Therefore, by shrinking the time we can apply the uniqueness in $\bar{B}_R$ for small times. To make sure that the uniqueness holds over the whole time interval $[0,T]$ we need to show that the same local existence time(up to a constant factor) can be kept. This can be done by using a continuity/bootstrap argument.*

Method 2: *The second method consists of applying Gronwall's lemma to the difference of two given solutions to (SNLS).*

1.2. **Appendix.** In this part we prove (1.4) and (1.10). Let us begin with (1.4). To this end, let $F(t) := \frac{\Psi(t)}{\|\phi\|_{HS(L^2;H^s)}}$, $t \in [0,T]$, where $T \leq 1$. By proceeding as in the computation of page 14 in Lec.2, since F is a Gaussian process, we obtain

$$\|F(t_1) - F(t_2)\|_{L^p(\Omega;H^s)} \lesssim p^{\frac{1}{2}} \|F(t_1) - F(t_2)\|_{L^2(\Omega;H^s)} \lesssim p^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}}. \tag{1.11}$$

Hence, from the Kolmogorv's continuity criterion we have

$$\mathbb{P}\left(\sup_{0<t\leq T} \frac{\|F(t) - F(0)\|_{H^s}}{|t|^{\frac{\alpha}{p}-\varepsilon}} \geq \lambda \right) \leq \frac{c_1}{\lambda^p} \tag{1.12}$$

where $p \gg 1$, $\alpha = \frac{p}{2} - 1$ and c_1 is a suitable constant. Therefore, from the Layer-Cake theorem we infer

$$\begin{aligned}
\|F\|_{L^2(\Omega;C_T H^s)} &= \int_0^{+\infty} \lambda \mathbb{P}\left(\|F\|_{C_T H^s} \geq \lambda\right) d\lambda \\
&= \int_0^{\infty} \lambda \mathbb{P}\left(\frac{\|F\|_{C_T H^s}}{T^{\frac{\alpha}{p}-\varepsilon}} \geq T^{\varepsilon - \frac{\alpha}{p}} \lambda \right) d\lambda \leq 1 + \int_1^{\infty} \lambda \mathbb{P}\left(\sup_{t\in[0,T]} \frac{\|F(t) - F(0)\|_{H^s}}{|t|^{\frac{\alpha}{p}-\varepsilon}} \geq \lambda \right) d\lambda \\
&\leq 1 + c_1 \int_1^{\infty} \lambda^{1-p}\, d\lambda < +\infty, \text{ provived } p \gg 1.
\end{aligned}$$

Thus, we have $\|F\|_{L^2(\Omega;C_T H^s)} < +\infty$ which implies (1.4). We now turn our attention to proving (1.9). Let Ψ_1, Ψ_2 be the stochastic convolutions with noise ϕ_1 and ϕ_2, respectively. From the linearity with respect to ϕ of the stochastic convolution, by proceeding as in the previous computation we obtain for any finite $p \geq 1$ the following

$$\left\| \|\Psi_1 - \Psi_2\|_{C([0,1];H^s)} \right\|_{L^p(\Omega)} \leq C_p \|\phi_1 - \phi_2\|_{HS(L^2;H^s)}, \tag{1.13}$$

where C_p is a positive constant depending on p. What we are wondering next is: can $C_p \simeq p^{\frac{1}{2}}$? The answer is yes but it takes more work to do. In what follows, for convenience we set $\Psi := \Psi_1 - \Psi_2$ and $\phi := \phi_1 - \phi_2$. To achieve this, for any given $k \in \mathbb{N}$, let $\left\{t_{l,k} \,:\, l = 0, 1, \cdots, 2^k\right\}$ be $2^k + 1$ equally spaced points in $[0,1]$. That is, $t_{0,k} = 0$ and $t_{l,k} - t_{l-1,k} = \frac{1}{2^k}$, for $1 \leq l \leq 2^k$. Once this is in place, we obtain

$$\Psi(t) = \sum_{k=1}^{+\infty} \Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1}) \tag{1.14}$$

for some $l_k = l_k(t) \in \{0, 1, \cdots, 2^k\}$. Such an expansion holds since any given $t \in [0,1]$ can be expanded into its binary expansion as follows $t = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$, where $b_j \in \{0,1\}$ for all $j \in \mathbb{N}$.

Hence, as $t_{l_k,k}$ we can take the binary expansion of t up to order k. Namely, $t_{l_k,k} = \sum_{j=1}^{k} \frac{b_j}{2^j}$,

$k \in \mathbb{N}$. Therefore, by taking the H^s-norm in (1.14) we obtain

$$\sup_{t\in[0,1]} \|\Psi(t)\|_{H^s} \leq \sum_{k=1}^{+\infty} \max_{0\leq l_k \leq 2^k} \left\|\Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1})\right\|_{H^s}.$$

Once we have this, if we fix $p \gg 1$ and let $\{q_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers such that $q_k \geq p$, for all $k \in \mathbb{N}$, then we have

$$\begin{aligned}
&\left\|\|\Psi\|_{C([0,1];H^s)}\right\|_{L^p(\Omega)} \leq \sum_{k=1}^{+\infty} \left\| \max_{0\leq l_k\leq 2^k} \left\|\Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1})\right\|_{H^s}\right\|_{L^p(\Omega)} \\
&\leq \sum_{k=1}^{+\infty} \left(\int_\Omega \sum_{l_k=0}^{2^k} \left\|\Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1})\right\|_{H^s}^{q_k} \, d\mathbb{P}\right)^{\frac{1}{q_k}} \\
&\leq \sum_{k=1}^{+\infty} \left(1 + 2^{\frac{k}{q_k}}\right) \max_{0\leq l_k\leq 2^k} \left\|\Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1})\right\|_{L^{q_k}(\Omega;H^s)} \\
&\lesssim \sum_{k=1}^{+\infty} 2^{-\frac{k}{2}} 2^{\frac{k}{q_k}} q_k^{\frac{1}{2}} \|\phi\|_{HS(L^2;H^s)}.
\end{aligned}$$

If we let $q_k = p + k$, for all $k \in \mathbb{N}$ then we obtain

$$\begin{aligned}
&\left\|\|\Psi\|_{C([0,1];H^s)}\right\|_{L^p(\Omega)} \lesssim \sum_{k=1}^{+\infty} 2^{-\frac{k}{2}} 2^{\frac{k}{p+k}} (p+k)^{\frac{1}{2}} \|\phi\|_{HS(L^2;H^s)} \\
&\lesssim \sum_{k=1}^{+\infty} 2^{-\frac{k}{2}} p^{\frac{1}{2}} k^{\frac{1}{2}} \|\phi\|_{HS(L^2;H^s)} \lesssim p^{\frac{1}{2}} \|\phi\|_{HS(L^2;H^s)}.
\end{aligned}$$

This shows that (1.13) holds. Once this is in place, by applying the exponentital Chebyshev inequality we obtain (1.9) from which we infer the validity of (1.10). Indeed, if (1.9) holds then (1.10) is equivalent to requiring that

$$\mathbb{P}\left(\omega \in \Omega \,:\, \exists K \in \mathbb{R} \text{ s.t. } \|\Psi_1(\omega) - \Psi_2(\omega)\|_{C([0,1];H^s)} \leq K\|\phi_1 - \phi_2\|_{HS(L^2;H^s)}\right) = 1.$$

Hence, we deduce that

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega \,:\, \exists K \in (0,+\infty) \text{ s.t. } \|\Psi_1(\omega) - \Psi_2(\omega)\|_{C([0,1];H^s)} \leq K\|\phi_1 - \phi_2\|_{HS(L^2;H^s)}\right) \\
&= 1 - \mathbb{P}\left(\omega \in \Omega \,:\, \forall n \in \mathbb{N} \text{ it holds } \|\Psi_1(\omega) - \Psi_2(\omega)\|_{C([0,1];H^s)} \geq n\|\phi_1 - \phi_2\|_{HS(L^2;H^s)}\right) \\
&= 1 - \lim_{n\to+\infty} \mathbb{P}\left(\omega \in \Omega \,:\, \|\Psi_1(\omega) - \Psi_2(\omega)\|_{C([0,1];H^s)} \geq n\|\phi_1 - \phi_2\|_{HS(L^2;H^s)}\right) = 1.
\end{aligned}$$

Where the last equality is a consequence of Chebyshev's inequality. Indeed, we have

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega \,:\, \|\Psi_1(\omega) - \Psi_2(\omega)\|_{C([0,1];H^s)} \geq n\|\phi_1 - \phi_2\|_{HS(L^2;H^s)}\right) \\
&\lesssim e^{-cn^2} \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

References

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1. Lecture 4 15/02/2022 (Billy Sumners)

We continue to consider the stochastic nonlinear Schrödinger equation (SNLS) with additive noise on $\mathbb{R}^d$

$$i\frac{\partial u}{\partial t} - \Delta u = |u|^{p-1}u + \Phi\xi$$
$$u|_{t=0} = u_0 \in H^s(\mathbb{R}^d),$$

where $\Phi \in \mathrm{HS}(L^2; H^s)$ is a Hilbert-Schmidt "smoothing" operator. In the previous lecture, we established local well-posedness in the simple case $s > \frac{d}{2}$ which let us use the property that $H^s(\mathbb{R}^d)$ is an algebra. Here, we consider more general s. As usual, we define the stochastic convolution to be

$$\Psi = \int_0^t S(t-t')\Phi \, \mathrm{d}W(t'),$$

where $S(t) = e^{it\Delta}$. Two simple properties of Ψ determined in previous lectures (in particular, a proposition stated in Lecture 2) are that

Proposition 1.1. *Let $s \in \mathbb{R}$. Then*

(1) $\Psi \in C_t H_x^s = C([0,\infty); H^s(\mathbb{R}^d))$ *a.s.,*
(2) *For all $q \in [1,\infty)$ and $r \in \left[2, \frac{2d}{d-2}\right)$ for $d \geq 3$, $r \in [2,\infty)$ for $d = 1, 2$, we have that $\Psi \in L_T^q W_x^{s,r} = L^q([0,T]; W^{s,r}(\mathbb{R}^d))$ a.s. for all $T > 0$.*

We use these properties voraciously going forward.

Example 1. *Consider the case $d = 1, p = 3$, and $s = 0$. Define the nonlinear "Duhamel formulation" operator Γ by*

$$\Gamma(u)(t) = \Gamma_{u_0,\Phi}(u)(t) = S(t)u_0 - i\int_0^t S(t-t')(|u(t')|^{p-1}u(t'))\,\mathrm{d}t' - i\Psi(t).$$

Recall that a pair of indices $(q,r) \in [2,\infty]^2$ is admissible *if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \text{ and } (q,r,d) \neq (2,\infty,2),$$

the importance being that we may use the Strichartz estimates (introduced in Lecture 3) for admissible pairs of indices. Since we will be applying the nonhomogeneous Strichartz estimate to $|u|^2u$, we would like (for notational cleanliness) to be sure that $r' = \frac{r}{r-1}$ has 3 in its denominator, where r is the index of spatial integrability, so that we may push the power of $|u|^3$ outside of the norm and obtain a whole number for L_x^r. This therefore gives us $r = 4$. The corresponding q for the admissible pair is then 8. Similarly, since $u_0 \in L^2$, we want u to retain this spatial integrability, and an admissible pair satisfying this is $(q,r) = (\infty, 2)$. Define the space

$$X(T) = C_T L_x^2 \cap L_T^8 L_x^4,$$

with norm given by the sum of the component norms (although we could always take the norm to be the ^{p} sum of the two norms without affecting anything since all these norms are equivalent). Note that the norm on C_T is the same as that on L_T^∞, which lets us use the appropriate Strichartz estimates on this space.

We establish local existence in the usual way through Picard iteration. First, by the triangle inequality, the homogeneous Strichartz estimate applied to the norm of $S(t)u_0$, and the *nonhomogeneous Strichartz estimate for the admissible pair* $(8,4)$ *applied to the norm of the nonlinearity, we have*

$$\|\Gamma(u)\|_{X(T)} \lesssim \|u_0\|_{L^2} + \||u|^2u\|_{L_T^{8/}\ L_x^{4/3}} + \|\Psi\|_{X(T)}.$$

As a small subtlety, we note that we cannot directly apply the nonhomogeneous Strichartz estimate to the norms $\|\cdot\|_{C([0,T];L^2(\mathbb{R}^d))}$ *and* $\|\cdot\|_{L^8([0,T];L^4(\mathbb{R}^d))}$*, as the estimate applies to the whole time interval (i.e. to the norms* $\|\cdot\|_{C([0,\infty);L^2(\mathbb{R}^d))}$ *and* $\|\cdot\|_{L^8([0,\infty);L^4(\mathbb{R}^d))}$ *respectively). This is resolved by noting that, say,*

$$\left\|\int_0^t S(t-t')F(t',x)\,\mathrm{d}t'\right\|_{L_T^qL_x^r} = \left\|\int_0^t S(t-t')(\mathbf{1}_{[0,T]}(t')F(t',x))\,\mathrm{d}t'\right\|_{L_t^qL_x^r},$$

so we may simply apply the estimate to the right hand side.

To deal with the norm of the nonlinearity, we write $\frac{7}{8} = \frac{3}{8} + \frac{4}{8}$ *and extract a power of* T *using Hölder's inequality:*

$$\begin{aligned}
\||u|^2u\|_{L_T^{8/}\ L_x^{4/3}} &= \|\mathbf{1}_{[0,T]}|u|^2u\|_{L_T^{8/}\ L_x^{4/3}} \\
&\le \|\mathbf{1}_{[0,T]}\|_{L_T^2}\||u|^2u\|_{L_T^{8/3}L_x^{4/3}} \\
&= T^{\frac{1}{2}}\|u\|^3_{L_T^8L_x^4} \\
&\le T^{\frac{1}{2}}\|u\|^3_{X(T)}.
\end{aligned}$$

Fix some $T \le 1$*. Then, using this estimate, we get*

$$\|\Gamma(u)\|_{X(T)} \lesssim (c_0\|u_0\|_{L^2} + \|\Psi\|_{X(1)}) + c_1T^{\frac{1}{2}}\|u\|^3_{X(T)}. \tag{1.1}$$

Also, in a similar manner to Lecture 3, we may find homogeneous polynomials $p(u,\overline{u},v,\overline{v})$ *and* $q(u,\overline{u},v,\overline{v})$ *of degree 2 such that*

$$|u|^2u - |v|^2v = p(u,\overline{u},v,\overline{v})(u-v) + q(u,\overline{u},v,\overline{v})(\overline{u}-\overline{v}),$$

Therefore, by noting we are in the additive noise case (so that the stochastic convolution Ψ *is independent of* u*) and using the nonhomogeneous Strichartz estimate, the triangle inequality, Young's inequality, and Hölder's inequality, we obtain*

$$\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X(T)} &= \left\|\int_0^t S(t-t')(|v|^2v-|u|^2u)\,\mathrm{d}t'\right\|_{X(T)} \\
&\le c_2\||u|^2u-|v|^2v\|_{L_T^{8/}\ L_x^{4/3}} \\
&\le c_2T^{\frac{1}{2}}\||u|^2u-|v|^2v\|_{L_T^{8/3}L_x^{4/3}} \\
&\le c_2T^{\frac{1}{2}}(\|p\|_{L_T^{8/2}L_x^{4/2}} + \|q\|_{L_T^{8/2}L_x^{4/2}})\|u-v\|_{L_T^8L_x^4} \\
&\le c_2T^{\frac{1}{2}}(\|u\|^2_{L_T^8L_x^4} + \|v\|^2_{L_T^8L_x^4})\|u-v\|_{L_T^8L_x^4} \\
&\le c_2T^{\frac{1}{2}}(\|u\|_{X(T)} + \|v\|_{X(T)})\|u-v\|_{X(T)},
\end{aligned} \tag{1.2}$$

where the constant c_2 *may change from line to line.*

Let $R = 2(c_0\|u_0\|_{L^2} + \|\Psi\|_{X(1)})$, which is finite a.s. by Proposition 1.1. Then, proceeding as in Lecture 3, we see by Eq. 1.1 that Γ maps the ball $\overline{B(0,R)} \subseteq X(T)$ to itself for T satisfying

$$\frac{R}{2} + c_1 R^3 T^{\frac{1}{2}} \leq R,$$

and by Eq. 1.2, Γ is a contraction on the ball for T satisfying

$$2c_2 T^{\frac{1}{2}} R^2 \leq \frac{1}{2}.$$

This establishes local existence and uniqueness. The argument of Lecture 3 carries over effectively verbatim to establish local stability, with the slight modification that we must again use the nonhomogeneous Strichartz estimate to bound the nonlinearity by something of the form seen on the right hand side of Eq. 1.2, as well as using the explicit bounds on the $L^8([0,1]; L^4(\mathbb{R}^d))$ norm of Ψ (used to show $\Psi \in L^q_T W^{s,r}_x$ a.s.) in addition to those of $C([0,1]; L^2(\mathbb{R}^d))$. This gives us local well-posedness.

Remark 1.2. *If $u_0 \in H^s(\mathbb{R}^d)$ and $\Phi \in \mathrm{HS}(L^2; H^s)$ for some $s \geq 0$, then the* fractional Leibniz rule

$$\|fg\|_{\dot{W}^{s,r}} \lesssim \|f\|_{\dot{W}^{s,p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|g\|_{\dot{W}^{s,q_2}}$$

for all $s \in (0,1)$ and $r, p_j, q_j \in (1,\infty)$ such that

$$\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}, j = 1,2$$

may be used to show that such regularity persists for all time, namely $u \in C_T H^s_x$. Indeed, a simple generalization of the fractional Leibniz rule is

$$\left\| \prod_{j=1}^{n} f_j \right\|_{\dot{W}^{s,r}} \lesssim \sum_{j=1}^{n} \|f_j\|_{\dot{W}^{s,p_{j,j}}} \prod_{i\neq j} \|f_i\|_{L^{p_{j,i}}}$$

where $\sum_{i=1}^{n} \frac{1}{p_{j,i}} = \frac{1}{r}$. Then, noting that $\frac{3}{4} = \frac{1}{8} + \frac{1}{8} + \frac{1}{2}$, we have that

$$\||u|^2 u\|_{\dot{W}^{s,4/3}} \lesssim \|u\|_{\dot{H}^s}\|u\|^2_{L^8}$$

for all $u \in \dot{H}^s \cap L^8$. The nonhomogeneous Strichartz estimate (applied to $|\nabla|^s(|u|^2u)$) then implies

$$\begin{aligned}
\left\|\int_0^t S(t-t')(|u|^2u)\,\mathrm{d}t'\right\|_{C_T\dot{H}^s} &\lesssim \||u|^2u\|_{L^{8/}_T\ \dot{W}^{s,4/3}_x} \\
&\lesssim \|\|u\|_{\dot{H}^s_x}\|u\|^2_{L^8_x}\|_{L^{8/}_T} \\
&\leq T^{\frac{1}{2}}\|\|u\|_{\dot{H}^s_x}\|u\|^2_{L^8_x}\|_{L^{1}_T\ /} \\
&\leq T^{\frac{1}{2}}\|u\|_{C_T\dot{H}^2_x}\|u\|^2_{L^{1}_T\ /3 L^8_x}.
\end{aligned}$$

This indicates that we may proceed as in the example, working instead in the space $X(T) = C_T H^s \cap L^8_T L^4_x \cap L^{16/3}_T L^8_x$.

1.1. **Scaling symmetry in the nonlinear Schrödinger equation.** Consider the *nonlinear Schrödinger equation*

$$i\frac{\partial u}{\partial t} - \Delta u = |u|^{p-1}u \tag{1.3}$$

A function u is a solution to Eq. 1.3 with initial data $u|_{t=0} = u_0$ if and only if, for all $\lambda > 0$, the scaled solution

$$u_\lambda(t,x) := \frac{1}{\lambda^{\frac{2}{p-1}}}u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \tag{1.4}$$

is a solution to Eq. 1.3 with initial data $u_\lambda|_{t=0} = u_{0,\lambda} := \lambda^{-\frac{2}{p-1}}u_0$. Indeed, this follows from the formal equalities

$$\frac{\partial u_\lambda}{\partial t} = \frac{\partial}{\partial t}\frac{1}{\lambda^{\frac{2}{p-1}}}u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) = \frac{1}{\lambda^{\frac{2p}{p-1}}}\frac{\partial u}{\partial t}\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right),$$

the spatial derivative

$$\Delta u_\lambda = \frac{1}{\lambda^{\frac{2p}{p-1}}}\Delta u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right),$$

and

$$|u_\lambda|^{p-1}u_\lambda = \frac{1}{\lambda^2}\left|u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)\right|^{p-1}\frac{1}{\lambda^{\frac{2}{p-1}}}u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) = \frac{1}{\lambda^2}(|u|^{p-1}u)\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

Now, for any $s \in \mathbb{R}$, we have

$$\begin{aligned}
\|u_{0,\lambda}\|_{\dot{H}^s} &= \||\xi|^s\widehat{u_{0,\lambda}}(\xi)\|_{L^2} \\
&= \lambda^{d-\frac{2}{p-1}}\||\xi|^s\widehat{u_0}(\lambda\xi)\|_{L^2} \\
&= \lambda^{-s+\frac{d}{2}-\frac{2}{p-1}}\|u_0\|_{\dot{H}^s} \\
&= \lambda^{s_{\text{crit}}-s}\|u_0\|_{\dot{H}^s},
\end{aligned} \tag{1.5}$$

where $s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$ is the *(scaling-)critical Sobolev regularity index.* Note that if $s = s_{\text{crit}}$, we have

$$\|u_{0,\lambda}\|_{\dot{H}^{s_{\text{crit}}}} = \|u_0\|_{\dot{H}^{s_{\text{crit}}}}$$

for all $\lambda > 0$.

The nonlinear Schrödinger equation conserves several quantities. Namely, the *mass*

$$M(u) := \int_{\mathbb{R}^d}|u|^2\,\mathrm{d}x = \|u\|_{L^2},$$

the *momentum*

$$P(u) := \operatorname{Im}\int_{\mathbb{R}^d}\overline{\nabla u}\cdot u\,\mathrm{d}x,$$

and the *energy*

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}x \quad \frac{1}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}x.$$

We say Eq. 1.3 is *mass-* or *L^2-critical* if $s_{\text{crit}} = 0$, and *energy-* or *H^1-critical* if $s_{\text{crit}} = 1$.

Given initial data $u_0 \in H^s(\mathbb{R}^d)$, the Cauchy problem for Eq. 1.3 is *subcritical* if $s > s_{\text{crit}}$, *critical* if $s = s_{\text{crit}}$, and *supercritical* if $s < s_{\text{crit}}$.The scaling symmetry may be used to determine if Eq. 1.3 is well-posed in each case. For the subcritical case, the $\dot{H}^s$ norm of $u_{0,\lambda}$, given in Eq. 1.5, shrinks as λ increases. At the same time, the expression of Eq. 1.4 indicates that a scaled solution u_λ to the Cauchy problem will exist for a longer period of time as

λ increases. This lets us infer that in the subcritical case, smaller initial data corresponds to existence over a longer period of time, and so we may expect the Cauchy problem to be well-posed. The supercritical case can be seen to have the opposite effect: the $\dot{H}^s$ norm of the scaled initial data grows as λ increases, and so larger initial data is required for existence over longer times. This means we can expect ill-posedness in the supercritical regime.

Example 2. *Let $p = 3$. If $d = 1$, then the critical index is $s_{crit} = \frac{d}{2} - \frac{2}{p-1} = -\frac{1}{2}$. Example 1 was therefore a subcritical problem (if we were to remove the stochastic term).*

If $p = 3$ and $d = 2$, then the critical index is $s_{crit} = 0$. Let's consider this problem. Take initial data $u_0 \in L^2(\mathbb{R}^2)$ and smoothing operator $\Phi \in \mathrm{HS}(L^2; L^2)$. In this case, two admissible pairs of indices are $(q, r) = (4, 4)$ and $(\infty, 2)$. Then, with Γ the Duhamel formulation operator as usual (cf. Example 1), we have, by the nonhomogeneous Strichartz estimate,

$$\begin{aligned}\|\Gamma(u)\|_{L^4_T L^4_x} &\leq \|S(t)u_0\|_{L^4_T L^4_x} + \|\Psi\|_{L^4_T L^4_x} + c\||u|^2 u\|_{L^{4/3}_T L^{4/3}_x} \\ &= \|S(t)u_0\|_{L^4_T L^4_x} + \|\Psi\|_{L^4_T L^4_x} + c\|u\|^3_{L^4_T L^4_x}.\end{aligned}$$

Note that unlike in previous examples, we cannot use Hölder's inequality to extract a term of the form T^θ. In the case $s > \frac{d}{2}$, we can extract this term using the product estimate in the Sobolev space H^s, and in Example 1, we do this using a clever discrepancy between the indices q and r. This allows us to have the radius of the ball on which we apply Picard iteration to depend on the norms of u_0 and Ψ, then pick T sufficiently small so that everything lies in that ball. In this instance, we must choose the radius to depend on T, then pick T sufficiently small such that everything lies in the ball, and this is why we do not immediately apply the homogeneous Strichartz estimate to the norm of $S(t)u_0$ like we have done previously.

Define the radius

$$R := 2(\|S(t)u_0\|_{L^4_T L^4_x} + \|\Psi\|_{L^4_T L^4_x})$$

Then, for $u \in \overline{B(0, R)} \subseteq L^4_T L^4_x$, we have

$$\|\Gamma(u)\|_{L^4_T L^4_x} \leq \frac{1}{2}R + cR^3,$$

which we can take to be less than R by taking R sufficiently small. Also,

$$\|\Gamma(u) - \Gamma(v)\|_{L^4_T L^4_x} \leq c' R^2 \|u - v\|_{L^4_T L^4_x},$$

which comes from a similar calculation to Eq. 1.2. Again, we may choose $c'R^2$ to be less than $\frac{1}{2}$ by choosing R sufficiently small. Now, we see that

$$\begin{aligned}\int_0^1 2(\|S(t)u_0\|^4_{L^4_x} + \|\Psi\|^4_{L^4_x})\,\mathrm{d}t &\leq 2(\|S(t)u_0\|^4_{L^4_t L^4_x} + \|\Psi\|^4_{L^4_1 L^4_x}) \\ &\lesssim 2(\|u_0\|^4_{L^2_x} + \|\Psi\|^4_{L^4_1 L^4_x}),\end{aligned}$$

which is almost surely finite by Proposition 1.1. It follows then from the dominated convergence theorem applied to the function $\mathbf{1}_{[T,1]}(t)f(t)$, where $f(t)$ is the above integrand, that

$$\lim_{T \downarrow 0} R = \lim_{T \downarrow 0} \int_0^T 2(\|S(t)u_0\|^4_{L^4_x} + \|\Psi\|_{L^4_x})\,\mathrm{d}t = 0$$

almost surely, so we can choose T sufficiently small such that R is small enough to satisfy the above conditions.

We have seen that Γ *is a contraction on* $\overline{B(0,R)} \subseteq L^4_T L^4_x$*, so there exists a unique local solution* $u \in \overline{B(0,R)}$*. We also want our solution to lie in* $C_T L^2_x$*. To see this, we write*

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')(|u|^2 u)(t')\,\mathrm{d}t' - i\Psi(t).$$

Now, Ψ *is in* $C_t L^2_x$ *a.s. by Proposition 1.1, and* $S(t)u_0$ *is in* $C_t L^2_x$ *by the theory of linear PDEs. We therefore just need to check the nonlinear term lies in* $C_T L^2_x$*. Note that it suffices to check boundedness of the norm - continuity arises naturally from the integration from 0 to* t*. By the nonhomogeneous Strichartz estimate,*

$$\begin{aligned}\left\| \int_0^t S(t-t')(|u|^2 u)(t')\,\mathrm{d}t' \right\|_{C_T L^2_x} &\lesssim \| |u|^2 u \|_{L^{4/3}_T L^{4/3}_x} \\ &= \|u\|^3_{L^4_T L^4_x},\end{aligned}$$

which is of course finite a.s.

Note that we don't have uniqueness in $C_T L^2_x$ *- we have to intersect it with the space* $L^4_T L^4_x$ *to establish uniqueness (shown at least for the ball* $\overline{B(0,R)}$ *above). This property is known as* conditional uniqueness. *In fact, unconditional uniquness never holds in* $C_T L^2_x$*, since in order for the nonlinear term above to make sense, we need* $|u|^2 u$ *to lie in* $L^1_{loc,\ x}$*, i.e. for* u *to lie in* $L^3_{loc,\ x}$*. Being in* L^2 *is not enough to gain* L^3_{loc} *integrability.*

1.2. **Stochastic nonlinear Schrödinger equation with multiplicative noise.** Consider the SPDE

$$i\frac{\partial u}{\partial t} - \Delta u = N(u) + \sigma(u)\Phi\xi \tag{1.6}$$

In all of our results, we will take the nonlinearity to be $N(u) = |u|^{p-1}u$, and the variance of the noise to be $\sigma(u) = |u|^{\gamma-1}u$. The stochastic convolution in this instance depends on u:

$$\Psi[u](t) := \int_0^t S(t-t')\sigma(u)\Phi\,\mathrm{d}W(t').$$

In contrast with the additive case, it is not sufficient that $\Phi \in \mathrm{HS}(L^2; H^s)$ for the important Proposition 1.1 to hold. Indeed, when we proved these properties, we wrote the cylindrical Wiener process W as

$$W(x,t) = \sum_{n=1}^{\infty} \beta_n(t) e_n(x)$$

for some independent Brownian motions β_n and an orthonormal basis e_n of L^2. Then the stochastic integrals

$$\int_0^t S(t-t')\langle\nabla\rangle^s \Phi(e_n)\,\mathrm{d}\beta_n(t')$$

were independent, which allowed us to use the Ito isometry and collapse everything into the Hilbert-Schmidt norm of Φ. This does not suffice in the multiplicative case as we are dealing with the stochastic integrals

$$\int_0^t S(t-t')(\langle\nabla\rangle^s(\sigma(u)\Phi(e_n)))\,\mathrm{d}\beta_n(t')$$

which, owing to the presence of $\sigma(u)$, are not independent in general.

We will introduce (somewhat informally) the notion of γ-*radonifying operators*, which are a certain generalization of Hilbert-Schmidt operators to the Banach space regime. To motivate this, let H be a separable Hilbert space. We would like for the set function

$$\mu(\mathrm{d}x) := e^{-\frac{1}{2}\|x\|_H}\,\mathrm{d}x$$

to be a measure on H, the canonical "Gaussian measure". In general, however, this is not countably additive if $\dim H = \infty$. Heuristically, countable unions of sets in H are somehow too large for the Gaussian measure to make sense. We must embed H in some larger Banach space so that these countable unions of sets in H are sufficiently small for the Gaussian measure to make sense when considered as subsets of B. Informally, then, an *abstract Wiener space* is a triple (H, B, μ) consisting of a Hilbert space H, a Banach space B into which H is continuously and densely imbedded, and the above Gaussian set function μ on H such that μ is a measure when pushed forward onto B under the imbedding. More information on abstract Wiener spaces can be found in the books [2, 3, 4], along with the original definition in [1].

Example 3. *Let $H = H^s(\mathbb{T}^d)$, and let μ_s be the set function given by*

$$\mu_s(\mathrm{d}u) = e^{-\frac{1}{2}\|u\|_{H^s}^2}\,\mathrm{d}u$$

In some sense, the $H^s(\mathbb{T}^d)$-valued random variable

$$u = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x} \tag{1.7}$$

has distribution μ_s, where the g_n are i.i.d. complex-valued standard Gaussian random variables. Indeed, if $u \in H^s(\mathbb{T}^d)$, we may formally derive

$$\begin{aligned} e^{-\frac{1}{2}\|u\|_{H^s}^2}\,\mathrm{d}u &= e^{-\frac{1}{2}\sum_{n\ \ \mathbb{Z}^d}\ n\ ^{2s}|u(n)|^2}\,\mathrm{d}u \\ &= \qquad e^{-\frac{1}{2}\ n\ ^{2s}|u(n)|^2}\,\mathrm{d}\hat{u}(n), \\ & {\scriptstyle n\in\mathbb{Z}^d} \end{aligned}$$

where $\mathrm{d}\hat{u}(n)$ is Lebesgue measure on $\mathbb{C}$. From this expression, we see that $\langle n\rangle^s\hat{u}(n)$ must be distributed as a standard complex-valued Gaussian random variable, which leads to Eq. 1.7.

Also, if u is the random variable defined in Eq. 1.7, we may compute

$$\mathbb{E}[\|u\|_{H}^2\ ] = \sum_{n\in\mathbb{Z}^d} \frac{\mathbb{E}[|g_n|^2]}{\langle n\rangle^{2s-2\sigma}},$$

which is finite if and only if $\sigma < s - \frac{d}{2}$. Therefore, the random variable u is well-defined a.s. in $H^\sigma(\mathbb{T}^d)$ - as, say, the H^σ limit of the truncated random variables

$$u_N := \sum_{|n|\le N} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x}$$

- if and only if $\sigma < s - \frac{d}{2}$. Since H^s is a dense subspace of H^σ in this case, we may conclude that (H^s, H^σ, μ_s) is an abstract Wiener space.

A similar derivation may be done to show that $(H^s, W^{\sigma,p}, \mu_s)$ is an abstract Wiener space for any $p \in [1, \infty]$.

Let's return to the cylindrical Wiener process

$$W(t) = \sum_{n\in\mathbb{Z}^d} \beta_n(t)e_n$$

For fixed t, $\beta_n(t)$ is a Gaussian random variable. A Hilbert-Schmidt operator $\Phi \in \mathrm{HS}(L^2; H^s)$ pushes this process to H^s. That is, $\Phi W(t) = \sum_{n\in\mathbb{Z}^d} \beta_n(t)\Phi e_n$ is a certain stochastic process in H^s. In a similar way, given a Banach space B, a γ-radonifying operator Φ from L^2 to B pushes $W(t)$ to B. More concretely, let H be a separable Hilbert space and B a Banach space. A bounded linear map $\Phi\colon H \to B$ is a *γ-radonifying operator* if the norm

$$\|\Phi\|_{\gamma(H;B)} := \left(\mathbb{E}\quad\left\|\sum_{n=1}^{\infty} g_n\Phi e_n\right\|_B^2\quad\right)^{\frac{1}{2}}$$

is finite, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H, and g_n are independent standard complex-valued Gaussian random variables. The space of all γ-radonifying operators from H to B is denoted $\gamma(H; B)$ (in some other texts, $M(H; B)$ or $R(H; B)$). If B is a Hilbert space, then the spaces $\gamma(H; B)$ and $\mathrm{HS}(H; B)$ coincide. Indeed, this can be seen from the calculation

$$\begin{aligned}
\|\Phi\|^2_{\gamma(H;B)} &= \mathbb{E}\quad\left\|\sum_{n=1}^{\infty} g_n\Phi e_n\right\|_B^2 \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} g_n g_m(\Phi e_n, \Phi e_m)_B\right] \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathbb{E}[g_n g_m](\Phi e_n, \Phi e_m)_B \\
&= \sum_{n=1}^{\infty} \|\Phi e_n\|_B^2 \\
&= \|\Phi\|^2_{\mathrm{HS}(H;B)}.
\end{aligned}$$

A certain extension of the Kahane-Khintchine inequality to Gaussian sums [6, 5] allows us to choose any index for the norm. That is,

$$\|\Phi\|_{\gamma(H;B)} \sim \left(\left[\left\|\sum_{n=1}^{\infty} g_n\Phi e_n\right\|^p\right]\right)^{\frac{1}{p}}$$

for any $p \in (1, \infty)$.

Notes. A more formal definition of an abstract Wiener space is as follows: let μ be the Gaussian set function on a Hilbert space H as defined before. A seminorm $|\cdot|$ on H is said to be *measurable* with respect to μ if, for all $\varepsilon > 0$, there exists a finite-dimensional orthogonal projection P_ε on H such that

$$\mu(|Px| > \varepsilon) < \varepsilon$$

for all finite-dimensional orthogonal projections P on H such that $P(H)$ is orthogonal to $P_\varepsilon(H)$. We then let B be the completion of H under the seminorm $|\cdot|$, and call (H, B, μ) an *abstract Wiener space*.

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1. SPDEs with Multiplicative Noise Lecture 5 (Typed by Erik Sätterqvist)

In this lecture we start by generalizing the definition of the space $\gamma(H, B)$ from last lecture. We then state the Burkholder-David-Gundy (BDG) inequality for stochastic integrals and then apply this to study the stochastic convolution for the SNLS.

1.1. The space $\gamma(H, B)$.

We start by giving a more general definition of the space $\gamma(H, B)$ from the last lecture. Let μ_H be such that $d\mu_H \sim e^{-\frac{1}{2}\|u\|_H^2} du$. The function $\Phi : H \to B$ is γ-radonifying (i.e. $\Phi \in \gamma(H, B)$) if and only if the pushforward

$$\Phi_{\#}\mu_H = \mu_H \circ \Phi^{-1}$$

has an extension to a countably additive (Gaussian probability) measure μ_Φ on B. By the *Fernique theorem* we then have that, for some $c > 0$,

$$\int_B e^{c\|u\|_B^2} \mu_\Phi(du) < \infty,$$

or equivalently

$$\mu_\Phi(\|u\|_B > \lambda) \leq Ce^{-c\lambda^2}, \quad \forall \lambda > 0.$$

This implies that

$$\|\Phi\|_{\gamma(H;B)} = \Big(\int_B \|u\|_B^2 \mu_\Phi(du) \Big)^{1/2},$$

is finite.

1.2. Burkholder-Davis-Gundy inequality.

Before we can state the BDG inequality we need some definitions.

Definition 1.1. *We say that a Banach space B is of* type p *if for any finite sequence*

$$\varepsilon_1, \dots, \varepsilon_N : \Omega \to \{-1, 1\}$$

of symmetric i.i.d. random variables and any finite sequence $u_1, \dots, u_N \in B$ there exists a $K > 0$ such that

$$\mathbb{E}\Big\| \sum_{n=1}^N \varepsilon_n u_n \Big\| \leq K \sum_{n=1}^N \|u_n\|.$$

Next recall that $\{f_n\}_{n=0}^\infty$ is a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n=0}^\infty \subset \mathcal{F}$ if $\mathbb{E}[f_n|\mathcal{F}_m] = f_m$ for $m \leq n$. Note that this implies $E[df_n|\mathcal{F}_m] = 0$ for all $m < n$.

Definition 1.2. *Let $1 \leq p \leq 2$. We say that a Banach space B is of* martingale type p *(M-type p) if*

$$\|f_N\|_{L^p(\Omega;B)} \leq C\Big(\sum_{n=0}^N \|f_n - f_{n-1}\|_{L^p(\Omega;B)}^p \Big)^{1/p}$$

for any B-valued L^p-martingales $\{f_n\}_{n=0}^N$, (here $f_{-1} \equiv 0$).

Definition 1.3. *We say that a Banach space B is has the unconditional martingale difference property (alternatively that it is a UMD space) if for any $p \in (1,\infty)$, any B-valued martingale difference $\{\xi_n\}_{n=1}^{\infty}$ (i.e. $\sum_{j=n}^{N} \xi_n$ is a martingale), any $\epsilon : \mathbb{N} \to \{-1,1\}$ and any $n \in \mathbb{N}$*

$$\mathbb{E}\Big|\sum_{n=1}^{N} \epsilon_n \xi_n\Big|^p \lesssim_{p,B} \mathbb{E}\Big|\sum_{n=1}^{N} \xi_n\Big|^p.$$

Note that M-type p implies type p. For the other direction we have that type p plus UMD implies M-type p. By [1] UMB is equivalent to Hilbert transforms of B-valued functions being bounded in $L^2(S;B)$.

Let us give some examples of Type p spaces. If B has M-type p (for some $1 \leq p \leq 2$) and A is a measure space, then $L^r(A;B)$ has M-type $p \wedge r$ for $1 < r < \infty$. Because Hilbert spaces are M-type 2 this implies that $L^r(\mathbb{R}^d;\mathbb{C})$ is of M-type 2 for $2 \leq r < \infty$. Thus $L^q(\mathbb{R}, L^r(\mathbb{R}^d;\mathbb{C})) = L_t^q L_x^r$ is of M-type 2 for $2 \leq q, r < \infty$.

Finally we need the notion of an accessible stopping time.

Definition 1.4. *We say that a stopping time τ is accessible if there exists an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that*

$$\tau_n < \tau \quad \textit{and} \quad \lim_{n\to\infty} \tau_n = \tau, \quad \textit{a.s.}$$

Theorem 1.5 (BDG inequality for stochastic integrals)**.** *Let $1 < p < \infty$ and let B be a Banach space of M-type 2 (this is also called* 2-smooth*). Then there exits a $C = C(p,B) > 0$ such that*

$$\mathbb{E}\Big[\sup_{0<t<\tau}\Big\|\int_0^t F(t')dW(t')\Big\|_B^p\Big] \leq C\mathbb{E}\Big[\Big(\int_0^\tau \|F(t)\|_{\gamma(K;B)}^2 dt\Big)^{p/2}\Big]$$

for any accessible stopping time τ and $\gamma(K;B)$-valued progressively measurable F.

Here W denotes a K-cylindrical Wiener process and for us $K = L^2(\mathbb{R}^d)$ or $K = L^2(\mathbb{T}^d)$. For proofs see [4], [5], [7] (optimal constant), [2] (stronger assumption: UMD & type 2). The BDG inequality will also be proved later in these notes.

1.3. **Back to SNLS with multiplicative noise.** Recall the stochastic convolution for the SNLS

$$\Psi(t) = \Psi[u](t) = \int_0^t S(t-t')\Big(\sigma(u)\Phi dW(t')\Big),$$

where

$$\sigma(u) = |u|^{\gamma-1}u, \quad \gamma \geq 1.$$

Suppose that $\gamma \in 2\mathbb{N}+1$ so that $\sigma(u)$ is algebraic, we then have the following proposition.

Proposition 1.6. *Let $s > \frac{d}{2}$ and $\Phi \in HS(L^2, H^s)$. Then, for any $u \in L_{ad}^{2\gamma}(\Omega; C_T H_x^s)$, we have $\Psi = \Psi[u] \in C_T H_x^s$ a.s..*

Proof. To prove the above proposition we will use the *factorization method* (Lemma 2.7 in [6])

Lemma 1.7. *Let $0 < \alpha < 1$ and $q > \frac{1}{\alpha}$ and suppose that $f \in L^q_T H^s_x$ for some $T > 0$. Then the function $F : [0,T] \to \mathbb{C}$ given by*

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')dt'$$

belongs to $C_T H^s_x$. Moreover, we have

$$\sup_{0 \le t \le T} \|F(t)\|_{H^s_x} \lesssim \|f\|_{L^q_T H^s_x}.$$

Let $0 < \alpha < 1$ and $0 \le \mu \le t \le$. Note that

$$\mathcal{B}(\alpha, 1-\alpha) = \int_\mu^t (t-t')^{\alpha-1}(t'-\mu)^{-\alpha},$$

where $\mathcal{B}$ is the beta function. Moreover we know for a fact that

$$\mathcal{B}(\alpha, 1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}.$$

Hence

$$\begin{aligned}
\Psi(t) &= \int_0^t S(t-\mu)\sigma(u)(\mu)\Phi dW(\mu) \\
&= \frac{\sin(\pi\alpha)}{\pi} \int_0^t \Big[\int_\mu^t (t-t')^{\alpha-1}(t'-\mu)^{-\alpha} dt'\Big] S(t-\mu)\sigma(u)(\mu)\Phi dW(\mu) \\
&= \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(t-t')(t-t')^{\alpha-1}\Big[\int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\sigma(u)(\mu)\Phi dW(\mu)\Big] dt' \\
&= \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')dt',
\end{aligned}$$

where

$$f(t') = \int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\sigma(u)(\mu)\Phi dW(\mu).$$

Thus in view of Lemma 1.7, it suffices to show $f \in L^q_T H^s_x$, for some $\frac{1}{\alpha} < q < \infty$. We want to show

$$\mathbb{E}\Big[\int_0^T \|f(t')\|^q_{H^s_x} dt'\Big] \le C(T, q, \Phi) < \infty. \tag{1.1}$$

By the BDG inequality 1.5 we have, for $1 \le q < \infty$,

$$\mathbb{E}\big[\|f(t')\|^q_{H^s_x}\big] \lesssim \mathbb{E}\Big[\Big(\int_0^{t'} \|S(t'-\mu)(t'-\mu)^{-\alpha}\sigma(u)(\mu)\Phi\|^2_{HS(L^2;H^s)} d\mu\Big)^{q/2}\Big]$$

Letting e_n be an orthonormal basis for L^2 we have for $0 < \alpha < \frac{1}{2}$ and $s > \frac{d}{2}$

$$\begin{aligned}
\int_0^{t'} \|S(t'-\mu)(t'-\mu)^{-\alpha}\sigma(u)(\mu)\Phi\|^2_{HS(L^2;H^s)} d\mu &= \int_0^{t'} (t'-\mu)^{-2\alpha}\|\sigma(u)(\mu)\Phi(e_n)\|^2_{l^2_n H^s_x} d\mu \\
&\lesssim \|\sigma(u)\|^2_{C_T H^s_x}\|\Phi\|^2_{HS(L^2;H^s)},
\end{aligned}$$

whence

$$\mathbb{E}\big[\|f(t')\|^q_{H^s_x}\big] \lesssim \mathbb{E}\big[\|u\|^{\gamma q}_{C_T H^s_x}\|\Phi\|^q_{HS(L^2;H^s)}\big] < \infty.$$

Now integrating from $t' = 0$ to T, (1.1) follows for any $1 \le q < \infty$, and in particular $q > \frac{1}{\alpha}$. Thus by Lemma 1.7, $\Psi \in C_T H^s_x$ a.s.. We need $\alpha < \frac{1}{2}$ so we can take any $\frac{1}{\alpha} < q < \infty$. $\square$

We end this section with two remarks:

(1) In proving (1.1) we viewed $S(t'-\mu)(\sigma(u)\Phi dW(t'))$ as $S(t-t')\circ M_{\sigma(u)(\mu)}\circ\Phi$ applied to $dW(t')$, where M_F denotes multiplication by a function F.

(2) With $\sigma(u)=|u|^{\gamma-1}u$ we need

$$\|\sigma(u)\|_{H^s}\lesssim\|u\|_{H^s}^{\gamma},\quad \text{for } s>\frac{d}{2}.$$

Hence we need the fact that $\gamma\in 2\mathbb{N}+1$ so that $\sigma(u)$ is algebraic (i.e. a product).

When $\gamma\notin 2\mathbb{N}+1$, we cannot consider $s\gg 1$ due to the lack of smootheness of $\sigma(\cdot)$. In general, given $s>\frac{d}{2}$ (so that $H^s\hookrightarrow L^\infty$), we need $\sigma\in C^k(\mathbb{C}\cong\mathbb{R}^2;\mathbb{C})$ with $k\geq[s]+1$. For example see Lemma A.9 in [8] or Lemma 4.10.2 in [3] (also see the fractional chain rule).

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LECTURE 6

YOUNES ZINE

We define the stochastic convolution $\Psi = \Psi[u]$ by

$$\Psi(t) := \int_0^t S(t-t')\sigma(u)\Phi dW(t'). \tag{0.1}$$

YZconv

with $S(t) = e^{-it\Delta}$ and $\sigma(u) = |u|^{\gamma-1}u$ for $\gamma \in 2\mathbb{N}+1$.

YZPROP:1 **Proposition 0.1.** *Let $s > \frac{d}{2}$ and $\Phi \in \mathrm{HS}(L^2; H^s)$. Then for $u \in L^{2\gamma}_{ad}\big(\Omega; C_T H^s_x\big)$, we have*

$$\Psi = \Psi[u] \in C_T H^s_x,$$

almost surely. Moreover, if $u \in L^{q\gamma}_{ad}\big(\Omega; C_T H^s_x\big)$ for some finite $q \geq 2$, we have

$$\mathbb{E}\big[\|\Psi\|^q_{C_T H^s_x}\big] \leq C(s,q) T^\theta\, \mathbb{E}\big[\|u\|^{q\gamma}_{C_T H^s_x}\big]\, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}$$

for some $\theta > 0$ and some constant $C(s,q) > 0$.

We will apply Proposition 0.1 to solve a fixed point argument for the following linear stochastic equation:

$$\begin{cases} i\partial_t u = \Delta u + u\Phi\xi \\ u|_{t=0} = u_0 \in H^s. \end{cases} \tag{0.2}$$

YZeq1

for $s > \frac{d}{2}$. By the Duhamel principle, u is said to solve (0.2) if u verifies the following integral equation:

$$\begin{aligned} u(t) &= S(t)u_0 - i\Psi[u](t) \\ &=: \Gamma_{u_0,\Phi}(u)(t) \end{aligned} \tag{0.3}$$

YZ100

where $\Psi[u]$ is given by (0.1) with $\sigma(u) = u$.

We prove the following result.

YZLEM:1 **Lemma 0.2.** *Fix $s > \frac{d}{2}$. The map $u \mapsto \Gamma_{u_0,\Phi}$ defined in (0.3) is a contraction on a ball in $L^2\big(\Omega; C_T H^s_x\big)$ for any $T > 0$ small enough..*

Proof. By Proposition 0.1, we estimate

$$\begin{aligned} \|\Gamma(u)\|_{L^2_\omega C_T H^s_x} &\leq \|u_0\|_{H^s_x} + CT^\theta \|u\|_{L^2_\omega C_T H^s_x} \\ \|\Gamma(u) - \Gamma(v)\|_{L^2_\omega C_T H^s_x} &\leq CT^\theta \|u\|_{L^2_\omega C_T H^s_x}. \end{aligned} \tag{0.4}$$

YZ1

Let $B_R \subset L^2\big(\Omega; C_T H^s_x\big)$ for $R \sim \|u_0\|_{H^s_x}$ be a closed ball of center 0 and radius R. Then, (0.4) show that Γ is a contraction on B_R by choosing $T > 0$ small enough. □

We now look at the following nonlinear equation:

$$\begin{cases} i\partial_t u = \Delta u + |u|^{k-1}u + |u|^{\gamma-1}u\Phi\xi \\ u|_{t=0} = u_0 \in H^s. \end{cases} \tag{0.5}$$

YZeq2

for $s > \frac{d}{2}$ and $k, \gamma \in 2\mathbb{N}+1$.

Again, a solution u to (0.5) verifies the following integral equation:

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')|u|^{k-1}u(t')dt' - i\Psi[u](t). \tag{0.6}$$ YZ2

with $\Psi[u]$ given by (0.1) for $\sigma(u) = |u|^{\gamma-1}u$.

We want to construct $u \in L^2\big(\Omega; C_T H_x^s\big)$ but in view of the high degree of $\sigma(u)$ we would a priori need $u \in L^{2\gamma}\big(\Omega; C_T H_x^s\big)$. In order to circumvent this issue we will use the truncation method. See [1].

Let η be a smooth cutoff function such that $\eta \equiv 1$ on $[0,1]$ and $\eta \equiv 0$ on $[2,\infty)$. For $R>0$, we define $\eta_R(u)$ by

$$\eta_R(u)(t) = \eta\Big(\frac{\|u\|_{C([0,t];H_x^s}}{R}\Big) \tag{0.7}$$ YZ3

We modify the integral equation to integrate the cutoff $\eta_R(u)$,

$$\begin{aligned} u(t) &= S(t)u_0 - i\int_0^t S(t-t')\eta_R(u)(t')|u|^{k-1}u(t')dt' \\ &\quad - i\int_0^t S(t-t')\eta_R(u)(t')|u|^{\gamma-1}u\,\Phi dW(t') \\ &=: S(t)u_0 + \mathrm{I}[u] + \mathrm{II}[u] := \Gamma_R[u]. \end{aligned} \tag{0.8}$$ YZ4

YZPROP:2 **Proposition 0.3.** *Fix $s > \frac{d}{2}$ and $R>0$. The map $u \mapsto \Gamma_R[u]$ defined in (0.8) is a contraction on a ball in $L^2\big(\Omega; C_T H_x^s\big)$ for $T>0$ small enough.*

Proof. We aim at showing that $\Gamma = \Gamma_R$ is a contraction on some ball. We now estimate the different terms $\mathrm{I}[u]$ and $\mathrm{II}[u]$. We have for any function $u \in C_T H_x^s$,

$$\begin{aligned} \|\mathrm{I}[u]\|_{C_T H_x^s} &\le \Big\| \int_0^t \eta_R(u)(t')\||u|^{k-1}u(t')\|_{H_x^s}dt'\Big\|_{C_T} \\ &\lesssim TR^k. \end{aligned} \tag{0.9}$$ YZ5

Fix two functions $u_1, u_2 \in C_T H_x^s$. We want to estimate the difference $\mathrm{I}[u_1] - \mathrm{I}[u_2]$ in $C_T H_x^s$. To this end, we introduce the stopping times $t_{j,R}$ $(j=1,2)$ by

$$t_{j,R} = \sup\{t \in [0,T] : \|u_j\|_{C([0,t];H_x^s)} \le 2R\}. \tag{0.10}$$ YZ6

Without loss of generality, we may assume $t_{1,R} \le t_{2,R}$. We now bound

$$\begin{aligned} \big\|\mathrm{I}[u_1] - \mathrm{I}[u_2]\big\|_{C_T H_x^s} &\le \Big\| \int_0^t S(t-t')\eta_R(u_1)(t')\big(N(u_1) - N(u_2)\big)(t')dt'\Big\|_{C_T H_x^s} \\ &\quad + \Big\| \int_0^t S(t-t')\big(\eta_R(u_1) - \eta_R(u_2)\big)(t')N(u_2)(t')dt'\Big\|_{C_T H_x^s} \\ &= \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{0.11}$$ YZ7

In the above, we used the shorthand notation $N(u) = |u|^{k-1}u$ for convenience.

• **Estimate of I_2:** By the mean value theorem, we have the following bound:

$$|\eta_R(u_1)(t') - \eta_R(u_2)(t')| \le \frac{\|\eta'\|_{L^\infty}}{R}\|u_1 - u_2\|_{C_T H_x^s}, \tag{0.12}$$ YZ8

for any $0 \leq t' \leq T$. Using (0.10) and (0.12), we estimate the term $\mathcal{I}_2$,

$$\begin{aligned}\|\mathrm{I}_2\|_{C_T H^s_x} &\leq T\,\|(\eta_R(u_1) - \eta_R(u_2))N(u_2)\|_{C_T H^s_x} \\ &= T\,\|(\eta_R(u_1) - \eta_R(u_2))N(u_2)\|_{C_{t_{2,R}} H^s_x} \\ &\lesssim T R^{k-1}\|u_1 - u_2\|_{C_T H^s_x}.\end{aligned} \tag{0.13}$$

• **Estimate of $\mathbf{I}_1$:** Similarly, by (0.10), we bound

$$\begin{aligned}\|\mathrm{I}_1\|_{C_T H^s_x} &\leq T\,\|\eta_R(u_1)(N(u_1) - N(u_2))\|_{C_T H^s_x} \\ &= T\,\|\eta_R(u_1)(N(u_1) - N(u_2))\|_{C_{t_{1,R}} H^s_x} \\ &\lesssim T R^{k-1}\|u_1 - u_2\|_{C_T H^s_x},\end{aligned} \tag{0.14}$$

where we used the bound

$$|N(u_1) - N(u_2)| \lesssim \max(|u_1|^{k-1}, |u_2|^{k-1})|u_1 - u_2|,$$

in the above.

Combining (0.13) and (0.14), we get

$$\|\mathrm{I}[u_1] - \mathrm{I}[u_2]\|_{C_T H^s_x} \lesssim T R^{k-1}\|u_1 - u_2\|_{C_T H^s_x}. \tag{0.15}$$

We now turn our attention to II. By applying BDG inequality/a modification of Proposition 0.1 (see Lecture 5), one can show

$$\begin{aligned}\|\mathrm{II}[u]\|_{L^2_\omega C_T H^s_x} &\lesssim T^\theta \|\Phi\|_{\mathrm{HS}(L^2;H^s)} R^\gamma \\ \|\mathrm{II}[u_1] - \mathrm{II}[u_2]\|_{L^2_\omega C_T H^s_x} &\lesssim T^\theta \|\Phi\|_{\mathrm{HS}(L^2;H^s)} R^{\gamma-1}\|u_1 - u_2\|_{L^2_\omega C_T H^s_x}.\end{aligned} \tag{0.16}$$

for any $u, u_1, u_2 \in L^2(\Omega; C_T H^s_x)$. Putting everything together, we get the following bounds on Γ from (0.8), (0.9), (0.15) and (0.16), for $0 < T \leq 1$,

$$\begin{aligned}\|\Gamma[u]\|_{L^2_\omega C_T H^s_x} &\leq \|u_0\|_{H^s_x} + C_1\big(R, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) \\ \|\Gamma[u_1] - \Gamma[u_2]\|_{L^2_\omega C_T H^s_x} &\leq C_2 T^\theta\big(1 + \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) R^{\min(k,\gamma)-1}\|u_1 - u_2\|_{L^2_\omega C_T H^s_x},\end{aligned} \tag{0.17}$$

for any $u, u_1, u_2 \in L^2(\Omega; C_T H^s_x)$.

Let $R_0 = \|u_0\|_{H^s_x} + C_1\big(R, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big)$ and B_{R_0} be the ball in $L^2(\Omega; C_T H^s_x)$ of center 0 and radius R_0. Then from (0.17), Γ is a contraction on B_{R_0} by choosing $T > 0$ small enough. More precisely, it suffices to choose T such that

$$T \ll \Big(\big(1 + \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) R^{\min(k,\gamma)-1}\Big)^{-\theta}. \tag{0.18}$$

□

We can globalize the solutions to (0.8) constructed in Proposition 0.3. This is the purpose of the next result.

Lemma 0.4. *Fix $s > \frac{d}{2}$ and $R > 0$. The solutions to (0.8) constructed in Proposition 0.3 exist globally in time. More precisely, for any $T > 0$, there exists $u_R \in L^2(\Omega; C_T H^s_x)$ which solves (0.8).*

Proof. Let T and u_R be the (random) time and the solution to (0.8) given by Proposition 0.3. We consider the problem (0.8) on $[T, 2T]$ with initial data $u_R(T)$ where u_R solves (0.8) on $[0, T]$. It reads

$$\begin{aligned} u(t) &= S(t-T)u_R(T) - i\int_T^t S(t-t')\eta_R(u)(t')|u|^{k-1}u(t')dt' \\ &\quad - i\int_0^t S(t-t')\eta_R(u)(t')|u|^{\gamma-1}u\,\Phi dW^T(t') \\ &:= \Gamma^T[u], \end{aligned} \tag{0.19}$$ YZ14

for any $T \le t \le 2T$. In (0.19), W^T denotes the shifted process $W^T(\cdot) := W(\cdot + T)$. Note that $\mathrm{Law}(W^T) = \mathrm{Law}(W)$.

It is then clear that $\Gamma^T[u]$ satisfies the same bounds (0.17) as $\Gamma[u]$ defined in (0.8) with u_0 replaced by $u_R(T)$. Hence, we have

$$\|\Gamma^T[u]\|_{L^2_\omega C_{I_2} H^s_x} \le \|u_R(T)\|_{L^2_\omega H^s_x} + C_1\big(R, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) \tag{0.20}$$ YZ15

$$\|\Gamma^T[u_1] - \Gamma^T[u_2]\|_{L^2_\omega C_{I_2} H^s_x} \le C_2 T^\theta \big(1 + \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) R^{\min(k,\gamma)-1} \|u_1 - u_2\|_{L^2_\omega C_{I_2} H^s_x}, \tag{0.21}$$ YZ16

for any $u, u_1, u_2 \in L^2(\Omega; C_{I_2} H^s_x)$, with $I_j = [(j-1)T, jT]$ for any $j \in \mathbb{N}$. By plugging the bound (0.17) into (0.20), we get

$$\|\Gamma^T[u]\|_{L^2_\omega C_{I_2} H^s_x} \le \|u_0\|_{H^s_x} + 2C_1\big(R, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big) \tag{0.22}$$ YZ17

Since T verifies (0.18), one can show, upon choosing $R_1\|u_0\|_{H^s_x} + 2C_1\big(R, \|\Phi\|_{\mathrm{HS}(L^2;H^s)}\big)$, that Γ^T in a contraction on the closed ball $B_{R_1} \subset L^2(\Omega; C_{I_2} H^s_x)$ of center 0 and radius R_1. By iterating this argument, we can construct solutions on I_j for any $j \ge 3$. $\square$

Remark 0.5. The solutions u_R that we constructed in Proposition 0.3 and Lemma 0.4 are adapted. Indeed, since we used a contraction argument to construct u_R, we can show by standard arguments that u_R is the limit (in $L^2\big(\Omega; C_T H^s_x\big)$, for any $T > 0$) of the Picard iterates which are adapted processes.

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Lecture 7

Last time, we constructed global solutions u_R to the truncated equations (SNLS$_R$). Recall

$$\eta_R(u_R)(t) = \eta\left(\frac{\|u_R\|_{C([0,t];H^s_x)}}{R}\right), \qquad \eta \equiv \begin{cases} 1 & \text{on } [0,1] \\ 0 & \text{on } [2,\infty) \end{cases} \tag{0.1}$$

Let $t_R = \inf\{t \geq 0 : \|u_R\|_{C([0,t];H^s)} \geq R\}$, which is a stopping time. Then, $u = u_R$ on $[0, t_R]$, where u is the solution to (SNLS).

Observe that t_R is non-decreasing in R (given $R < R'$, we have $u_R = u_{R'} = u$ on $[0, t_R]$).

Set $t_* = \lim\limits_{R\to\infty} t_R$, which is random, and define u on $[0, t_*]$ by setting $u = u_R$ on $[0, t_R]$. Thus, u is a solution to (SNLS) on $[0, t_*]$.

Blow-up alternative: If $t_* < \infty$, then by (0.1) and in light of the definition of t_*, we get $\lim\limits_{t\nearrow t_*} \|u(t)\|_{H^s} = \infty$.

On the algebra property of H^s and smoothness of a non-linearity.

Let $N(u)$ be a non-linearity in u and $\bar{u}$, which is homogeneous of degree p.

By the fundamental theorem of calculus, we have

$$\begin{aligned} N(u(x)) - N(u(y)) = \int_0^1 [\ &\partial_z N(u(y) + \theta\,(u(x) - u(y)))\,(u(x) - u(y)) \\ &+ \partial_{\bar{z}} N\,(u(y) + \theta\,(u(x) - u(y)))\,(\overline{u(x) - u(y)})\ ]\, d\theta, \end{aligned}$$

where

$$\begin{cases} \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \end{cases} \quad \text{with } \ z = x + iy.$$

Thus, we have

$$|N(u(x)) - N(u(y))| \lesssim \underbrace{\left\|N'(u)\right\|_{L^\infty_x}}_{\underset{\text{assume}}{\lesssim} \|u\|^{p-1}_{L^\infty_x}} |u(x) - u(y)|. \tag{0.2}$$

Hence, for $0 < s < 1$

$$\begin{aligned} \|N(u)\|_{\dot{H}^s} &= \left(\int_M \int_M \frac{|N(u(x)) - N(u(y))|^2}{|x-y|^{d+2s}}\, dx\, dy\right)^{\frac{1}{2}} \qquad M = \mathbb{R}^d \textit{ or } \mathbb{T}^d \\ &\overset{\text{by}(0.2)}{\lesssim} \|u\|^{p-1}_{L^\infty_x} \underbrace{\left(\int_M \int_M \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy\right)^{\frac{1}{2}}}_{=\|u\|_{\dot{H}^s}} \\ &\underset{\text{if } s>\frac{d}{2}}{\overset{\text{Sobolev}}{\lesssim}} \|u\|^p_{H^s}. \end{aligned}$$

For further studying when $M=\mathbb{R}^d$ mentioned in the first line above, see the book by Stein [7].

Then, assuming $N(u)$ behaves like u^p and the derivative of $N(u)$ behaves like u^{p-1}, we have

$$\|N(u)\|_{L^2} \underset{\substack{\uparrow \\ \text{assume}}}{\lesssim} \|u\|_{L^{2p}}^p \overset{\text{Sobolev}}{\lesssim} \|u\|_{H^s}^p \qquad s \geq d\left(\frac{1}{2}-\frac{1}{p}\right).$$

Since $H^s = L^2 \cap \dot{H}^s$, we conclude

$$\|\underset{\substack{\uparrow \\ C^1}}{N}(u)\|_{H^s} \lesssim \|u\|_{H^s}^p, \qquad \left.\begin{array}{r} 0<s<1 \\ s>\frac{d}{2} \end{array}\right\} \Rightarrow d=1.$$

In order to study a non-linear PDE, say SNLS, we need to estimate the difference $N(u)-N(v)$.

Once again, by FTC, we write

$$N(u)-N(v)=\int_0^1 \Big[\overbrace{\overbrace{\partial_z N(v+\theta\,(u-v))}\overbrace{(u-v)}}^{\text{product}} + \partial_{\bar{z}} N\,(v+\theta(u-v))\,(\overline{u-v})\Big]\,d\theta.$$

For $s>\frac{d}{2}$, by Minkowski inequality, making use of some algebra properties, and then using previous computation and the triangle inequality, we have

$$\|N(u)-N(v)\|_{H^s} \overset{\text{Mink.}}{\lesssim} \int_0^1 \underbrace{\big\|N'(v+\theta\,(u-v))\big\|_{H^s}}_{\substack{\lesssim \|v+\theta(u-v)\|_{H^s}^{p-1} \\ \lesssim \|u\|_{H^s}^{p-1}+\|v\|_{H^s}^{p-1}}} \|u-v\|_{H^s}\,d\theta, \qquad 0<s<1 \text{ and } N\in C^2.$$

where we used the notation N' for the relevant terms $\partial_z N$ and $\partial_{\bar{z}} N$ in the previous lines.
Note that in the above computation, we needed $N\in C^2$ and we used the condition $s<1$, which is not very useful except for one dimension.

What if $s>1$?

- Write $s=[s]+\{s\}$, $[\cdot]$: integer part, $\{\cdot\}$: fractional part.
- First, compute $\partial_x^{[s]}N(u)$.
- Then, repeat the previous computation to compute the $H^{\{s\}}$-norm of $\partial_x^{[s]}N(u)$, for which we need $N\in C^{[s]+1}$.
- Then, for the difference estimate, we need $N\in C^{[s]+2}$.

Stochastic non-linear wave equation (SNLW) with multiplicative space-time white noise on $\mathbb{T}^d$ (on $\mathbb{R}^d$, spacial white noise $W(t)$ is "unbounded").

As we see later, $d=1$.

Consider

$$\text{(SNLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u = N(u)+\sigma(u)\xi \\ (u,\partial_t u)\,|_{t=0} = (u_0,u_1), \end{cases}$$

where ξ is the space-time white noise.

<u>Duhamel formulation</u> (= mild formulation):

$$u(t) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1 + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}N(u)(t')dt' + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}\big(\sigma(u)(t')\,dW(t')\big).$$

With $S(t) = \frac{\sin(t|\nabla|)}{|\nabla|}$, $|\nabla| = \sqrt{-\Delta}$, $\widehat{S(t)f}(n) = \begin{cases} \frac{\sin(t|n|)}{|n|}\hat{f}(n), & n \neq 0 \\ t\hat{f}(0), & n = 0 \end{cases}$, we have

$$u(t) = \partial_t S(t)u_0 + S(t)u_1 + \int_0^t S(t-t')N(u)(t')dt' + \underbrace{\int_0^t S(t-t')\sigma(u)(t')dW(t')}_{=:\Psi[u](t)}.$$

Write $S(t) = S_+(t) - S_-(t)$ (No need to do this at the zeroth frequency), where $S_\pm(t) = \frac{e^{\pm it|\nabla|}}{2i|\nabla|}$. Now, write $\Psi[u] = \Psi_+[u] - \Psi_-[u]$.

Then, by BDG inequality,

$$\mathbb{E}\left[\sup_{0\le t\le T}\|\Psi_\pm[u](t)\|_{H^s}^p\right] \lesssim \mathbb{E}\left[\left(\int_0^T \underbrace{\big\|S_\pm(-t')\sigma(u)(t')\big\|_{HS(L^2;H^s)}^2 dt'}_{\text{When is this finite?}}\right)^{\frac{p}{2}}\right]$$

Compute

$$\begin{aligned}\big\|S_\pm(-t')\sigma(u)(t')\big\|_{HS(L^2;H^s)} &= \left(\sum_k \big\|S_\pm(-t')\circ M_{\sigma(u)(t')}(e_k)\big\|_{H^s}^2\right)^{\frac12} \qquad e_k = \mathrm{e}^{2\pi i k\cdot x} \\ &\sim \left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s-2}\underbrace{\left|\sum_{n=n_1+n_2}\widehat{\sigma(u)}(n_1)\delta_{n_2k}\right|^2}_{\left|\widehat{\sigma(u)}(n-k)\right|^2}\right)^{\frac12} \qquad \text{Sum in k,} \xrightarrow{} \text{then in n.} \\ &\sim \|\sigma(u)\|_{L^2}.\end{aligned}$$

In the above, $M_{\sigma(u)(t)}$ is multiplication by $\sigma(u)(t)$, and we replaced $S_\pm(-t')$ by $|\nabla|^{-1}$ (we do not care about the oscillation part because of unitarity). Furthermore, note that considering the term $\langle n\rangle^{2s-2}$, for obtaining convergence, we need $2s-2 < -d$ meaning that $s < -\frac{d}{2}+1$. On the other hand, we want the solution to be in H^s, so we want $s > 0$ or $s = \frac12 - \varepsilon$. Thus, we should have $d = 1$.

This computation shows that we can make sense of the stochastic convolution in one dimension. Now, in the case $d = 1$, take $s = \frac12 - \varepsilon$ for small $\varepsilon > 0$. Then

$$\|\underset{\substack{\shortparallel \\ u^\gamma}}{\sigma(u)}\|_{L^2} = \|u\|_{L^{2\gamma}}^\gamma \overset{Sobolev}{\lesssim} \|u\|_{H^s}^\gamma, \qquad (s \ge \frac12 - \frac{1}{2\gamma}).$$

And also, as for the non-linear part,

$$\left\| \int_0^t S(t-t') \underset{\substack{\uparrow \\ u^k}}{N(u)}(t')dt' \right\|_{C_T H_x^s} \lesssim T \left\| N(u) \right\|_{C_T H_x^{s-1(<0)}}$$

$$\overset{Sobolev}{\lesssim} \quad T \left\| N(u) \right\|_{C_T L_x^r} \qquad 1-s \geq \tfrac{1}{r} - \tfrac{1}{2} \Rightarrow -\tfrac{1}{r} \geq s - \tfrac{3}{2} \quad (*)$$

$$\overset{N(u)=u^k}{=} \quad T \|u\|^k_{C_T L_x^{kr}} \qquad s \geq \tfrac{1}{2} - \tfrac{1}{kr} \Rightarrow ks - \tfrac{k}{2} \geq -\tfrac{1}{r} \quad (**)$$

$$\overset{Sobolev}{\lesssim} \quad T \|u\|^k_{C_T H_x^s} \qquad ks - \tfrac{k}{2} \overset{(*),(**)}{\geq} s - \tfrac{3}{2} \Rightarrow s \geq \tfrac{k-3}{2(k-1)} \; (< \tfrac{1}{2})$$

Use the truncation method and construct global solutions $(u_R, \partial_t u_R) \in C(\mathbb{R}_+; \mathcal{H}^{\min(s, \frac{1}{2}-\varepsilon)}(\mathbb{T}))$, where $\mathcal{H}^s(\mathbb{T}) = H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$.

Thus, we get LWP of (SNLW) in $\mathcal{H}^s(\mathbb{T})$, $s \geq \max\left(\frac{1}{2} - \frac{1}{2\gamma}, \frac{k-3}{2(k-1)}\right)$.
(i.e. given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T})$, $\exists!$ solution $(u, \partial_t u)$ to (SNLW) in $C([0, t_*); \mathcal{H}^{s_0}(\mathbb{T}))$, where $s_0 = \min(s, \frac{1}{2} - \varepsilon)$).

Comment: We talked about local well-posedness. Now, the question is about global well-posedness.

- Set $N(u) \equiv 0$ and consider

$$\partial_t^2 u - \Delta u = \sigma(u)\xi, \qquad \text{on } \mathbb{T}.$$

  $-$GWP? Yes. when $\sigma(u)$ is locally Lipschitz and if $|\sigma(u)| \lesssim \langle u \rangle \log(2 + |u|)$, it does not grow very fast, then global well-posedness is known. This is done by Mueller (Ann Prob '97, see [4]).

  $-$Open question: $|\sigma(u)| \sim |u|^\gamma$ for $\gamma > 1$. finite time blow-up with positive probability?

- Similar question for stochastic heat equation:

$$\partial_t u - \Delta u = \sigma(u)\xi, \qquad \text{on } \mathbb{T} \qquad \sigma(u) \sim u^\gamma.$$

  $-$ Finite time blow-up for $\gamma \gg 1$ and $\gamma > \frac{3}{2}$ is known (for the former, see also [5] by Mueller-Sowers, PTRF '93, and for the latter, see [6] by Mueller, Ann Prob '00).
  $-$ GWP for $\gamma < \frac{3}{2}$ is known by Mueller, PTRF '91 (see [3]).

Lecture 8, pages 1-10

Stochastic non-linear heat equation with multiplicative space-time white noise

$$\text{(SNLH)} \quad \begin{cases} \partial_t u - \Delta u = N(u) + \sigma(u)\xi \\ u|_{t=0} = u_0 \end{cases} \qquad \text{on } \mathbb{T}^d,$$

where ξ is space-time white noise, i.e. $\Phi = Id$.

<u>Duhamel formulation</u> (= mild formulation):

$$u(t) = P(t)u_0 + \int_0^t P(t-t')N(u)(t')dt' + \underbrace{\int_0^t P(t-t')\sigma(u)(t')dW(t')}_{=:\Psi[u](t)}, \quad P(t) = e^{t\Delta}.$$

<u>Schauder estimate</u>: For $1 \le p \le q \le \infty,\ s \ge 0$,

$$\|P(t)f\|_{W^{s,q}} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{s}{2}}\|f\|_{L^p} \tag{0.3}$$

holds for any $t > 0$ on $\mathbb{R}^d$ and for any $0 < t \le 1$ on $\mathbb{T}^d$. For $\hat{P}(t) = e^{t(\Delta-1)}$, then (0.3) holds for any $t > 0$ even on $\mathbb{T}^d$.

Besov spaces and Hölder-Besov spaces:

Consider the classical Hölder spaces defined via the Hölder norm:

$$\|u\|_{\mathcal{C}^s} = \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s}, \quad 0 < s < 1.$$

Note that this is not a norm (because this is equal to zero for constant functions). In fact, this is a semi-norm.

Now, we want to introduce a different function space. To this end, we consider Littlewood-Paley frequency projector:

Let P_j = "smooth" frequency projection onto frequencies $\begin{cases} \{|n| \sim 2^j\}, & j \ge 1 \\ \{|n| \lesssim 1\}, & j = 0 \end{cases}$.

Take a smooth function $\varphi \in C_c^\infty(\mathbb{R}; [0,1])$ with

$$\begin{cases} \varphi(\xi) \equiv 1, & \text{for} \quad |\xi| \le 1 \\ \varphi(\xi) \equiv 0, & \text{for} \quad |\xi| \ge 2 \end{cases}$$

(the numbers used to define the cut-off function do not really matter, and if you like to get more useful numbers, see chapter 6 of the book [2] by Grafakos or the book by Bahouri-Chemin-Danchin [1]. These books are written for the analysis essentially on $\mathbb{R}^d$. On $(T)^d$, sometimes, you need to do something.)

Now, set

$$\varphi_j(\xi) = \begin{cases} \varphi(|\xi|), & j = 0, \\ \varphi(\frac{|\xi|}{2^j}) - \varphi(\frac{|\xi|}{2^{j-1}}), & j \ge 1. \end{cases} \qquad \left(\equiv \begin{cases} 1 \text{ for} & |\xi| \sim 2^j, \\ 0 \text{ for} & |\xi| \ll 2^j \text{ or } |\xi| \gg 2^j. \end{cases}\right)$$

Then, we set $\widehat{P_j u}(n) = \varphi_j(n)\hat{u}(n)$.

Theorem 0.1. *(Littlewood-Paley Theorem) Consider* $1 < p < \infty$. *We have*

$$\Bigg\| \underbrace{\left(\sum_{j=0}^\infty |P_j(u)|^2\right)^{\frac{1}{2}}}_{\textit{square function}} \Bigg\|_{L^p} \sim \|u\|_{L^p}.$$

Here, note that although the definition of the projector depends on the cut-off function φ, the result holds for any such cut-off function, but the constant depends on the choice of φ.

Littlewood-Paley characterization of H^s:

$$\|u\|_{H^s} \sim \left\| 2^{js} \|P_j(u)\|_{L^2_x} \right\|_{\ell^2_{j\geq 0}}, \qquad s \in \mathbb{R}.$$

Besov space $B^s_{p,q}$ (or $B^{s,p}_q$):

$$\|u\|_{B^s_{p,q}} = \left\| 2^{js} \|P_j(u)\|_{L^p_x} \right\|_{\ell^q_{j\geq 0}}, \qquad 1 \leq p, q \leq \infty, \quad s \in \mathbb{R}.$$

On $\mathbb{R}^d$, by setting $\psi_j(\xi) = \varphi(\frac{|\xi|}{2^j}) - \varphi(\frac{|\xi|}{2^{j-1}}),\ j \in \mathbb{Z}$ and $\widehat{Q_j u}(\xi) = \psi_j(\xi)\hat{u}(\xi)$, we can define the homogeneous Besov space $\dot{B}^s_{p,q}$ by

$$\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\| 2^{js} \|Q_j(u)\|_{L^p_x} \right\|_{\ell^q_{j\in\mathbb{Z}}}, \qquad 1 \leq p, q \leq \infty, \quad s \in \mathbb{R}.$$

Then, we have $\|u\|_{\mathcal{C}^s} = \|u\|_{\dot{B}^s_{\infty,\infty}},\ 0 < s < 1$.
If we set $\Lambda^s = C^s \cap L^\infty$, then, we have $\|u\|_{\Lambda^s} = \|u\|_{B^s_{\infty,\infty}},\ 0 < s < 1,$ on $\mathbb{R}^d$ and $\mathbb{T}^d$ (on $\mathbb{T}^d$, we need to modify a bit about the distance between elements on the torus).
Note also that Λ^s is a Lipschitz space, and $C^s = \dot{\Lambda}^s$. Furthermore, note that the right-hand side of the relation $\|u\|_{\mathcal{C}^s} = \|u\|_{B^s_{\infty,\infty}}$ makes sense for any $s \in \mathbb{R}$, which we now make use of for extending the definition.

Hölder-Besov spaces: $\mathcal{C}^s = B^s_{\infty,\infty},\ s \in \mathbb{R}$,

- $\|u\|_{\mathcal{C}^s} = \|u\|_{B^s_{\infty,\infty}} = \sup_{j\geq 0} 2^{js}\|P_j(u)\|_{L^\infty_x}$,
- $\mathcal{C}^s \supset W^{s,\infty}$,
- $s > 0,\ \mathcal{C}^s$ is an algebra,
- Schauder: $\|P(t)f\|_{\mathcal{C}^{s_2}} \lesssim t^{-\frac{s_2-s_1}{2}} \|f\|_{\mathcal{C}^{s_1}},\ s_2 \geq s_1$.

We want to study the stochastic convolution

$$\Psi[u](t) = \int_0^t P(t-t')\sigma(u)(t')dW(t')$$

via BDG inequality in $C_T\mathcal{C}^s_x$ (since we want to construct the stochastic convolution as an object which is continuous in time and takes values in $\mathcal{C}^s_x$).

Ideal property of γ-radonifying operator:

$$B_1 \xrightarrow{S} B_2 \xrightarrow{\Phi} B_3 \xrightarrow{T} B_4$$

$\Phi \in \gamma(B_2, B_3)$ is γ-radonifying operator, and $S \in L(B_1, B_2)$ and $T \in L(B_3, B_4)$ are linear and bounded operators.

Then $T \circ \Phi \circ S \in \gamma(B_1, B_4)$ with $\|T \circ \Phi \circ S\|_{\gamma(B_1,B_4)} \lesssim \|T\|_{L(B_3,B_4)}\|\Phi\|_{\gamma(B_2,B_3)}\|S\|_{L(B_1,B_2)}$.

Now, we want to use BDG inequality for continuos functions with values in $\mathcal{C}^s$ on the torus. $\mathcal{C}^s(\mathbb{T}^d) \underset{\text{Young}}{\supset} W^{s,\infty}(\mathbb{T}^d) \underset{\text{Sobolev}}{\supset} W^{s+\varepsilon,r}(\mathbb{T}^d), \quad \varepsilon r > d.$ Take $r \gg 1$ s.t. $\varepsilon = 0+$.

Fix $t \geq 0$. Then, by Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}\left[\|\Psi[u](t)\|^p_{\mathcal{C}^s_x}\right] &\leq \mathbb{E}\left[\sup_{0\leq t_0 \leq t}\left\|\int_0^{t_0} P(t-t')\sigma(u)(t')dW(t')\right\|^p_{W_x^{s+\varepsilon,r}}\right] \\ &\overset{BDG}{\lesssim} \mathbb{E}\left[\underbrace{\left(\int_0^t \left\|P(t-t_0)\circ M_{\sigma(u)(t')}\right\|^2_{\gamma(L^2;W^{s+\varepsilon,r})} dt'\right)^{\frac{p}{2}}}_{\text{When is this finite?}}\right].\end{aligned}$$

<u>Note:</u> $Id \in \gamma(L^2; W^{\alpha,r})$ iff $\alpha < -\frac{d}{2}$. $1 \leq r \leq \infty$.

1) $\sigma(u) \equiv 1$ (i.e. additive case)

$$\begin{aligned}\left\|P(t-t')\circ Id\right\|_{\gamma(L^2;W^{s+\varepsilon,r})} &\lesssim \left\|P(t-t')\right\|_{L(W^{-\frac{d}{2}-\varepsilon,r},W^{s+\varepsilon,r})} \underbrace{\|Id\|_{\gamma(L^2;W^{-\frac{d}{2}-\varepsilon,r})}}_{\lesssim 1} \\ &\overset{\text{Schauder}}{\lesssim} \left(t-t'\right)^{-\frac{s+\frac{d}{2}+2\varepsilon}{2}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_0^t \left\|P(t-t')\circ Id\right\|^2_{\gamma(L^2;W^{s+\varepsilon,r})} dt' &\lesssim \int_0^t \left(t-t'\right)^{-(s+\frac{d}{2}+2\varepsilon)}dt' \\ &< \infty \iff s+\frac{d}{2}+2\varepsilon < 1 \iff s < 1-\frac{d}{2}-2\varepsilon.\end{aligned}$$

On the other hand, we need $s > 0$ (s.t. $\mathcal{C}^s$ is an algebra) to handle the non-linearity $N(u) = u^k$. Hence, $0 < s < 1 - \frac{d}{2} - 2\varepsilon \Rightarrow d = 1$.

Remark 0.2. *(i) In the additive case ($\sigma(u) \equiv 1$), we do <u>not</u> need to work in $L^2(\Omega; C_T\mathcal{C}^s_x)$ with the BDG inequality and the truncation method. Instead, we can directly prove <u>path-wise</u> local well-posedness.*

(ii) We imposed the condition $s > 0$ s.t. u is a function (in x). In the additive case, the solution theory can be built for higher dimensions ($d = 2, 3$) even when $u(t)$ is only a distribution (in x). In this case, we need to introduce a <u>renormalization</u> to give a proper meaning to the non-linearity $N(u)$. See my course note from Spring 2021.

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Lecture 8, pages 11-29

Recall the stochastic non-linear heat equation with multiplicative space-time white noise:

$$\text{(SNLH)} \qquad \begin{cases} \partial_t u - \Delta u = N(u) + \sigma(u)\xi \\ u|_{t=0} = u_0, \end{cases}$$

where ξ is space-time white noise, i.e. $\Phi = Id$.
Duhamel formulation (or mild formulation) is given by:

$$u(t) = P(t)u_0 + \int_0^t P(t-t')N(u)(t')dt' + \int_0^t P(t-t')\sigma(u)(t')dW(t'),$$

where $P(t) = e^{t\Delta}$. We define $\Psi[u](t)$ as the stochastic convolution:

$$\Psi[u](t) := \int_0^t P(t-t')\sigma(u)(t')dW(t').$$

2) General case $\sigma(u) = u^\gamma$.

Correction: In this case, we can Not close the argument in $C_T\mathcal{C}_x^s$, simply using the BDG inequality. We instead work in $C_T W_x^{s,r}$ for some $s \geq 0,\ 2 \leq r < \infty$.

By BDG inequality, we need to study

$$\Big\| P(t-t') \circ M_{\sigma(u)(t')} \Big\|_{\gamma(L^2; W^{s,r})} \sim \bigg(\underset{\text{for}\{g_k\}}{\underset{\uparrow}{\mathbb{E}'}} \bigg[\Big\| \sum_k g_k \langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k) \Big\|_{L_x^r}^2 \bigg] \bigg)^{\frac{1}{2}},$$

where $\mathbb{E}'$ is the expectation with respect to $\{g_k\}$, and $e_k = e^{2\pi i k x}$. Also, note the solution u on Ω and $\{g_k\}$ on Ω' are independent.

$$\begin{aligned}
\big\| P(t-t') \circ M_{\sigma(u)(t')} \big\|_{\gamma(L^2; W^{s,r})} &\sim \Big\| \Big\| \sum_k g_k \langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k) \Big\|_{L_x^r} \Big\|_{L^r(\Omega')} \\
&= \Big\| \Big\| \sum_k g_k \langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k) \Big\|_{L^r(\Omega')} \Big\|_{L_x^r} \\
&\sim \Big\| \Big\| \sum_k g_k \langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k) \Big\|_{L^2(\Omega')} \Big\|_{L_x^r} \\
&\sim \Big\| \Big(\sum_k \big| \langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_x^r}.
\end{aligned}$$

In the above, we first replace the second moment by the rth moment (since it's Gaussian, every moments are equivalent) and then, after switching the order, we replace the rth moment by the second moment.

In general, $\|\Phi\|_{\gamma(L^2;L^r)} \sim \big\| \big(\sum_k |\Phi(e_k)|^2\big)^{\frac{1}{2}} \big\|_{L_x^r}$.

We have

$$\begin{aligned}
&\Big\|\Big(\sum_k \big|\langle\nabla\rangle^s P(t-t')\circ M_{\sigma(u)(t')}(e_k)\big|^2\Big)^{\frac12}\Big\|_{L^r_x}\\
&\qquad = \Big\|\Big(\sum_k \Big|\sum_{n\in\mathbb{Z}^d} e^{-(t-t')|n|^2}\langle n\rangle^s \widehat{\sigma(u)(t')}(n-k)e_n(x)\Big|^2\Big)^{\frac12}\Big\|_{L^r_x}\\
&\qquad \underset{r\geq 2}{\overset{\text{Minkowski}}{\leq}} \Big\|\Big\|\sum_{n\in\mathbb{Z}^d} e^{-(t-t')|n|^2}\langle n\rangle^s \widehat{\sigma(u)(t')}(n-k)e_n(x)\Big\|_{L^r_x}\Big\|_{\ell^2_k}\\
&\qquad \overset{\text{Hausdorff-Young}}{\leq} \Big\|\Big\|e^{-(t-t')|n|^2}\langle n\rangle^s \widehat{\sigma(u)(t')}(n-k)\Big\|_{\ell^{r'}_n}\Big\|_{\ell^2_k}, \qquad (r'\leq 2)\\
&\qquad \overset{\text{Minkowski}}{\leq} \Big\|e^{-(t-t')|n|^2}\langle n\rangle^s \underbrace{\Big\|\widehat{\sigma(u)(t')}(n-k)\Big\|_{\ell^2_k}}_{=\|\sigma(u)(t')\|_{L^2_x}}\Big\|_{\ell^{r'}_n}.
\end{aligned}$$

Now, we estimate $\big\|e^{-(t-t')|n|^2}\langle n\rangle^s\big\|_{\ell^{r'}_n}$, by using the bound

$$e^{-r'(t-t')|n|^2} \lesssim \min\Big(\frac{1}{(t-t')|n|^2},1\Big)^{\alpha}, \quad \text{for any } \alpha\geq 0.$$

We have

$$\begin{aligned}
\big\|e^{-(t-t')|n|^2}\langle n\rangle^s\big\|_{l^{r'}_n} &= \Big(\sum_{n\in\mathbb{Z}^d} e^{-r'(t-t')|n|^2}\langle n\rangle^{sr'}\Big)^{\frac{1}{r'}}\\
&\lesssim \Big(\sum_{n\in\mathbb{Z}^d}\frac{1}{(t-t')^{\alpha}\langle n\rangle^{2\alpha-sr'}}\Big)^{\frac{1}{r'}}\\
&\lesssim \frac{1}{(t-t')^{\frac{\alpha}{r'}}}, \quad \text{provided that } 2\alpha-sr'>d \Leftrightarrow s<\frac{2}{r'}\alpha-\frac{d}{r'}
\end{aligned}$$

Under $s<\frac{2}{r'}\alpha-\frac{d}{r'}$, we then have

$$\begin{aligned}
&\int_0^t \big\|P(t-t)\circ M_{\sigma(u)(t')}\big\|^2_{\gamma(L^2;W^{s,r})}\,dt'\\
&\qquad\lesssim \underbrace{\int_0^t \frac{1}{(t-t')^{\frac{2\alpha}{r'}}}dt'}_{\lesssim 1 \text{ iff } 2\alpha<r'}\times\|\sigma(u)\|^2_{C_TL^2_x}, \quad 0\leq t\leq T\\
&\qquad\lesssim \|\sigma(u)\|^2_{C_TL^2_x}.
\end{aligned}$$

If we put all conditions together, we get

$$\begin{cases} s<\frac{2}{r'}\alpha-\frac{d}{r'}\\ 2\alpha<r'\end{cases} \quad\Rightarrow\quad 0\leq s<1-\frac{d}{r'}\ \Big(\Leftrightarrow 0\leq s<\frac{1}{r}\ \text{ when }\ \alpha=1\Big)\ , \quad 1<r'\leq 2 \qquad (0.1)$$

which shows $d=1$.

With $\sigma(u) = u^\gamma$, we have

$$\|\sigma(u)\|_{C_T L^2_x} = \|u\|^\gamma_{C_T L^{2\gamma}_x}$$

$$\lesssim \|u\|^\gamma_{C_T W^{s,r}_x}, \qquad \begin{array}{l} s \geq 0 \ \text{ if } \ \text{r} \geq 2\gamma \\ \text{otherwise, by Sobolev } (d=1), \quad s \geq \frac{1}{r} - \frac{1}{2\gamma} \ \left(< \frac{1}{r}\right) \end{array}$$

Note: The BDG inequality with the computations above shows

$$\mathbb{E}\left[\|\Psi[u](t)\|^p_{W^{s,r}}\right] \lesssim \mathbb{E}\left[\|u(t)\|^{\gamma p}_{C_T W^{s,r}_x}\right],$$

for any <u>fixed</u> $0 \leq t \leq T$. <u>But</u>, we need to insert $\sup\limits_{0<t<T}$ under the expectation on LHS (see pages below).

As for the non-linear part, we have

$$\left\| \int_0^t P(t-t')N(u)(t')dt' \right\|_{C_T W^{s,r}_x}$$

$$\leq \left\| \int_0^t \left\|P(t-t')N(u)(t')\right\|_{W^{s,r}} dt' \right\|_{C_T} \qquad \text{(by Minkowski's integral inequality)}$$

$$\overset{\text{Schauder}}{\leq} \left\| \int_0^t (t-t')^{-\frac{s}{2} - \frac{1}{2}(1-\frac{1}{r})} \left\|N(u)(t')\right\|_{L^1_x} dt' \right\|_{C_T}$$

Considering $N(u) = u^k$, we have

$$\left\|N(u)(t')\right\|_{L^1_x} \leq \|u\|^k_{C_T L^k_x} \lesssim \|u\|^k_{C_T W^{s,r}_x} \quad \text{for} \begin{cases} s \geq 0, \ \text{if } r \geq k \\ s \geq \frac{1}{r} - \frac{1}{k} \ (< \frac{1}{r}) \end{cases}$$

Also, note that $\frac{s}{2} + \frac{1}{2}(1 - \frac{1}{r}) < 1$ since $0 < s < \frac{1}{r} < 1$. Thus, we have

$$\left\| \int_0^t P(t-t')N(u)(t')dt' \right\|_{C_T W^{s,r}_x} \lesssim T^\theta \|u\|^k_{C_T W^{s,r}_x}.$$

With a truncation, we can perform a contraction argument in $L^p_{ad}(\Omega; C_T W^{s,r}_x)$ to construct u_R ($p \gg 1$, see (0.3)) by choosing

1) $0 \leq s \leq \frac{1}{r}$
2) $r \geq 2\gamma$ or $s \geq \frac{1}{r} - \frac{1}{2\gamma}$
3) $r \geq k$ or $s \geq \frac{1}{r} - \frac{1}{k}$

Then, we have LWP of (SNLH) in $W^{s,r}_x(\mathbb{T})$.

Back to $\mathbb{E}\big[\sup\limits_{0<t<T} \|\Psi[u](t)\|^p_{W^{s,r}}\big]$, set $T = 1$. We repeat the argument in lecture 3.

Let $t_{l,k} = \frac{l}{2^k}$, $l = 0, 1, 2, \dots, 2^k$, and write

$$\Psi[u](t) = \sum_{k=1}^\infty \left(\Psi[u](t_{l_k,k}) - \Psi[u](t_{l_{k-1},k-1})\right) \tag{0.2}$$

for some $l_k = l_k(t) \in \{0, 1, \dots, 2^k\}$.

Note: In (0.2) we used continuity (in t) of $\Psi[u]$ (which we want to show). Strictly speaking, we perform the following argument to $\Pi_N u$ in place of u and take $N \to \infty$ (Π_N = smooth

frequency projection onto frequencies $\{\|n\| \le N\}$).
We then have

$$\sup_{0\le t\le 1} \|\Psi[u](t)\|_{W_x^{s,r}} \le \sum_{k=1}^{\infty} \max_{0\le l_k \le 2^k} \left\| \Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1}) \right\|_{W_x^{s,r}}$$

where $|t_{l_k,k} - t_{l'_{k-1},k-1}| \le 2^{-k}$. Thus,

$$\begin{aligned}
&\Big\| \sup_{0\le t\le 1} \|\Psi[u](t)\|_{W^{s,r}} \Big\|_{L^p(\Omega)} \\
&\qquad \le \sum_{k=1}^{\infty} \left\| \Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1}) \right\|_{L^p(\Omega;\ell^p_{l_k} W_x^{s,r})} \\
&\qquad \lesssim \sum_{k=1}^{\infty} 2^{\frac{k}{p}} \max_{0\le l_k\le 2^k} \left\| \Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1}) \right\|_{L^p(\Omega;W_x^{s,r})}
\end{aligned}$$

Claim 0.1.

$$\sup_{0\le t_1<t_2\le T} \big\|\Psi[u](t_2) - \Psi[u](t_1)\|_{L^p(\Omega;W_x^{s,r})} \lesssim (t_2-t_1)^{\theta} \|\sigma(u)\big\|_{L^p(\Omega;C_T L_x^2)}$$

for some $\theta > 0$ independent of p.

By assuming Claim, we have

$$\left\| \sup_{0\le t\le 1} \|\Psi[u](t)\|_{W^{s,r}} \right\|_{L^p(\Omega)} \lesssim \underbrace{\sum_{k=1}^{\infty} 2^{\frac{k}{p}} 2^{-k\theta}}_{\substack{\lesssim 1 \\ \text{by choosing } p\gg 1}} \|\sigma(u)\|_{L^p(\Omega;C([0,1];L_x^2))} \tag{0.3}$$

Proof of Claim. For $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
\Psi[u](t_2) - \Psi[u](t_1) &= \int_{t_1}^{t_2} P(t_2-t')\sigma(u)(t')dW(t') \\
&\quad + \int_0^{t_1} \left[P(t_2-t') - P(t_1-t')\right] \sigma(u)(t')dW(t') \\
&=: \mathrm{I}(t_1,t_2) + \mathrm{II}(t_1,t_2).
\end{aligned}$$

By BDG inequality and repeating the computations on previous pages

$$\|\mathrm{I}(t_1,t_2)\|_{L^p(\Omega;W_x^{s,r})} \lesssim \underbrace{\left(\int_{t_1}^{t_2} \frac{1}{(t_2-t')^{1-\frac{2\theta}{p}}} dt' \right)^{\frac{p}{2}}}_{\sim (t_2-t_1)^{\theta}} \|\sigma(u)\|_{L^p(\Omega;C_T W_x^{s,r})},$$

for some small $\theta > 0$.
As for II, first note that

$$\begin{aligned}
\left|\mathcal{F}_x\left(\left[P(t_2-t') - P(t_1-t')\right] f\right)(n)\right| &= \left|e^{-(t_2-t_1)|n|^2} - 1\right| e^{-(t_1-t')|n|^2} |\hat{f}(n)| \\
&\overset{\text{MVT}}{\lesssim} (t_2-t_1)^{\theta} |n|^{2\theta} e^{-(t_1-t')|n|^2} |\hat{f}(n)|.
\end{aligned} \tag{0.4}$$

Using (0.4), we repeat similar previous computation

$$
\begin{aligned}
&\Big\| \Big(e^{-(t_2-t')|n|^2} - e^{-(t_1-t')|n|^2} \Big) \langle n \rangle^s \Big\|_{\ell_n^{r'}} \\
&\quad \lesssim (t_2-t_1)^\theta \bigg(\sum_{n\in\mathbb{Z}^d} \frac{1}{(t_1-t')^\alpha} \frac{1}{\langle n\rangle^{2\alpha - sr' - 2\theta}} \bigg) \\
&\quad \lesssim (t_2-t_1)^\theta \frac{1}{(t_1-t')^{\frac{\alpha}{r'}}} \, , \qquad \text{if } \; 2\alpha - sr' - 2\theta > d \quad (A)
\end{aligned}
$$

Also note

$$
\int_0^{t_1} \frac{1}{(t_1-t')^{\frac{2\alpha}{r'}}} dt' \lesssim 1 \, , \qquad \text{if } \; \frac{2\alpha}{r'} < 1 \quad (B)
$$

By choosing $\theta > 0$ sufficiently small, the conditions (A) and (B) are satisfied in view of the conditions (0.1). $\square$

Remark 0.2. *In the argument LWP of (SNLH), we put $\sigma(u)(t')$ in L_x^2 when this term has more regularity. Thus, we can improve the argument a bit but it seems that we can* <u>*not*</u> *close the argument in $C_T\mathcal{C}_x^s$ via the BDG inequality used in the additive case (since we would get two contradictory conditions $s > \frac{1}{2} - \frac{1}{r} \underset{\epsilon r > 1}{>} \frac{1}{2} - \varepsilon$ and $s < \frac{1}{2} - \varepsilon$. (This ε comes from the embedding $W^{s+\varepsilon, r} \hookrightarrow \mathcal{C}^s$)).*

In the following, we directly show $\Psi[u]$ in $C_T\mathcal{C}_x^{s_0}$ for some $s_0 > 0$, where $u \in L_{ad}^p C_T H_x^{\frac{1}{2}-}$, $p \gg 1$.

In view of Kolmogorov's continuity criterion, it suffices to show

$$
\mathbb{E}\big[\, |\Psi[u](t_1,x_1) - \Psi[u](t_2,x_2)|^p \, \big] \lesssim |(t_1,x_1) - (t_2,x_2)|^{1+\theta} \quad \text{for some } \; \theta > 0 \; \text{ and } \; p \gg 1
$$

We have

$$
\begin{aligned}
\Psi[u](t_1,x_1) - \Psi[u](t_2,x_2) &= \big(\Psi[u](t_1,x_1) - \Psi[u](t_2,x_1) \big) \\
&\quad + \big(\Psi[u](t_2,x_1) - \Psi[u](t_2,x_2) \big).
\end{aligned}
$$

For the first term on the right-hand side, we can use the ideas from the proof of Claim . So we focus only on

$$
\mathbb{E}\big[\, |\Psi[u](t,x_1) - \Psi[u](t,x_2)|^p \, \big].
$$

We have

$$
\begin{aligned}
&\Psi[u](t,x_1) - \Psi[u](t,x_2) \\
&\quad = \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \int_0^t e^{-(t-t')|n|^2} \widehat{\sigma(u)(t')}(n-k) d\beta_k(t') \big(e_n(x_1) - e_n(x_2) \big) \\
&\quad = \sum_{k\in\mathbb{Z}^d} \int_0^t \sum_{n\in\mathbb{Z}^d} e^{-(t-t')|n|^2} \widehat{\sigma(u)(t')}(n-k) \big(e_n(x_1) - e_n(x_2) \big) d\beta_k(t').
\end{aligned}
$$

By the BDG inequality for scalar martingales, we have

$$
\begin{aligned}
&\mathbb{E}\big[\,|\Psi[u](t,x_1)-\Psi[u](t,x_2)|^p\,\big]\\
&\quad\lesssim \mathbb{E}\Bigg[\bigg(\sum_{k\in\mathbb{Z}^d}\int_0^t\Big|\sum_{n\in\mathbb{Z}^d}e^{-(t-t')|n|^2}\widehat{\sigma(u)(t')}(n-k)\big(e_n(x_1)-e_n(x_2)\big)\Big|^2dt'\bigg)^{\frac{p}{2}}\Bigg].
\end{aligned}
$$

By mean value theorem, we have $|e_n(x_1)-e_n(x_2)|\lesssim |n|^\delta|x_1-x_2|^\delta$, for any $0\le\delta\le 1$. Thus,

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\int_0^t\Big|\sum_{n\in\mathbb{Z}^d}e^{-(t-t')|n|^2}\widehat{\sigma(u)(t')}(n-k)\big(e_n(x_1)-e_n(x_2)\big)\Big|^2dt'\\
&\quad\lesssim |x_1-x_2|^{2\delta}\int_0^t\|a_n*b_n\|^2_{\ell^2_n}\,dt',
\end{aligned}
$$

where $a_n=e^{-(t-t')|n|^2}|n|^\delta$, $b_n=\widehat{\sigma(u)(t')}(n)$, and we have $a_n\lesssim\dfrac{1}{(t-t')^{2\theta}\langle n\rangle^{2\theta-\delta}}$,

$$
\begin{aligned}
&\overset{\text{Young}}{\lesssim}\ |x_1-x_2|^{2\delta}\underbrace{\int_0^t\frac{1}{(t-t')^{2\theta}}dt'}_{\substack{\lesssim 1\\ \text{for }\ \theta<\frac12}}\times\underbrace{\Big\|\frac{1}{\langle n\rangle^{2\theta-\delta}}\Big\|^2_{\ell^{1+}_n}}_{\substack{\lesssim 1\\ \text{by choosing }\theta=\frac12-,\ \delta>0\text{ small}}}\ \big\|\widehat{\sigma(u)(t')}(n)\big\|^2_{\ell^{2-}_n}\qquad (\tfrac12+1=\tfrac{1}{1+}+\tfrac{1}{2-})\\
&\overset{\text{Hölder}}{\lesssim}\ |x_1-x_2|^{2\delta}\big\|\sigma(u)\big\|^2_{C_TH^\alpha_x},\quad \alpha=0+\,.
\end{aligned}
$$

Lastly, by the fractional Leibniz rule with $\sigma(u)=u^\gamma$

$$
\begin{aligned}
\big\|\sigma(u)\big\|_{C_TH^\alpha_x}&\lesssim\|u\|^\gamma_{C_TW^{\alpha,2\gamma}_x}\\
&\overset{\text{Sobolev}}{\lesssim}\ \|u\|^\gamma_{C_TH^s_x}\,,\qquad s\ge\underset{0+}{\overset{\shortparallel}{\alpha}}+\frac12-\frac{1}{2\gamma}(<\frac12).
\end{aligned}
$$

Hence, we proved

$$
\begin{aligned}
\mathbb{E}\big[\,|\Psi[u](t,x_1)-\Psi[u](t,x_2)|^p\,\big]&\lesssim|x_1-x_2|^{\delta p}\,\mathbb{E}\big[\,\|u\|^{\gamma p}_{C_TH^s_x}\,\big]\\
&\lesssim|x_1-x_2|^{1+\theta}\,\mathbb{E}\big[\,\|u\|^{\gamma p}_{C_TH^s_x}\,\big],\qquad\text{by choosing }p\gg 1.
\end{aligned}
$$

As for the non-linear part,

$$
\begin{aligned}
\Big\|\int_0^tP(t-t')N(u)(t')dt'\Big\|_{C_T\mathcal{C}^{s_0}_x}&\lesssim\Big\|\int_0^t\big\|P(t-t')N(u)(t')\big\|_{W^{s_0,\infty}_x}dt'\Big\|_{C_T}\\
&\overset{\text{schauder}}{\lesssim}\ \underbrace{\Big\|\int_0^t(t-t')^{-\frac12-\frac{s_0}{2}}\,dt'\Big\|_{C_T}}_{\lesssim T^\theta}\underbrace{\|N(u)\|_{C_TL^1_x}}_{\le\|u\|^k_{C_TL^k_x}}\\
&\lesssim T^\theta\|u\|^k_{C_TH^s},\qquad s\ge\frac12-\frac1k\,(<\frac12).
\end{aligned}
$$

LECTURE NOTES FOR STOCHASTIC PDES WITH MULTIPLICATIVE NOISES

USAMA NADEEM

Contents

1. Back to SNLS with multiplicative noise

Recall that the Stochastic Non-linear Schrödinger equation with multiplicative noise:

$$\begin{cases} i\partial_t u - \nabla u = \mathcal{N}(u) + \sigma(u)\Phi\xi \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d) \end{cases} \tag{SNLS}$$

In analysing (SNLS) we will require the following "stochastic" Strichartz estimate ([1] and [2]):

Proposition 1.1 (Stochastic Strichartz Estimates)**.** *Let $F : \mathbb{R}_+ \to HS(L^2; H^s)$, τ an accessible stopping time (i.e. it is covered by by a sequence of predictable times), and (q, r) an admissible pair in the sense that: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 2)$. Then the stochastic integral,*

$$I_{[0,\tau)}F(t) = \int_0^t \mathbf{1}_{[0,\tau)}(t)S(t-t')F(t')dW(t')$$

admits the following bound, except for $r = \infty \Rightarrow d = 1$:

$$\left\| \|I_{[0,\tau)}F\|_{L^q_T W^{s,r}_x(\mathbb{R}^d)} \right\|_{L^p(\Omega)} \leq C(p,q,r,T) \left\| \left(\int_0^\tau ||F(t)||^2_{HS(L^2;H^s)} dt \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

Remark 1.2. *The above result is only for finite times.*

Remark 1.3. *Compare with the usual deterministic Strichartz estimate on the non-homogeneous part is given:*

$$\left\| \int_0^t S(t-t')\mathcal{N}(t')dt' \right\|_{L^q_t W^{s,r}_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\mathcal{N}\|_{L^{\tilde{q}'}_t W^{s,\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}$$

where $(q, r), (\tilde{q}, \tilde{r})$ are admissible and by the primes we mean the Hölder conjugates.

Proof. Let $0 \leq t \leq T$

$$\begin{aligned} I_{[0,\tau)}F(t) &= \int_0^T \mathbf{1}_{[0,\tau)}(t')G_{t'}(t)dW(t') \\ &=: \left(\int_0^T \mathbf{1}_{[0,\tau)}(t')G_{t'}dW(t') \right)(t) \end{aligned}$$

where $G_{t'} : [0,T] \ni t \mapsto \mathbf{1}_{[t',T]}(t)S(t-t')F(t')$ with $t' \in [0,T]$. Notice that the second step is nothing but repurposing the integral as a function of t, with which we write:

$$\|I_{[o,\tau)}F\|_{L^q_T W^{s,r}_x} = \left\| \int_0^T \mathbf{1}_{[0,\tau)}(t')G_{t'}dW(t') \right\|_{L^q_t([0,T];W^{s,r}_x)} \tag{1.1}$$

Assume now that $q < \infty$ and $r < \infty$. This we want to do because the space $L^q_T W^{s,r}_x$ for $2 \leq q, r < \infty$ is M-type 2 and hence with the BDG inequality and (1.1):

$$\left\| \|I_{[0,\tau)}F\|_{L^q_T W^{s,r}_x(\mathbb{R}^d)} \right\|_{L^p(\Omega)} \overset{\text{BDG}}{\lesssim} \left\| \left(\int_0^T \mathbf{1}_{[0,\tau]}(t')\|G_{t'}\|^2_{\gamma(L^2:L^q_T W^{s,r}_x)} dt' \right)^{1/2} \right\|_{L^p(\Omega)} \tag{1.2}$$

To bound the above quantity then we first establish the following bound

$$\begin{aligned}
\|G_{t'}\|^q_{\gamma(L^2;L^q_T W^{s,r}_x)} &\sim \mathbb{E}\int_0^T \Big\|\sum_n g_n G_{t'}(e_n)(t)\Big\|^q_{W^{s,r}_x} dt \\
&= \mathbb{E}\int_{t'}^T \left\|\sum_n g_n \left(S(t-t')\circ F(t')\right)(e_n)(t)\right\|^q_{W^{s,r}_x} dt \\
&= \mathbb{E}\int_{t'}^T \left\|S(t)\circ S(-t')\left(\sum_n g_n F(t')(e_n)\right)\right\|^q_{W^{s,r}_x} dt \\
&\leq \mathbb{E}\left[\|S(t)(\cdots)\|^q_{L^q_t(\mathbb{R};W^{s,r}_x)}\right] \\
&\overset{(\diamond)}{\lesssim} \mathbb{E}\left[\|\sum_n g_n F(t')(e_n)\|^q_{H^s}\right] \\
&\sim \|\sum_n g_n F(t')(e)n)\|^q_{L^2(\Omega;H^s)} \\
&= \|F(t')\|^q_{\gamma(L^2;H^s} = \|F(t')\|^2_{\mathrm{HS}(L^2;H^s)}
\end{aligned}$$

where $\{e_n\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. The initial steps are just from the norm on our space and the definition of $G_{t'}$, the bound $(\diamond)$ is from the usual strichartz estimate, and the last two equalities are again just by definition. The required bound is then achieved by putting the above bound into (1.2).

Consider now the case $q=\infty$ for which we know that $r=2$. The quantity of interest then is:

$$\begin{aligned}
\big\|\|I_{[0,\tau)}F\|_{L^\infty_T H^s_x}\big\|_{L^p(\Omega)} &= \left\|\sup_{0\leq t\leq T}\left\|\int_0^t \mathbf{1}_{[0,\tau)](t')S(-t')F*t')dW(t')}\right\|_{H^s}\right\|_{L^p(\Omega)} \\
&\overset{BDG}{\lesssim} \|\left(\int_0^\tau \|S(-t')F(t')\|^2_{HS(L^2;H^s)}dt'\right)^{1/2}\|_{L^p(\Omega)}
\end{aligned}$$

The first equality is gotten by recalling the definition of $I_{[0,\tau)}F(t)$, the fact $S(t-t') = S(t)\circ S(-t')$ and the unitarity of $S(t)$ in H^s_x. Notice that the BDG is now applied on H^s space and not spacetime as we did in the previous case. Recall then that $\|S(-t')F(t')\|^2_{\mathrm{HS}(L^2;H^s)} = \|S(t')F(t')(e_n)\|_{\ell^2_n H^s_x}$ and hence it too we are able to drop. The result follows. $\square$

Remark 1.4. *The first paper that handles the* (SNLS) *in a modern way is* [3]*, and then in* [4] *they handle the H^1 subcritical case. To handle the stochastic convolution they used first the BDG inequality and then the dispersive estimate (which is an ingredient to the strichartz estimate that we have used).*

$$\|S(t)f\|_{L^r_x} \lesssim \frac{1}{|t|^{\frac{d}{2}-\frac{d}{r}}}\|f\|_{L^2_x}$$

2. Stochastic Convolution

As is usual, we have to handle the stochastic convolution first:

$$\Psi[u](t) = \int_0^t S(t-t')\sigma(u)(t')\Phi dW(t') \tag{2.1}$$

where $\sigma(u) \sim u^\gamma$, $\gamma \geq 1$.

As we are interested in studying the (SNLS) in L^2 setting we assume that:

$$\Phi \in \mathrm{HS}(L^2;L^2) \cap \gamma(L^2;L^\infty)$$

and then to bound (2.1), in light of Proposition 1.1 (with $F(t')$ chosen to be $\sigma(u)(t')\Phi$), we only need the bound:

$$\begin{aligned}\|\sigma(u)\Phi\|_{L^2_T \mathrm{HS}(L^2;L^2)} &= \left\| \Big(\sum_n \|\sigma(u)(t)\Phi(e_n)\|^2_{L^2_x} \Big)^{1/2} \right\|_{L^2_T} \\ &\leq \|\sigma(u)\|_{L^2_T L^2_x} \|\Phi\|_{\gamma(L^2;L^\infty)}\end{aligned}$$

where for the inequality we use Hölder's inequality on the normed quantity in the summand, independence of $\sigma(u)(t)$ from n, and then just the definition of $\gamma(L^2;L^\infty)$. By our assumption on Φ, the second term in the last inequality is finite. Also:

$$\|\sigma(u)\|_{L^2_T L^2_x} = \|u^\gamma\|_{L^2_T L^2_x} = \|u\|^\gamma_{L^{2\gamma}_T L^{2\gamma}_x} \overset{?}{\lesssim} \|u\|^\gamma_{L^q_T L^r_x}$$

The inequality marked by (?) is what we would like to see for some admissible pair (q,r). By applying Hölder's in time we can take $q \geq 2\gamma$ but in this setup (with L^∞ used in the previous step) the only viable choice is $r = 2\gamma$. For admissibility we require that:

$$\frac{d}{2} = \frac{2}{q} + \frac{d}{r} \leq \frac{1}{\gamma}(d+2)$$

but the upper bound on in terms of the γ and d comes from our choice of q and r. Hence the condition to be satisfied for all our requirements to be met is given by $\gamma \leq 1 + \frac{2}{d}$.

Remark 2.1. *For the non-linear part $\mathcal{N}(u) = |u|^{k-1}u$, we need the L^2-(sub)criticality: $s_{\mathrm{crit}} \geq 0$ in order to study SNLS with $u_0 \in L^2_x$. This condition translates to $k \leq 1 + \frac{4}{d}$.*

Now equipped with Proposition 1.1, we get for any admissible pair $(\tilde{q},\tilde{r})$ with $\tilde{r} < \infty$:

$$\begin{aligned}\Big\| \|\Psi[u](t)\|_{L^{\tilde{q}}_T L^{\tilde{r}}_x} \Big\|_{L^p(\Omega)} &\lesssim \Big\| \|\sigma(u)\Phi\|_{L^2_T \mathrm{HS}(L^2;L^2)} \Big\|_{L^p(\Omega)} \\ &\lesssim \|\Phi\|_{\gamma(L^2;L^\infty)} \Big\| \|u\|_{L^q_T L^r_x} \Big\|_{L^p(\Omega)}\end{aligned} \tag{2.2}$$

where $r = 2\gamma$, $1 \leq \gamma \leq 1 + \frac{2}{d}$ and (q,r) admissible.

Remark 2.2. *Notice that while we require $\tilde{r} < \infty$, $\tilde{q}$ can be very well be infinite, and in this case we would have $\tilde{r} = 2$. In particular the argument works with $\tilde{q} = q$ and $\tilde{r} = r$.*

Now by a truncation argument we can prove the local well-posedness of the (SNLS) in $L^(\mathbb{R}^d)$ when $1 \leq \gamma \leq 1 + \frac{2}{d}$ and $1 < k \leq 1 + \frac{4}{d}$. One has to be careful that now we are working in the intersection of two spaces and hence the truncation will depend on the two respective norms. The cutoff we will use now has the following form:

$$\eta_R(u)(t) = \eta\left(\frac{\|u\|_{C([0,t];L^2x)} + \|u\|_{L^q([0,t];L^r_x)}}{R}\right)$$

and (2.1) is handled via Proposition 1.1. Everything else should carry over from the additive case.

Example 1. *Take $d = 1$, $k = 3$, $\gamma = 2$, that is the 1-d cubic* (SNLS)*. This implies that $(q, r) = (8, 4)$, which is the same pair as the additive case in Lecture 4.*

Remark 2.3. *When $\gamma < 1 + \frac{2}{d}$, we can relax the condition $\Phi \in \gamma(L^2; L^\infty)$. In* (2)*, if we check the $L^2_T L^\alpha_x$-norm (with $\alpha > 2$):*

$$\|\sigma(u)\|_{L^2_T L^\alpha_x} = \|u\|^\gamma_{L^{2\gamma}_T L^{\alpha\gamma}_x}$$

for $r = \alpha\gamma$ and $q \geq 2\gamma$. Then, we put Φ in $\gamma(L^2; L^{\frac{2\alpha}{\alpha-2}})$ instead of $\gamma(L^2; L^\infty)$. Notice that $\frac{1}{2} = \frac{1}{\alpha} + \frac{\alpha-2}{2\alpha}$

Now for r and q as before and admissible, we get:

$$\frac{d}{2} = \frac{2}{q} + \frac{d}{r} \leq \frac{1}{\gamma} + \frac{d}{r} \Rightarrow \frac{1}{\alpha} = \frac{\gamma}{r} \geq \frac{\gamma}{2} - \frac{1}{\alpha}$$
$$\Rightarrow \frac{1}{\beta} = \frac{1}{2} - \frac{1}{\alpha} \leq \frac{1}{2} - \frac{\gamma}{2} + \frac{1}{\alpha} = \frac{1}{2}\left(1 + \frac{2}{d} - \gamma\right)$$

and hence it is enough to assume $\Phi \in \gamma(L^2; L^\beta)$, with $\beta \geq \frac{2}{1+\frac{2}{d}-\gamma} \vee 2$.

3. Blowup Alternative

If the maximal time of existence (which is random) $T_{\max} = T_{\max}(\omega) < \infty$, then:

$$\lim_{t \nearrow T_{\max}} \left(\|u\|_{C([0,t];H^s_x)} + \|u\|_{L^q([0,t];L^r_x)}\right) = \infty \tag{3.1}$$

In some cases we are able to reduce this to:

$$\lim_{t \nearrow T_{\max}} \|u\|_{C([0,t];H^s_x)} + \|u\|_{L^q([0,t];L^r_x)} = \infty \tag{3.2}$$

Of course (3.2) is not automatic and requires a proof.

Example 2. *Let $d = 1$, $k = 3$, $\gamma = 1$. The solution is then constructed in $C_T L^2_x \cap L^8_T L^4_x$. Indeed by using Strichartz, followed by Hölder's inequality, one has:*

$$\begin{aligned}\left\|\int_0^t S(t-t')|u|^2 u(t')dt'\right\|_{L^8_T L^4_x} &\lesssim \left\||u|^2 u\right\|_{L^{4/3}_t L^1_x} \\ &\leq T^{1/2}\|u\|_{L^\infty_T L^2_x}\|u\|^2_{L^8_T L^4_x}\end{aligned}$$

From the Duhamel formulation then we have the following:

$$\begin{aligned}\|u\|_{L^8_T L^4_x} &\leq C\|u_0\|_{L^2} + CT^{1/2}\|u\|_{L^\infty_T L^2_x}\|u\|^2_{L^8_T L^4_x} + \|\Psi[u]\|_{L^8_T L^4_x} \\ &\overset{\diamond}{\leq} C_0\left(1 + \|u\|_{L^\infty_T L^2_x}\right)^2 + \|\Psi[u]\|_{L^8_T L^4_x} + C_1 T\|u\|^4_{L^8_T L^4_x}\end{aligned}$$

where $\diamond$ is from Cauchy's inequality ($ab \leq \frac{a^2}{2} + \frac{b^2}{2}$). We assume that the $\|u\|_{L^\infty_T L^2_x}$ is finite.

Fixing a stopping time $\tau < T_{\max} = T_{\max}(\omega)$, we get:

$$\|u\|_{L_I^8 L_x^4} \leq C_0 \left(1 + \|u\|_{L_\tau^\infty L_x^2}\right)^2 + \|\Psi[u]\|_{L_\tau^8 L_x^4} + C_1 |I| \|u\|_{L_I^8 L_x^4}^4 \tag{3.3}$$

for any interval $I \subset [0, \tau)$.

Through a continuity argument with $|I| \gg 1$ (which we spell out in Subsection 3.1) we have:

$$\|u\|_{L_I^8 L_x^4} \leq 2C_0 \left(1 + \|u\|_{L_\tau^\infty L_x^2}\right)^2 + 2\|\Psi[u]\|_{L_\tau^8 L_x^4} \quad =: K(w) \tag{3.4}$$

3.1. **Continuity Argument.** Suppose that a continuous function $X(t)$ satisfies: $X(t) \leq A + B\varepsilon X^4(t)$ for any $t \in [t_0, t_1]$ and $X(t_0) \leq A$. Then a $\varepsilon < 0$ can be chosen such that the initial condition is not violated and $X(t) \leq 2A$ for any $t \in [t_0, t_1]$.

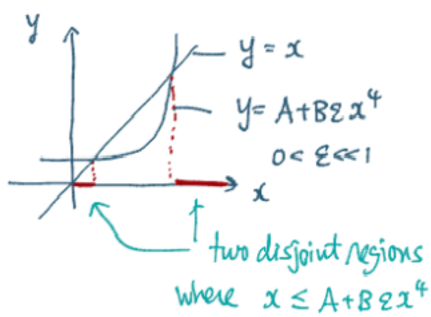

FIGURE 1. Continuity Argument

Essentially the argument is that in the above plot there are two disjoint regions depicted in red where the line is dominated by the degree 4 polynomial. The additional condition assures us that it actually must be in the left region and not the other unbounded region. Due to continuity we know that it cannot 'jump' and hence it must stay there.

Instead of making this more rigorous we prefer to get the same conclusion from a bootstrap argument:

Proposition 3.1. *For a continuous function $X(t)$ if the following is true:*

$$X(t_0) \leq A$$
$$X(t) \leq A + B\varepsilon X^4(t)$$

then for a sufficiently small choice of ε and for all $t \in I = [t_0, t_1]$, one has:

$$X(t) \leq 2A$$

$\forall t \in I$.

Proof. The proof is inductive in nature. We begin with $X(t_0) \leq A$ and use the continuity of X to conclude that $X(t) \leq 10A$, for all $t \in [t_0, t_0 + \delta_1]$. Putting this into the second part of the hypothesis and then choosing ε sufficiently small (say $\varepsilon \sim A^{-3}$) yields: $X(t) \leq A + B\varepsilon(10A)^4 \leq 2A$, for all $t \in [t_0, t_0 + \delta_1]$, and so in particular $X(t_0 + \delta_1) \leq 2A$.

Again by continuity we are able to push a bit forward, in the sense that $X(t) \leq 10A$, for all $t \in [t_0, \delta_1, t_0 + \delta_1 + \delta_2]$, which as before implies that $X(t) \leq A + B\varepsilon(10A)^4 \leq 2A$, for all $t \in [t_0 + \delta_1, t_o + \delta_1 + \delta_2]$. We repeat this algorithm until the whole of the interval is covered. $\square$

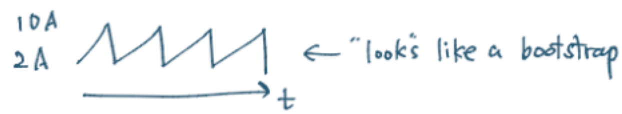

FIGURE 2. Bootstrapping the bound

With this proposition in hand, if one chooses $|I| \sim K(\omega)^{-3}$ (which comes from how we chose ε in the proof), then (3.3) implies (3.4).

Fix now some $T_\star \gg 1$. By writing $[0, T_\star \wedge \tau] = \cup I_j$, where $|I_j| = I$ (except for the last interval), and τ is any reasonable stopping time like say $T_{\max} - \delta$ for δ small, we have:

$$\begin{aligned} \|u\|_{L^8([0,T_\star\wedge\tau];L^4_X x} &= \left(\sum_j \|u\|^8_{L^8_{I_j} L^4_x}\right)^{1/8} \\ &\lesssim \left(\frac{T_\star}{|I|}\right)^{1/8} K(w) \\ &< T_\star^{1/8} K^{5/8}(\omega) \end{aligned} \tag{3.5}$$

where the first inequality is from the fact that we know each summand to be less than $2K$ and the number of such summand (or intervals) is given by fraction. For the second inequality above we have used the choice of $|I|$.

Our goal now is to show that (3.2) holds. Assume for contradiction that it does not. Then we have:

$$\mathbb{P}\left(\sup_{t\leq T_{\max}} \|u(t)\|_{L^2} < \infty \text{ and } T_{\max} < \infty\right) > 0$$

By chosing $T_\star \gg 1$ and the fact that the blow-up time is finite, we get:

$$\mathbb{P}\left(\sup_{t\leq T_{\max}} \|u(t)\|_{L^2} < \infty \text{ and } T_{\max} < \infty\right) > 0$$

For some given $R \gg 1$, we define the stopping time:

$$t_R = \inf\{t_0 \in [0, T_\star) : \|u(t_0)\|]\}_{L^2} \geq R \text{ or } t_0 \geq T_{\max}$$

Which gives us:

$$\mathbb{P}(\underbrace{t_R = T_{\max}}_{=:A_R}) > 0 \tag{3.6}$$

by choosing $R >> 1$.

We also see from (3.5) and $\tau = t_R$:

$$\mathbb{E}\left[\|u\|^p_{L^8_{t_R} L^4_x}\right] \lesssim T_\star^{1/8}\mathbb{E}\left[K(w)^{3p/8}\right]$$

Further (3.4) can be put together with t_R, do give:

$$K(w) \leq 2C_0 (1+R)^2 + \|\Psi[u]\|_{L^8_{t_R} L^4_x}$$

.

And from (2.2), we have:

$$\Big\| \|\Psi[u]\|_{L^8_{t_R} L^4_x} \Big\|_{L^p(\Omega)} \lesssim \|\Phi\|_{\gamma(L^2;L^\infty)} \Big\| \underbrace{\|u\|_{L^\infty_{t_R} L^2_x}}_{\leq R} \Big\|_{L^p(\Omega)}$$

One see that by use of the stopping time we have finiteness for the RHS in the last inequality, which feeds into the penultimate inequality, which further feeds into the antepenultimate inequality.

Putting these three together then we get:

$$\mathbb{E}\left[\|u\|^p_{L^8_{t_R} L^4_x} \right] \leq C(T_\star, R)$$

Finally, from (3.6), and the inequality above, we get:

$$\mathbb{E}\left[\mathbf{1}_{\{A_R\}} \|u\|^p_{L^8_{t_R} L^4_x} \right] \leq C(T_\star, R)$$

Which implies on A_R, that:

$$\|u\|_{L^\infty_{T_{\max}} L^2_x} + \|u\|_{L^8_{T_{\max}} L^4_x} < \infty$$

But $T_{\max} < \infty$, which is contradictory to (3.1).

We can conlude from this discussion that for $d = 1$, $k = 3$, $r = 1$, if $T_{\max} < \infty$, then $\lim_{t \nearrow T_{\max}} \|u(t)\|_{L^2} = \infty$

Hence, global well-posedness follows once we prove:

$$\sup_{0 \leq t \leq T_{\max}} \|u(t)\|_{L^2} < \infty$$

which we will prove by applying Ito's lemma to the mass $M(u) = \int |u|^2 dx$ which is conserved under the deterministic NLS (i.e. $\Phi \equiv 0$)):

$$i\partial_t u - \Delta u = |u|^{k-1} u$$

For smooth solutions, the proof of conservation of mass is straightforward:

$$\partial_t M(u) = 2\mathrm{Re} \int u \, \overline{\partial_t u} \, dx \overset{\text{IBP}}{=} -2\mathrm{Re} i \int |\nabla u|^2 dx + 2\mathrm{Re} i \int |u|^{k+1} dx = 0$$

For (SNLS) we do not expect this to be conserved but we will hope to get some control via Ito's lemma.

Finally we collect some references for the (SNLS):

- GWP of the (SNLS) with multiplicative noise:
  - de Bouard-Debusshe [3], [4] ($\gamma = 1$)
  - Hormung [2] ($k < 1 + \frac{4}{d}$, $1 \leq \gamma \leq \gamma_k$).
  - mass-critical case ($\gamma = 1$). In the case of $d = 1$ and $k = 5$, we cannot conclude that if the solution blows up then the L^2 norm blows up. The argument for this, as in Fan-Xu [5] is more subtle, like as in [6]. The idea is to write the Duhamel formula with a linear, non-linear, and stochastic convolution part. You try to view the stochastic convolution as a perturbation and one can use the GWP of the deterministic NLS and combine it with a perturbation argument to conclude.
- Well-posedness on Π^d:

– Need to use the Fourier restriction norm method
– Cheung-Mosincat [7]

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LECTURE 10 BURKHOLDER-DAVIS-GUNDY INEQUALITY

MAN HO SUEN

Contents

1. Burkholder-Davis-Gundy Inequality

1.1. **Introduction.** We first define B for Banach space, K for separable (real) Hilbert space and $(\Omega, \mathcal{A}, \mathbb{P})$ as the probability space.

Definition 1.1 ($\mathcal{H}$-isonormal process)**.** Given a (real) separable Hilbert space $\mathcal{H}$, we say $W : \mathcal{H} \to \mathrm{L}^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ is a $\mathcal{H}$-isonormal process if $\{W(h) : h \in \mathcal{H}\}$ is a centred (jointly) Gaussian family indexed by $\mathcal{H}$, with

$$\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle \quad \forall h_1, h_2 \in \mathcal{H}$$

Remark 1.2. Uncorrelation within a jointly Gaussian family implies independence.

Example 1 (Jointly Gaussian family)**.** We illustrate the above remark with this example.

$$G \sim \mathrm{N}(0, 1)$$

where ϵ is a symmetric Rademacher random variable, i.e. ± 1 with probability $\frac{1}{2}$. We can see that G and ϵ are independent and $\epsilon G \sim \mathrm{N}(0, 1)$ is uncorrelated with G but ϵG is not independent of G.

Example 2 (Wiener integral (see Lecture 1))**.** Let $(Y_t : t \geq 0)$ follow a standard real Brownian motion in a separable Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+; \mathbb{R})$. $\{W(h) = \int_0^\infty h(t)dY_t,\ h \in \mathcal{H}\}$ is $L^2(\mathbb{R}_+; \mathbb{R})$-isonormal(i.e. $\mathcal{H}$-isonormal).

Definition 1.3 (K-cylindrical Wiener process)**.** W is a K-cylindrical Wiener process if W is $L^2(\mathbb{R}_+; K)$-isonormal. Then, we have $W_t(h) \equiv W(\mathbb{I}_{[0,t]} \otimes h)$ is a centred Gaussian with variance (i.e. second moment) $||\mathbb{I}_{[0,t]} \otimes h)||^2 = t||h||_K^2$. And,

$$\mathbb{E}[W_t(h_1)W_s(h_2)] = \langle \mathbb{I}_{[0,t]} \otimes h_1, \mathbb{I}_{[0,t]} \otimes h_2 \rangle_{L^2(\mathbb{R}_+;K)} = (s \wedge t)\langle h_1, h_2 \rangle_K \quad \forall t, s \in \mathbb{R}_+ \forall h_1, h_2 \in K.$$

That is, $\{W_t(h) : t \geq 0\}$ is a multiple of Brownian motion [1, 2].

Remark 1.4. *If either $(s, t] \cup (s', t'] = \phi$ or $\langle h_1, h_2 \rangle_k = 0$, we have $W(\mathbb{I}_{(s,t]} \otimes h_1)$ is independent of $W(\mathbb{I}_{(s',t']} \otimes h_2)$.*

Definition 1.5 (γ-Radonifying operators)**.** .

(i)

$$K \otimes B = \{\underbrace{\sum_{j=1}^{N} h_j \otimes b_j}_{\substack{\text{finite-rank} \\ \text{operator} \\ \text{from } K \text{ to } B}} : h_j \in K, b_j \in B, N \in \mathbb{N}\}.$$

where $(h \otimes b)(\varphi) = \langle h, \varphi \rangle_K b \ \in B$, in which $h \ \in K$, $b \in B$ and $\varphi \in K$

(ii) $\gamma(K, B)$, the space of γ-Radonifying operators from K to B, is the completion of $K \otimes B$ under the norm

$$||\sum_{n=1}^{N} h_n \otimes b_n||_{\gamma(K,B)} := \Big(\mathbb{E}\big[||\sum_{n=1}^{N} G(h_n)b_n||^2\big]\Big)^{\frac{1}{2}},$$

where G is K-isonormal or equivalently, by assuming $\{h_n\}_{n=1}^N$ is an orthonormal system in K,

$$\text{LHS} = \Big(\mathbb{E}\big[||\sum_{n=1}^{N} g_n b_n||^2\big]\Big)^{\frac{1}{2}}$$

with $(g_n)_n$ iid real N$(0,1)$. We say $T \in \mathcal{L}(K,B)$ is γ-Radonifying if $||T||_\gamma(K,B)$ is finite.

Remark 1.6. *When B is a Hilbert space, $\gamma(K,B) = HS(K,B)$, i.e. the space of Hilbert–Schmidt operators from K to B.*

Definition 1.7. *For $p \in (0,2]$, a Banach space B is of type p if $\exists \mu \in (0,\infty)$ s.t.*

$$\mathbb{E}[||\sum_{n=1}^{N} \epsilon_n b_n||_B^P] \leq \mu \sum_{n=1}^{N} ||b_n||_B^P \tag{1.1}$$

for any finite sequence (b_n) in B, where $(\epsilon_n)_n$ is iid symmetric Rademacher random variables, i.e. ± 1 with probability $\frac{1}{2}$. In the case of cotype p for $p \in [2,\infty]$ [10],

$$\mathbb{E}[||\sum_{n=1}^{N} \epsilon_n b_n||_B^P] \geq \tilde{\mu} \sum_{n=1}^{N} ||b_n||_B^P.$$

Remark 1.8. We consider other cases of p.

(i) For $p > 2$, we can take $b_1 = \cdots = b_N \neq 0$ in 1.1.

$$\begin{aligned} N^{\frac{p}{2}} = (\mathbb{E}[|\sum_{n=1}^{N} \epsilon_n|^2])^{\frac{p}{2}} &\leq \mathbb{E}[|\sum_{n=1}^{N} \epsilon_n|^p] \quad \text{(by Jensen's inequality)} \\ &\leq \mu N \end{aligned}$$

However, this fails for $N >> 1$ due to contradiction.

(ii) Every Banach space is of type 1.

(iii) Type p implies Type q for $q \leq p$. Indeed,

$$\begin{aligned} (\mathbb{E}[||\sum_{n=1}^{N} \epsilon_n b_n||_B^q])^{\frac{1}{q}} &\leq \mathbb{E}[||\sum_{n=1}^{N} \epsilon_n b_n||_B^p]^{\frac{1}{p}} \quad \text{(by Jensen's inequality)} \\ &\leq \mu^{\frac{1}{p}}(||\sum_{n=1}^{N} b_n||_B^p)^{\frac{1}{p}} \quad \text{(Type p)} \\ &\leq \mu^{\frac{1}{p}}(||\sum_{n=1}^{N} b_n||_B^q)^{\frac{1}{q}}, \end{aligned}$$

since $||(a_n)||_{L_n^p} \leq ||(a_n)||_{l_n^q}$ for $p \geq q$.

Remark 1.9. We have further remarks.

(i) Banach space of type p and cotype q measures "how far" it is from being Hilbert space.

(ii) Von Neumann stated that parallelogram law only holds on Hilbert spaces

$$\begin{aligned} ||x+y||_B^2 + ||x-y||_B^2 = 2||x||_B^2 + 2||y||_B^2 \quad \forall x,y \in B \\ \Leftrightarrow \text{B is a Hilbert space} \end{aligned}$$

(iii) '72 Kwapién proved that B cotype 2 and type 2 B is isometrically a Hilbert space [4].
(iv) $K \overset{\gamma}{\to} B$ implies true Radon probability on B (see Lecture 4) [5, 6].

1.2. K-cylindrical Wiener process.

Now we fix W to be a K-cylindrical Wiener process.

Definition 1.10. *We say $\phi : \mathbb{R}_+ \to K \otimes B$ is (deterministic) simple if*

$$\phi = \textit{linear combination of } \mathbb{I}_{(s,t]} \otimes (h \otimes b).$$

Define

$$\int_0^\infty \phi \, dW := \textit{linear combination of } W(\mathbb{I}_{(s,t]} \otimes h)b,$$

where $W(\mathbb{I}_{(s,t]} \otimes h)$ is real Gaussian random variable. This is a Banach space B-valued Gaussian random variable [7, 8].

Lemma 1.11. (i) (B type 2 and ϕ (deterministic) simple) *Then,*

$$\mathbb{E}\big[|| \int_0^\infty \phi \, dW||_B^2\big] \lesssim \int_0^\infty ||\phi||^2_{\gamma(K,B)} dt \tag{1.2}$$

Remark 1.12. *After obtaining 1.2, one can extend $\int_0^\infty \phi \ dW$ for deterministic $\phi \in L^2(\mathbb{R}_+; \gamma(K,B))$ by a standard density argument and 1.2 is still valid for such general integrand.*

Proof. Without loss of generality, consider

$$\phi = \sum_{n=1}^N \mathbb{I}_{(t_{n-1},t_n]} \otimes \sum_{j=1}^k h_j \otimes b_{jn},$$

where (h_j) is orthonormal in K, $b_{jn} \in B$, $0 = t_0, < t_1 < \cdots < t_N < \infty$. By definition,

$$\begin{aligned}\int_0^\infty \phi \, dW &= \sum_{n=1}^N \sum_{j=1}^k \underbrace{W(\mathbb{I}_{(t_{n-1},t_n]} \otimes h_j)}_{\substack{=:\sqrt{t_n - t_{n-1}} g_{jn} \\ \text{where } (g_{j,n})_{j,n} \sim \\ \text{iid N(0,1)}}} b_j n \\ &= \sum_{n=1}^N \sum_{j=1}^k \sqrt{t_n - t_{n-1}} g_{jn} b_{jn}\end{aligned}$$

Note that the variance $(t_n - t_{n-1})||h_j||_K^2 = t_n - t_{n-1}$ and g_{jn} is independent of $g_{j',n'}$ for $(j,n) \neq (j',n')$.

Therefore, LHS of 1.2$= \mathbb{E}\big[|| \sum_{n=1}^N \sum_{j=1}^k \sqrt{t_n - t_{n-1}} g_{jn} \epsilon_n b_{jn}||_B^2\big]$, where (ϵ_n) is iid symmetric Rademacher independent of $(g_{jn})_{jn}$. Adding (ϵ_n) does not change the law of random object inside $||\cdot||_B^2$. Hence, first integrate out the randomness of (ϵ_n) and use type 2 definition,

$$\begin{aligned} E\big[||\sum_{n=1}^{N}\sum_{j=1}^{k}\sqrt{t_n-t_{n-1}}g_{jn}\epsilon_n b_{jn}||_B^2\big] &\lesssim \sum_{n=1}^{N}\mathbb{E}\big[||\sum_{j=1}^{k}\sqrt{t_n-t_{n-1}}g_{jn}b_{jn}||_B^2\big] \\ &= \sum_{n=1}^{N}(t_n-t_{n-1})\mathbb{E}\big[||\sum_{j=1}^{k}g_{jn}b_{jn}||_B^2\big] \\ &= \sum_{n=1}^{N}(t_n-t_{n-1})||\sum_{j=1}^{k}h_j\otimes b_{jn}||_{\gamma(K,B)}^2 \\ &= \int_0^\infty ||\phi(t)||_{\gamma(K,B)}^2 dt \end{aligned}$$

□

Definition 1.13. *For $p\in[1,2]$, we say B has martingale type p [denote MT_p for short]. If $\exists\mu_p\in(0,\infty)$ s.t.*

$$\mathbb{E}\big[||\sum_{n=1}^{N}d_n||_B^2\big]\le\mu_p\sum_{n=1}^{N}\mathbb{E}\big[||d_n||_B^p\big]$$

for any finite sequence $\{d_n\}_{n=1}^N$ of martingale difference, i.e. $\mathbb{E}[d_n|d_1,\dots,d_{n-1}]=0 \quad \forall n\ge1$, in $L^p(\omega,\mathcal{A},\mathbb{P};B)$.

Remark 1.14. Digression to martingale [9].

(i) $\mathrm{Fin}(B)=\{$random variable on Ω that take only finitely many values$\}$

(ii) We say $f:\Omega\to B$ is (Bochner) measurable if $\exists f_n\in\mathrm{Fin}(B)$ s.t.

$$f_n(\omega)\overset{n\uparrow\infty}{\longrightarrow}f(\omega)\quad\forall w\in\omega$$

for $f\in\mathrm{Fin}(B)$, $||f||_{L^p(\Omega;B)}=\big(\int_\Omega||f(w)||_B^P\big)^{\frac{1}{p}}$ is well-defined. When $p=\infty$, it is essential supremum norm. $L^p(\Omega;B)$ refers to completion of $\mathrm{Fin}(B)$ under $L^p(\Omega;B)$-norm

(iii) For $X\in L^1(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})$ and $\mathcal{G}\subseteq\mathcal{A}$ σ-algebra, $\mathbb{E}^{\mathcal{G}}[X]$ is the conditional expectation of X given $\mathcal{G}$ defines a norm-1 operator on $L^p(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})$ and positive $\mathbb{E}^{\mathcal{G}}[X]\ge0$ for $x\ge0$ a.s.. Then [see section 1.2 in Pisier 2016 book [10]], $(\mathbb{E}^{\mathcal{G}}\otimes\mathbb{I}_B)(X\otimes b)=\mathbb{E}^{\mathcal{G}}[X]b$ extends to a bounded linear operator on $L^p(\Omega,\mathcal{A},\mathbb{P};B)$ where $X\in L^p(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})$ and $b\in B$.

(iv) Useful property: For $p\in[1,\infty]$,

$$\mathbb{E}^{\mathcal{G}}[XY]=Y\mathbb{E}^{\mathcal{G}}[X] \tag{1.3}$$

for any $X\in L^p(\Omega,\mathcal{A},\mathbb{P};B)$, $Y\in L^p(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})$, $\mathcal{G}\subseteq\mathcal{A}$.

1.3. **Filtration $\mathbb{F}$ generated by the K-cylindrical Wiener process W.** $\mathbb{F}=\{\mathcal{F}_t:t\ge0\}$ where $\mathcal{F}_t=\sigma$-algebra generated by $\{W(\mathbb{I}_{[0,s]}\otimes h):s\le t,h\in k\}$. Clearly $\mathcal{F}_t\subseteq\mathcal{F}_{t'}\quad\forall t\le t'$.

Definition 1.15. *We say (a random time) $\tau:\Omega\to[0,\infty]$ is a $\mathbb{F}$-stopping time if*

$$\{\tau\le t\}=\{\omega\in\Omega:\tau(\omega)\le t\}\in\mathcal{F}_t,\quad\forall t\ge0.$$

We call τ predictable if $\exists$ τ_n $\mathbb{F}$-stopping times s.t. $\tau_n<\tau_{n+1}<\tau\quad\forall n$ and $\tau_n\uparrow\tau$ as $n\uparrow\infty$.

Definition 1.16. *We say $\phi : \mathbb{R}_+ \times \Omega \to K \otimes B$ is a $\mathbb{F}$-adapted and simple process if $\phi(s)$ is $\mathcal{F}_s$-measurable $\forall s \in \mathbb{R}_+$ and $\phi(\omega)$ is a simple function.*

$$\Leftrightarrow \phi = \textit{linear combination of } \mathbb{I}_{(s,t]\times A} \otimes (h \otimes b)$$

where $s < t$, $h \in K$, $b \in B$, and the event $A \in \mathcal{F}_s$. Define

$$\int_0^\infty \phi \, dW = \textit{linear combination of } \mathbb{I}_A \, W(\mathbb{I}_{(s,t]} \otimes h)b,$$

where $\mathbb{I}_A$ and $W(\mathbb{I}_{(s,t]} \otimes h)$ are independent because $A \in \mathcal{F}_s$ is independent over disjoint interval.

Lemma 1.17 (B has MT_2 and ϕ is simple $\mathbb{F}$-adapted.)**.** Then

$$\mathbb{E}\big[|| \int_0^\infty \phi \, dW||^2 - B\big] \lesssim \mathbb{E} \int_0^\infty ||\phi(t)||^2_{\gamma(K,B)} dt \tag{1.4}$$

Proof. Consider $\phi = \sum_{n=1}^N \mathbf{1}_{(t_{n-1},t_n]} \sum_{m=1}^M \mathbb{I}_{F_{mn}} \otimes \sum_{j=1}^k h_j \otimes b_{jmn}$ where $0 = t_0 < \cdots < t_N < \infty$, h_j orthonormal in K and $b_{jmn} \in B$.

For each $n \in \{1, \dots, N\}$, the events $\{F_{mn}\}_{m=1}^M$ are mutually disjoint and in $\mathcal{F}_{t_{n-1}}$ (1.5)

$$\begin{aligned}
\int_0^\infty \phi \, dW &= \sum_{n=1}^N \sum_{m=1}^M \sum_{j=1}^k \mathbb{I}_{F_{mn}} W(\mathbb{I}_{(t_{n-1},t_n]} \otimes h_j) b_{jmn} \\
&= \sum_{n=1}^N \sum_{m=1}^M \sum_{j=1}^k \mathbb{I}_{F_{mn}} (t_{n-1} - t_n)^{\frac{1}{2}} g_{jn} b_{jmn} =: \sum_{n=1}^N d_n
\end{aligned}$$

with $d_n = (t_{n-1} - t_n)^{\frac{1}{2}} \sum_{n=1}^N \sum_{j=1}^k \mathbb{I}_{F_{mn}} g_{jn} b_{jmn}$ is $\mathcal{F}_{t_n}$-measurable, in which $\mathbb{I}_{F_{mn}}$ is $\mathcal{F}_{t_{n-1}}$-measurable and g_{jn} is independent of $\mathcal{F}_{t_{n-1}}$.

Remark 1.18. *$d_1, \dots, d_n$ adapted to $\mathcal{F}_{t_1}, \dots, \mathcal{F}_{t_n}$ is a martingale difference. Using 1.3, one has $\mathbb{E}[d_n|\mathcal{F}_{t_1}] = \mathbb{E}[g_{jn}|\mathcal{F}_{t_1}] = 0$.*

Therefore,

$$\begin{aligned}
\text{LHS of 1.4} &= \mathbb{E}\big[|| \sum_{n=1}^N d_n||^2_B\big] \\
&\lesssim \sum_{n=1}^N \big[||d_n||^2_B\big] \\
&= \sum_{n=1}^N (t_{n-1} - t_n) \mathbb{E}\big[|| \sum_{m=1}^M \mathbb{I}_{F_{mn}} \sum_{j=1}^k g_{jn} b_{jmn}||^2_B\big] \quad (\text{by } MT_2) \\
&= \sum_{n=1}^N \sum_{m=1}^M \mathbb{E}[\mathbb{I}_{F_{mn}}] \mathbb{E}\big[|| \sum_{j=1}^k g_{jn} b_{jmn}||^2_B\big] \quad (\text{by 1.5}) \\
&= \sum_{n=1}^N \sum_{m=1}^M \mathbb{E}[\mathbb{I}_{F_{mn}}] || \sum_{j=1}^k h_j \otimes b_{jmn}||^2_{\gamma(K,B)} = \text{RHS of 1.4}
\end{aligned}$$

$\square$

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LECTURE 11 (05/04/22) - BDG INEQUALITY IN MARTINGALE TYPE 2 SPACE

MARTIN ULMER

Contents

1. Lecture 11 - BDG inequality in martingale type 2 space

1.1. **Dool's maximal inequality.** We continue with the notation from Lecture 10, and we recall Lemma **??** (Lemma 2 from Lecture 10). Therefore, let K be a separable (real) Hilbert space and B a Banach space. Let W be a $K-$Cylindrical Wiener process that gives rise to a filtration $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ with the "usual conditions", i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$ and $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. Furthermore, recall that if $\phi : \mathbb{R}_+ \times \Omega \to K \otimes B$ is $\mathbb{F}$-adapted and simple, we can write

$$\phi = \underset{s<t, A\in\mathcal{F}_s, h\in K, b\in B}{\text{lin.Comb.}} \mathbb{I}_{(s,t]\times A} \otimes (h \otimes b).$$

For such ϕ we also have

$$\int_0^\infty \phi dW := \text{lin. Comb. } \mathbb{I}_A W(\mathbb{I}_{(s,t]} \otimes h)b.$$

Lemma **??** (Lemma 2 from Lecture 10) now states that if B has martingale type 2 and ϕ is simple $\mathbb{F}$-adapted, then

$$\mathbb{E}\Big[\| \int_0^\infty \phi dW\|_B^2\Big] \lesssim \mathbb{E}\int_0^\infty \|\phi\|^2_{\gamma(K,B)} dt. \tag{1.1}$$

This lemma can be improved to the following Theorem (1.2). Before being able to state the theorem we need the next definition.

Definition 1.1. *We say $\{\phi(s,\omega)\}_{s\in\mathbb{R}_+,\omega\in\Omega}$ is a progressively measurable process, if for all $T \in (0,\infty)$*

$$(s,\omega) \in [0,T] \times \Omega \mapsto \phi(s,\omega) \in \gamma(K,B) \text{ is } B([0,T]) \otimes \mathcal{F}_T \text{ measurable.}$$

Theorem 1.2. *[Doob's maximal inequality in the MT_2 setting]*

(1) *For a simple and $\mathbb{F}$-adapted ϕ it holds:*

$$\mathbb{E}\Big[\sup_{t\geq 0} \| \int_0^t \phi dW\|_B^2\Big] \lesssim \mathbb{E}\int_0^\infty \|\phi\|^2_{\gamma(K,B)} dt. \tag{1.2}$$

(2) *For a progressively measurable process with* $\mathbb{E}\int_0^\infty \|\phi\|^2_{\gamma(K,B)}dt < \infty$ *we also have that inequality (1.2) holds, and hence*

$$M_t := \int_0^t \mathbb{I}_{[0,t]}\phi dW \in L^2(\Omega; C(\mathbb{R}_+; B)).$$

Remark 1.3.

- *Simple,* $\mathbb{F}$*-adapted processes are progressively measurable. To see this, note that they generate the so-called predictable* σ*-algebra that is equivalent to the one generated by all adapted and left-continuous processes (more information in* [4]*). Since left-continuous processes are a dense subset of the set of progressivly-measurable processes, simple,* $\mathbb{F}$*-adapted processes are also a sense subset. To be precise here, we have that simple,* $\mathbb{F}$*-adapted processes are dense in* $L^2(prog) = \{$*progressively measurable process with* $\mathbb{E}\int_0^\infty \|\phi\|^2_{\gamma(K,B)}dt < \infty\}$*.*
- *The first part of Theorem 1.2 can be considered as an improvement of Lemma (**??**) (Lemma 2 from Lecture 10) to the inequality (1.2), while the second part is an extension of this inequality to the wider class of progressively measurable processes.*
- *Later we will use this theorem to prove Theorem (1.5). In fact, this theorem is already the BDG inequality in Theorem (1.5) for the special case* $p = 2$*.*

Proof. First we proof (i). From the proof of Lemma (**??**) (Lemma 2 from Lecture 10) we get that we can write

$$\int_0^\infty \phi dW = \sum_{n=1}^N d_n,$$

where $d_1, ..., d_N$ is a martingale difference with respect to the filtration generated by themselves, i.e. $\mathbb{E}[d_k; \sigma\{d_1, ..., d_{k-1}\}] = 0$. Then, by definition of $\int_0^\infty \phi dW$ for simple, adapted processes, we can see that

$$\int_0^\infty \phi dW = \int_0^\infty \mathbb{I}_{[0,t]}(s)\phi(s)dW(s) = \sum_{n=1}^{N_t} d_n,$$

where N_t is some integer that is nondecreasing in t. Hence, if we can show that there exists a constant C independent of N such that for each $N \geq 1$

$$\mathbb{E}\sup_{n\leq N} \|\sum_{k=1}^n d_k\|_B^2 \leq C\sum_{k=1}^N \mathbb{E}[\|d_k\|_B^2] \tag{1.3}$$

holds, we can conclude

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\geq 0}\Big\|\int_0^t \phi dW\Big\|_B^2\Big] &= \mathbb{E}\Big[\sup_{t\geq 0}\|\sum_{k=1}^{N_t} d_k\|_B^2\Big] \\
&= \lim_{N\to\infty}\mathbb{E}\Big[\sup_{n\leq N}\|\sum_{k=1}^{n} d_k\|_B^2\Big] \\
&\lesssim \lim_{N\to\infty}\sum_{k=1}^N \mathbb{E}[\|d_k\|_B^2] \\
&= \mathbb{E}\Big[\int_0^\infty \|\phi\|^2_{\gamma(K,B)}dt\Big].
\end{aligned}$$

To show (1.3) we notice first that

$$\{X_n := \|\sum_{i=1}^{n} d_k\|_B : n \geq 1\}$$

is a real-valued sub-martingale. To see this we can calculate

$$\begin{aligned}\mathbb{E}\big[X_{n+1}|d_1, ..., d_n\big] &= \mathbb{E}\big[\|\sum_{i=1}^{n+1} d_k\|_B|d_1, ..., d_n\big] \\ &\geq \Big\|\mathbb{E}\big[\sum_{i=1}^{n+1} d_k|d_1, ..., d_n\big]\Big\|_B = X_n\end{aligned}$$

The inequality we used here is from page 8 of [1]. Then, we can apply Theorem 26.3 from [2], Doob's L^2-inequality, and we get

$$\mathbb{E}[\max_{n\leq N} X_n^2] \leq \big(\frac{2}{2-1}\big)^2\mathbb{E}[X_n]^2,$$

which implies using the martingale type 2 property

$$\mathbb{E}\Big[\sup_{n\leq N} \|\sum_{k=1}^{n} d_k\|_B^2\Big] \leq 4\mathbb{E}\Big[\|\sum_{k=1}^{N} d_k\|_B^2\Big] \lesssim \sum_{k=1}^{N} \mathbb{E}[\|d_k\|_B^2].$$

Let us now proof (ii). We are using a "localization" or "truncation" approach to use the density of simple, adapted processes. Let ϕ be progressively measurable with $\int_0^\infty \|\phi(t)\|_{\gamma(K,B)}^2 dt < \infty$ almost surely.

(1) Define the stopping time

$$\tau_n = \inf\Big\{t \geq 0 : \int_0^t (1 + \|\phi(t)\|_{\gamma(K,B)}^2)dt \geq n\Big\} < \infty.$$

Due to our assumption we have that $\tau_n \uparrow \infty$ almost surely for $n \to \infty$.

(2) Define

$$\int_0^T \phi dW := \int_0^\infty \mathbb{I}_{[0,\tau_n]}\phi dW \qquad \text{on } \{T \leq \tau_n\}.$$

First note that $\mathbb{I}_{[0,\tau_n]}$ is adapted. Thus, $\mathbb{I}_{[0,\tau_n]}\phi$ is progressively measurable and hence the stochastic integral is welldefined. Since $\tau_n \uparrow \infty$ almost surely, the probability of the event $\{T \leq \tau_n\}$ is going to 1 for $n \to \infty$. Thus, this definition is almost surely well-defined.

(3) We set $M_t := \int_0^t \phi dW$. Then we have

$$M_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} \phi dW = \int_0^\infty \mathbb{I}_{[0,t\wedge\tau_n]}\phi dW \in L^2(\Omega; B).$$

Moreover, for each n $(M_{t\wedge\tau_n} : t \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_{t\wedge\tau_n} : t \geq 0)$. Since τ_n is a stopping time also $t \wedge \tau_n$ is a stopping time and for a stopping time τ we have that $\mathcal{F}_\tau = \{A : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$ is a σ-algebra which contains all information up to the random stopping time τ. Since $\tau_n \uparrow \infty$ the theorem follows.

□

The next lemma can be found as Lemma 3.6 in [3]

Lemma 1.4. *Let τ be a $\mathbb{F}$-stopping time, ϕ progressively measurable in $\gamma(K,B)$, such that*
$$\mathbb{E}\Big[\int_0^\infty \|\phi(s)\|^2_{\gamma(K,B)} ds\Big] < \infty.$$
Then for $t \geq 0$
$$M_t = \int_0^t \phi(s) dW$$
defines a martingale and the definition
$$M_{t\wedge\tau} = \int_0^{t\wedge\tau} \mathbb{I}_{[0,\tau]}\phi dW$$
is well-defined almost surely.

1.2. **BDG inequality in martingale type 2.**

Theorem 1.5 (BDG in MT_2 setting)**.** *Let $0 < p < \infty$. Let B be a separable Banach space of martingale type 2. There exists $C = C_{p,B} < \infty$ such that for any $\mathbb{F}-$adapted stopping time τ and for every in $\gamma(K,B)$ progressively measurable process F we have*
$$\mathbb{E} \sup_{0\leq t\leq\tau} \Big\| \int_0^t F(t')dW(t')\Big\|^p_B \leq C_{p,B}\mathbb{E}\Big[\Big(\int_0^\tau \|F(t')\|^2_{\gamma(K,B)} dt'\Big)^{\frac{p}{2}}\Big].$$
Here W is a $K-$cylindrical Wiener process.

Proof. We begin the proof by defining certain notations. We define for $r \geq 0$
$$M(r) = \Big\| \int_0^r F(t')dW(t')\Big\|_B$$
and its running maximum
$$M^*(r) = \sup_{s\leq r} M(s).$$
Furthermore, set
$$N(r) = \Big[\int_0^r \|F(t')\|^2_{\gamma(K,B)} dt'\Big]^{\frac{1}{2}}.$$
The goal is to show a "good λ inequality" and we need the variables $\beta > 1, \delta > 0, \lambda > 0, t \geq 0$ which will be chosen later. We can then define
$$\begin{aligned}
\tau_1 &= \inf\{r \geq 0 : M(r) \geq \beta\lambda\},\\
\tau_2 &= \inf\{r \geq 0 : M(r) \geq \lambda\},\\
\sigma &= \inf\{r \geq 0 : N(r) \geq \delta\lambda\},\\
\rho_n &= \inf\{r \geq 0 : M(r) \geq n\}.
\end{aligned}$$

From these definitions we can clearly see that $\tau_1 \leq \tau_2$ and that $M^*(t\wedge\rho_n) \leq n$. Furthermore we define the two sets

$$\begin{aligned}
A_1 &= \{t \geq 0 : M^*(t\wedge\rho_n) \geq \beta\lambda \text{ and } N(t) < \delta\lambda\},\\
A_2 &= \Big\{t \geq 0 : \Big\| \int_0^{t\wedge\tau_1\wedge\sigma\wedge\rho_n} FdW - \int_0^{t\wedge\tau_2\wedge\sigma\wedge\rho_n} FdW\Big\|_B \geq \lambda(\beta-1)\Big\}.
\end{aligned}$$

For $t \in A_1$ we can observe from the definitions that $\tau_2 \leq \tau_1 \leq t\wedge\rho_n \leq t \leq \sigma$ which implies

$$\begin{cases} t \wedge \tau_1 \wedge \sigma \wedge \rho_n = \tau_1 \\ t \wedge \tau_2 \wedge \sigma \wedge \rho_n = \tau_2 \end{cases},$$

and hence

$$\begin{aligned}
&\Big\| \int_0^{t\wedge\tau_1\wedge\sigma\wedge\rho_n} F dW - \int_0^{t\wedge\tau_2\wedge\sigma\wedge\rho_n} F dW \Big\|_B \\
&\quad = \Big\| \int_0^{\tau_1} F dW - \int_0^{\tau_2} F dW \Big\|_B \\
&\quad \geq M(\tau_1) - M(\tau_2) = (\beta - 1)\lambda.
\end{aligned}$$

As a consequence we have $A_1 \subset A_2$.

Next, we use the fact $\tau_2 < \tau_1$ and the application of part (i) in Theorem (1.2) to calculate

$$\begin{aligned}
&\mathbb{E}\Big[\Big\| \int_0^{t\wedge\tau_1\wedge\sigma\wedge\rho_n} F dW - \int_0^{t\wedge\tau_2\wedge\sigma\wedge\rho_n} F dW \Big\|_B^2\Big] \\
&= \mathbb{E}\Big[\Big\| \int_0^t \mathbb{I}_{(\tau_2\wedge\sigma\wedge\rho_n, \tau_1\wedge\sigma\wedge\rho_n]} F dW \Big\|_B^2\Big] \\
&\lesssim \mathbb{E}\Big[\int_0^t \mathbb{I}_{(\tau_2\wedge\sigma\wedge\rho_n, \tau_1\wedge\sigma\wedge\rho_n]}(s) \|F(s)\|_{\gamma(K,B)}^2 ds\Big].
\end{aligned}$$

The integrand is nonzero only when $\tau_2 \leq \sigma \wedge \rho_n \wedge \tau_1 \wedge \rho_n$, which means in particular $\tau_2 \leq t \wedge \rho_n$. But for all these t we then have $M^*(t \wedge \rho_n) \geq \lambda$ and we can add an indicator function to get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \mathbb{I}_{(\tau_2\wedge\sigma\wedge\rho_n, \tau_1\wedge\sigma\wedge\rho_n]}(s) \|F(s)\|_{\gamma(K,B)}^2 ds\Big] \\
&\leq \mathbb{E}\Big[\int_0^{t\wedge\sigma} \|F(s)\|_{\gamma(K,B)}^2 ds \mathbb{I}_{M^*(t\wedge\rho_n)\geq\lambda}\Big] \\
&\leq N(\sigma)^2 \mathbb{P}(M^*(t\wedge\rho_n) \geq \lambda) \\
&\lesssim \delta^2\lambda^2 \mathbb{P}(M^*(t\wedge\rho_n) \geq \lambda).
\end{aligned}$$

Therefore, by Chebyshev's inequality

$$\mathbb{P}(A_1) \leq \mathbb{P}(A_2) \leq \frac{C\delta^2\lambda^2\mathbb{P}(M^*(t\wedge\rho_n) \geq \lambda)}{\lambda^2(\beta-1)^2} \leq \frac{C\delta^2\mathbb{P}(M^*(t\wedge\rho_n) \geq \lambda)}{(\beta-1)^2},$$

and

$$\begin{aligned}
\mathbb{P}(M^*(t\wedge\rho_n) \geq \beta\lambda) &\leq \mathbb{P}(M^*(t\wedge\rho_n) \geq \beta\lambda, N(t) < \delta\lambda) + \mathbb{P}(N(t) \geq \delta\lambda) \\
&\leq \mathbb{P}(N(t)\delta^{-1} \geq \lambda) + \mathbb{P}(A_1) \\
&\leq \mathbb{P}(N(t)\delta^{-1} \geq \lambda) + \frac{C\delta^2}{(\beta-1)^2}\mathbb{P}(M^*(t\wedge\rho_n) \geq \lambda).
\end{aligned}$$

Recall the layer cake representation for nonnegative random variables. For $Z \geq 0$ and $p > 0$ we can write

$$\mathbb{E}[Z^p] = \mathbb{E}\Big[\int_0^Z p\lambda^{p-1} d\lambda\Big] = \mathbb{E}\Big[\int_0^\infty \mathbb{I}_{[0,Z]}(\lambda) p\lambda^{p-1} d\lambda\Big] = \mathbb{E}\Big[\int_0^\infty \mathbb{P}(Z \geq \lambda) p\lambda^{p-1} d\lambda\Big].$$

Applying this representation we obtain

$$\begin{aligned}
\mathbb{E}\big[M^*(t \wedge \rho_n \beta^{-1})^p\big] &= \mathbb{E}\Big[\int_0^\infty \mathbb{P}(M^*(t \wedge \rho_n \geq \beta\lambda) p\lambda^{p-1} d\lambda\Big] \\
&\leq \mathbb{E}\Big[\int_0^\infty \mathbb{P}(N(t)\delta^{-1} \geq \lambda) p\lambda^{p-1} d\lambda\Big] + \frac{C\delta^2}{(\beta-1)^2}\mathbb{E}\Big[\int_0^\infty \mathbb{P}(M^*(t \wedge \rho_n) \geq \lambda) p\lambda^{p-1} d\lambda\Big] \\
&= \delta^{-p}\mathbb{E}\Big[\int_0^\infty \mathbb{P}(N(t) \geq \lambda) p\lambda^{p-1} d\lambda\Big] + \frac{C\delta^2}{(\beta-1)^2}\mathbb{E}\Big[\int_0^\infty \mathbb{P}(M^*(t \wedge \rho_n) \geq \lambda) p\lambda^{p-1} d\lambda\Big] \\
&= \delta^{-p}\mathbb{E}\big[(N(t))^p\big] + \frac{C\delta^2}{(\beta-1)^2}\mathbb{E}\big[M^*(t \wedge \rho_n)^p\big].
\end{aligned}$$

Since also from the layer cake representation we have the equality $\mathbb{E}\big[(M^*(t \wedge \rho_n \beta^{-1})^p\big] = \beta^p \mathbb{E}\big[(M^*(t \wedge \rho_n)^p\big]$ we have together

$$\mathbb{E}\big[M^*(t \wedge \rho_n)^p\big] = \beta^p \delta^{-p}\mathbb{E}\big[(N(t))^p\big] + \frac{C\delta^2 \beta^p}{(\beta-1)^2}\mathbb{E}\big[M^*(t \wedge \rho_n)^p\big].$$

Now we can choose β and δ such that $\frac{C\delta^2\beta^p}{(\beta-1)^2} < \frac{1}{2}$, so that we can hide the second summand on the left side. In total we get

$$\mathbb{E}\big[M^*(t \wedge \rho_n)^p\big] \lesssim \mathbb{E}\big[(N(t))^p\big]$$

where the implicit constant is independent of n. Since $\rho_n \uparrow \infty$ as $n \uparrow \infty$ and $M^*(t \wedge \rho_n) \leq M^*(t)$ we have $M^*(t \wedge \rho_n) \uparrow M^*(t)$. By the monotone convergence theorem we obtain

$$\mathbb{E}\big[M^*(t)^p\big] \lesssim \mathbb{E}\big[(N(t))^p\big].$$

Plugging in the definitions for M^* and N yields in

$$\mathbb{E}\Big[\sup_{0 \leq t \leq \tau} \Big\| \int_0^t F(t') dW(t') \Big\|_B^p\Big] \leq C_{p,B}\mathbb{E}\Big[\Big(\int_0^\tau \|F(t')\|_{\gamma(K,B)}^2 dt'\Big)^{\frac{p}{2}}\Big]$$

what we wanted to show.

□

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LECTURE 12

RUOYUAN LIU

In Lecture 9 Page 11 (in the slides), we studied the following 1D cubic SNLS:

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u + u\Phi\xi \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}) \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{0.1}$$

where ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ and $\Phi \in \gamma(L^2_x; L^\infty_x)$ (see Lecture 4 Page 15 for the definition of γ-radonifying operators and see Lecture 9 Page 12 for a discussion on relaxing the condition $\Phi \in \gamma(L^2_x; L^\infty_x)$).

To prove local well-posedness of (0.1), we used the following truncation function:

$$\eta_R(u)(t) = \eta\bigg(\frac{\|u\|_{C([0,t];L^2_x)} + \|u\|_{L^8([0,t];L^4_x)}}{R}\bigg), \tag{0.2}$$

where η is a smooth and nonnegative cutoff function on $\mathbb{R}_+$ such that $\eta \equiv 1$ on $[0,1]$ and $\eta \equiv 0$ on $[2,\infty)$. Here, $(8,4)$ is a Schrödinger admissible pair (see Lecture 3 Page 2). From Lecture 9 Page 13 - 21, we constructed a local-in-time solution of (0.1) in $C_T L^2_x \cap L^8_T L^4_x$.

Let $T_{\max}$ be defined as the maximal time of existence of solutions to (0.1). Note that if $T_{\max} < \infty$, we have

$$\lim_{t \nearrow T_{\max}} \|u(t)\|_{L^2_x} = +\infty.$$

The main goal in this lecture is to show

$$\sup_{0 \le t \le T_{\max}} \|u(t)\|_{L^2_x} < +\infty, \tag{0.3}$$

which implies global well-posedness of (0.1).

1. The Ito formula

In order to prove (0.3), we use the Ito formula [2, 3]. Consider a d-dimensional Ito process $X_t = (X^{(1)}_t, \dots, X^{(d)}_t) \in \mathbb{R}^d$ with

$$X^{(j)}_t = X^{(j)}_0 + \int_0^t Y^{(j)}_s ds + \int_0^t Z^{(j)}_s dB^{(j)}_s, \quad j = 1, \dots, d. \tag{1.1}$$

Here, $Y^{(j)}$ and Z^j are adapted processes, and $(B^{(1)}_t, \dots, B^{(d)}_t)_{t \ge 0}$ is a d-dimensional Brownian motion. The Ito formula says that, for $G \in C^2(\mathbb{R}^d)$, we have

$$G(X_t) - G(X_0) = \int_0^t \langle \nabla G(X_s), dX_s \rangle_{\mathbb{R}^d} + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij} G(X_s) d\langle X^{(i)}, X^{(j)} \rangle_s. \tag{1.2}$$

Note that (1.1) can be written as

$$dX^{(j)}_s = Y^{(j)}_s ds + Z^{(j)}_s dB^{(j)}_s, \tag{1.3}$$

and also we have

$$d\langle X^{(i)}, X^{(j)}\rangle_s = \begin{cases} 0 & i \neq j \\ |Z_s^{(j)}|^2 ds & i = j. \end{cases} \tag{1.4}$$

Thus, by plugging in (1.3) and (1.4), we note that (1.2) becomes

$$\begin{aligned} G(X_t) - G(X_0) &= \sum_{j=1}^{d} \int_0^t \partial_j G(X_s)(Y_s^{(j)} ds + Z_s^{(j)} dB_s^{(j)}) + \frac{1}{2}\sum_{j=1}^{d} \int_0^t \partial_{jj} G(X_s)|Z_s^{(j)}|^2 ds \\ &= \int_0^t \Big(\langle \nabla G(X_s), Y_s\rangle_{\mathbb{R}^d} + \frac{1}{2}\sum_{j=1}^{d} \partial_{jj} G(X_s)|Z_s^{(j)}|^2 \Big) ds \\ &\quad + \sum_{j=1}^{d} \int_0^t \partial_j G(X_s) Z_s^{(j)} dB_s^{(j)}. \end{aligned} \tag{1.5}$$

Note that on the right-hand side of (1.5), the first integral is a Lebesgue integral with random integrand, and the second integral is an Ito integral.

Let us also mention the Stratonovich integrals [3, Page 143], which is similar and related to the Ito integrals. For an adapted process Y and a Brownian motion B, the Stratonovich integral of Y against B is defined as the following limit of Riemann sums:

$$\int_0^T Y_s \circ dB_s = \lim_{\substack{\text{mesh size} \\ \text{of } [0,T] \to 0}} \sum_i \frac{Y_{t_i} + Y_{t_{i+1}}}{2}(B_{t_{i+1}} - B_{t_i}),$$

while the Ito integral of Y against B is defined as

$$\int_0^T Y_s \cdot dB_s = \lim_{\substack{\text{mesh size} \\ \text{of } [0,T] \to 0}} \sum_i Y_{t_i}(B_{t_{i+1}} - B_{t_i}).$$

Using the Stratonovich integrals, we have the following chain rule:

$$g(B_t) - g(B_0) = \int_0^t g'(B_s) \circ dB_s.$$

Also, (1.5) can be rewritten as

$$G(X_t) - G(X_0) = \int_0^t \langle \nabla G(X_s), Y_s\rangle_{\mathbb{R}^d} ds + \sum_{j=1}^{d} \int_0^t \partial_j G(X_s) \circ dU_s^{(j)},$$

where $(U_t^{(j)})_{t\geq 0}$ is an Ito process given by $U_t^{(j)} = \int_0^t Z_s^{(j)} dB_s^{(j)}$.

2. Back to 1D Cubic SNLS

2.1. **The Stratonovich-1D Cubic SNLS and the Ito-1D Cubic SNLS.** Let us consider the following 1D cubic Stratonovich SNLS:

$$\begin{cases} i\partial_t u = \Delta u + |u|^2 u + u \circ \Phi\xi \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}). \end{cases} \tag{2.1}$$

Here, $\circ$ denotes the Stratonovich product. Let us assume that the noise term $\Phi\xi$ is real-valued.

NLS has wide applications in modeling, for example, the wave propagation in fiber optics (medium for telecommunication and computer networking, etc). The deterministic NLS preserves the mass. The above SNLS (2.1) can be viewed as NLS over random medium (corresponding to the stochastic forcing $u \circ \Phi\xi$). Note that when the noise is real, SNLS (2.1) also preserves the mass $M(u)(t) = \|u(t)\|^2_{L^2_x}$, which is explained in the following (assuming that u is smooth):

$$\begin{aligned}
\partial_t M(u)(t) &= \partial_t \int_{\mathbb{R}} u(t)\overline{u}(t)dx \\
&= \int_{\mathbb{R}} \partial_t\big(u(t)\overline{u}(t)\big)dx \\
&= 2\,\mathrm{Re}\int_{\mathbb{R}} \overline{u}(t)\partial_t u(t)dx \\
&= 2\,\mathrm{Re}\int_{\mathbb{R}} \overline{u}\big(-i\Delta u - i|u|^2u - iu\Phi\xi\big)dx \\
&= 2\,\mathrm{Re}\, i\int_{\mathbb{R}} \big((\partial_t u)^2 - |u|^2u - u\Phi\xi\big)dx \\
&= 0,
\end{aligned}$$

where in the second last line we used integration by parts.

We can write the Stratonovich SNLS (2.1) in the following Ito formulation:

$$\begin{cases} i\partial_t u = \Delta u + |u|^2u + u\cdot\Phi\xi - i\frac{1}{2}uF_\Phi \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}), \end{cases} \tag{2.2}$$

where F_Φ is defined as

$$F_\Phi(x) := \sum_{k=0}^{\infty}(\Phi e_k)^2(x),$$

and $\Phi\xi$ is the destributional derivative $\partial_t\Phi W$, where

$$\Phi W(t,x) = \sum_{k=0}^{\infty}\beta_k(t)(\Phi e_k)(x).$$

Here, $(e_k)_{k\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R};\mathbb{R})$, $(\beta_k)_{k\geq 0}$ is a sequence of i.i.d real Brownian motions, and $\Phi \in \gamma(L^2(\mathbb{R};\mathbb{R}); L^\infty(\mathbb{R};\mathbb{R}))$. More details can be found in [1].

Note that F_Φ does not depend on the particular choice of the orthonormal basis. Indeed, if $(h_k)_{k\geq 0}$ is another orthonormal basis, we have

$$\begin{aligned}
\sum_{k=0}^{\infty}(\Phi h_k)^2 &= \sum_{k=0}^{\infty}\left(\Phi\sum_{j=0}^{\infty}\langle h_k, e_j\rangle e_j\right)^2 \\
&= \sum_{k=0}^{\infty}\sum_{j,\ell=0}^{\infty}\langle h_k, e_j\rangle\langle h_k, e_\ell\rangle(\Phi e_j)(\Phi e_\ell) \\
&= \sum_{j=0}^{\infty}(\Phi e_j)^2.
\end{aligned}$$

We now state a few facts.

Fact (1). Let Θ be the Fourier multiplier with symbol $\theta : \mathbb{R} \to [0,1]$ being an even function, that is,

$$\widehat{\Theta v}(\xi) = \theta(\xi)\widehat{v}(\xi).$$

Then, Θ is a bounded linear operator from $L^2(\mathbb{R};\mathbb{R})$ to $L^2(\mathbb{R};\mathbb{R})$. Indeed, by Plancherel's theorem,

$$\|\Theta v\|_{L^2_x} = \|\theta(\xi)\widehat{v}(\xi)\|_{L^2_\xi} \le \|\widehat{v}(\xi)\|_{L^2_\xi} = \|v\|_{L^2_x}.$$

Also, for $v \in \mathcal{L}^2(\mathbb{R};\mathbb{R})$, since $\widehat{v}(-\xi) = \overline{v}(\xi)$ and $\theta(-\xi) = \theta(\xi)$, we have

$$(\Theta v)(x) = \int_{\mathbb{R}} \theta(\xi)\widehat{v}(\xi)e^{ix\cdot\xi}d\xi \in \mathbb{R}.$$

Fact (2). Let Θ be as in above and assume that $\Phi \in \gamma(L^2(\mathbb{R};\mathbb{R}); L^\infty(\mathbb{R};\mathbb{R})) \cap \gamma(L^2; L^2)$ (see Lecture 9 Page 12). Then, we have

$$F_{\Phi\Theta}(x) := \sum_{k=0}^{\infty}(\Phi\Theta e_k)^2(x) \in L^\infty(\mathbb{R}).$$

By definition and letting $\{g_k\}_{k\ge 0}$ be a sequence of i.i.d real-valued standard Gaussian random variables, we have

$$F_{\Phi\Theta}(x) = \sum_{k=0}^{\infty}(\Phi\Theta e_k)^2(x) = \mathbb{E}\bigg|\sum_{k=0}^{\infty} g_k\Phi\Theta e_k(x)\bigg|^2,$$

so that for $2 \le q < \infty$, we use Minkowski's inequality and Hölder's inequality to obtain

$$\begin{aligned}\|F_{\Phi\Theta}\|_{L^q(\mathbb{R})} &\le \mathbb{E}\bigg[\bigg\|\sum_{k=0}^{\infty} g_k\Phi\Theta e_k\bigg\|^2_{L^{2q}(\mathbb{R})}\bigg] \\ &\le \|\Phi\Theta\|^{2/q}_{\gamma(L^2;L^2)}\|\Phi\Theta\|^{2(q-1)/q}_{\gamma(L^2;L^\infty)}.\end{aligned}$$

By letting $q \to \infty$, we get

$$\|F_{\Phi\Theta}\|_{L^\infty(\mathbb{R})} \le \|\Phi\Theta\|_{\gamma(L^2;L^\infty)} \lesssim \|\Phi\|_{\gamma(L^2;L^\infty)},$$

where the last inequality follows from the L^2-boundedness of Θ in Fact (1) and the ideal property in Lecture 8 Page 7.

Let us also mention the following Ito-1D cubic SNLS as in previous lectures.

$$\begin{cases} i\partial_t u = \Delta u + |u|^2 u + u \cdot \Phi\xi \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}). \end{cases} \tag{2.3}$$

One can compare (2.3) with the Stratonovich SNLS (2.2).

2.2. **Global well-posedness of the 1D Cubic SNLS with real-valued noise.** In this subsection, we prove global well-posedness of the Ito-1D cubic SNLS (2.3) with real-valued noise. Note that our steps below can also be applied to show global well-posedness of the Stratonovich-1D cubic SNLS (2.1).

As in Lecture 7 Page 1, we let $u = u_R$ be the solution to the truncated version of (2.3):

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')\eta_R(u)(t')\big(|u|^2u\big)(t')dt' - i\int_0^t S(t-t')\eta_R(u)(t')u(t')\Phi dW(t')$$

for $t \in [0, t_R]$ for some stopping time t_R. See also Lecture 6 Page 6.

In order to apply the Ito formula, we need some further regularization. Recall that η is a smooth and nonnegative cutoff function on $\mathbb{R}_+$ such that $\eta \equiv 1$ on $[0,1]$ and $\eta \equiv 0$ on $[2,\infty)$. We define the following Fourier multipliers:

$$\widehat{\Theta_k v}(\xi) = \eta\Big(\frac{|\xi|}{k}\Big)\widehat{v}(\xi), \quad k \in \mathbb{N} = \{1, 2, \dots\},$$
$$\widehat{S_k(t)v}(\xi) = \widehat{\Theta_k S(t)v}(\xi) = \eta\Big(\frac{|\xi|}{k}\Big)e^{it|\xi|^2}\widehat{v}(\xi).$$

Let $u = u_{R,m,N}$ with $m = (m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ be the solution to the following regularized equation (in Duhamel formulation):

$$\begin{aligned} u(t) = S_{m_1}(t)u_0 - i\int_0^t S_{m_1}(t-t')\eta_R(u)(t')\Theta_{m_2}\big(|\Theta_{m_2}u|^2\Theta_{m_2}u\big)(t')dt' \\ - i\int_0^t S_{m_1}(t-t')\eta_R(u)(t')u(t')\Phi\Theta_{m_2}dW_N(t'), \end{aligned} \tag{2.4}$$

where W_N is the noise term with finitely many components $\{\beta_0, \dots, \beta_N\}$. Here, the stochastic integral is understood as

$$-i\sum_{k=0}^N \int_0^t S_{m_1}(t-t')\eta_R(u)(t')u(t')\big(\Phi\Theta_{m_2}e_k\big)(x)d\beta_k(t'),$$

which is an Ito integral since, due to $\sum_{k=0}^N (\Phi\Theta_{m_2}e_k)^2 \in L^\infty_x$ and Fubini's theorem,

$$\sum_{k=0}^N \int_{\mathbb{R}}\int_0^t \Big|S_{m_1}(t-t')\eta_R(u)(t')u(t')\big(\Phi\Theta_{m_2}e_k\big)(x)\Big|^2 dt'dx \lesssim 1.$$

Note that, with $du(t)$ denoted as the Ito differential for $u = u_{R,m,N}$, (2.4) can be written as

$$du(t) = -i\Theta_{m_1}\Delta u - i\eta_R(u)\Theta_{m_2}\big(|\Theta_{m_2}u|^2\Theta_{m_2}u\big) - i\eta_R(u)u\Phi\Theta_{m_2}dW_N(t).$$

Let us recall that the mass is defined as

$$M(u)(t) = \int_{\mathbb{R}} u(t)\overline{u}(t)dx.$$

We apply the Ito formula for $|u(t,x)|^2$:

$$u(t,x)\overline{u}(t,x) = |u_0(x)|^2 + 2\,\mathrm{Re}\int_0^t u(t',x)d^{\mathrm{Ito}}u(t',x) + \langle u(x), \overline{u}(x)\rangle_t. \tag{2.5}$$

Here, for the second term on the right-hand side of (2.5), we have

$$2\,\mathrm{Re}\int_0^t u(t',x)d^{\mathrm{Ito}}u(t',x) \tag{2.6}$$

$$= \mathrm{Re}\int_0^t u(t',x)\Big[i\Theta_{m_1}\Delta u(t',x) + i\eta_R(u)(t')\Theta_{m_2}\big(|\Theta_{m_2}u|^2\Theta_{m_2}u\big)(t')\Big]dt' \tag{2.7}$$

$$+ \mathrm{Re}\sum_{k=0}^N i\int_0^t |u(t',x)|^2\eta_R(u)(t')\overline{\big(\Phi\Theta_{m_2}e_k\big)}(x)d\beta_k(t'), \tag{2.8}$$

where the last term is zero when the noise is real ($\Phi\Theta_{m_2}e_k$ is real. See Fact (1) in the previous subsection). Also, the last term on the right-hand side of (2.5) is a co-variation process given by

$$\langle u(x), \overline{u}(x)\rangle_t = \sum_{k=0}^{N}\int_0^t |u(t')|^2 \big(\Phi\Theta_{m_2}e_k\big)^2(x)dt' \tag{2.9}$$

$$= \sum_{k=0}^{N}\big(\Phi\Theta_{m_2}e_k\big)^2(x)\int_0^t |u(t',x)|^2 dt'. \tag{2.10}$$

Therefore, in the real-valued noise setting, we have

$$\begin{aligned} M(u)(t) = M(u_0) + \int_0^t\int_{\mathbb{R}}\sum_{k=0}^{N}\big(\Phi\Theta_{m_2}e_k\big)^2(x)|u(t',x)|^2 dxdt' \\ + 2\,\mathrm{Re}\int_0^t\int_{\mathbb{R}} u(t',x)\Big[i\overline{\Theta_{m_1}\Delta u}(t',x) + i\eta_R(u)(t')\overline{\Theta_{m_2}\big(|\Theta_{m_2}u|^2\Theta_{m_2}u\big)(t')}\Big]dxdt' \end{aligned} \tag{2.11}$$

Note that

$$\int_{\mathbb{R}}\overline{u}(t',x)\Theta_{m_1}\Delta u(t',x)dx = -\int_{\mathbb{R}}\Theta_{m_1}(\xi)|\xi|^2|\widehat{u}(t',\xi)|^2 d\xi \in \mathbb{R}$$

and

$$\eta_R(u)(t')\int_{\mathbb{R}}\overline{u}(t',x)\Theta_{m_2}\big(|\Theta_{m_2}u|^2\Theta_{m_2}u\big)(t')dx = \eta_R(u)(t')\int_{\mathbb{R}}|\Theta_{m_2}u|^4 dx \in \mathbb{R},$$

so that the last term on the right-hand side of (2.11) vanishes. Here, we remark that the second term on the right-hand side of (2.11) would be canceled in the Stratonovich-SNLS (see [1]). As a result, we obtain

$$\begin{aligned} M(u)(t) &= M(u_0) + \int_0^t\int_{\mathbb{R}}\sum_{k=0}^{N}\big(\Phi\Theta_{m_2}e_k\big)^2(x)|u(t',x)|^2 dxdt' \\ &\le M(u_0) + C\int_0^t M(u)(t')dt', \end{aligned}$$

where the constant C does not depend on R, m_1, m_2, or N. By Grönwall's inequality (which is valid because of the cutoff $\eta_R(u)$ defined in (0.2)), we get

$$M(u)(t) \le M(u_0)e^{Ct}.$$

Finally, we send $m_1 \to +\infty$, then $m_2 \to +\infty$, $N \to +\infty$, and send $R \to +\infty$, we achieve our goal for establishing the bound (0.3).

2.3. **Global well-posedness of the 1D Cubic SNLS with non-conservative noise.** In this subsection, we prove global well-posedness of the Ito-1D cubic SNLS (2.3) when the noise is not real-valued.

By (2.5), (2.8), and (2.10), we have

$$M(u)(t) = M(u_0) + \mathrm{Re}\, i \sum_{k=0}^{N} \int_{\mathbb{R}} \int_0^t |u(t',x)|^2 \eta_R(u)(t') \overline{\big(\Phi\Theta_{m_2} e_k\big)}(x) d\beta_k(t') dx \tag{2.12}$$

$$+ \int_0^t \int_{\mathbb{R}} \sum_{k=0}^{N} \big(\Phi\Theta_{m_2} e_k\big)^2(x) |u(t',x)|^2 dx dt' \tag{2.13}$$

$$=: M(u_0) + A(t) + \int_0^t \int_{\mathbb{R}} \sum_{k=0}^{N} \big(\Phi\Theta_{m_2} e_k\big)^2(x) |u(t',x)|^2 dx dt'. \tag{2.14}$$

Recall that we have

$$\Big\| \sum_{k=0}^{N} \big(\Phi\Theta_{m_2} e_k\big)^2(x) \Big\|_{L^\infty_x} \lesssim 1.$$

Fix any $q \in [2,\infty)$. Note that $(a+b+c)^q \lesssim a^q + b^q + c^q$ for all $a,b,c \in \mathbb{R}_+$. Thus, we obtain from (2.14) and Jensen's inequality that that

$$\begin{aligned} M(u)(t)^q &\lesssim M(u_0)^q + |A(t)|^q + \bigg(t \int_0^t \frac{1}{t} \eta_R(u)(t') M(u)(t') dt' \bigg)^q \\ &\le M(u_0)^q + |A(t)|^q + t^{q-1} \int_0^t \eta_R(u)^q(t') M(u)^q(t) dt'. \end{aligned}$$

For any finite time $T > 0$, by the Burkholder-Davis-Gundy inequality, Minkowski's inequality, the fact that the L^∞_x norm of $\sum_{k=0}^{N} (\Phi\Theta_{m_2} e_k)^2(x)$ is bounded by 1, and Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\le t\le T} |A(t)|^q \Big] &\le \mathbb{E}\bigg[\bigg| \sup_{t\le T} \sum_{k=0}^{N} \int_0^t \eta_R(u)(t') \int_{\mathbb{R}} \overline{\big(\Phi\Theta_{m_2} e_k\big)}(x) |u(t',x)|^2 dx d\beta_k(t') \bigg|^q \bigg] \\ &\lesssim_q \mathbb{E}\bigg[\bigg| \int_0^T \eta_R(u)^2(t') \sum_{k=0}^{N} \Big(\int_{\mathbb{R}} \overline{\big(\Phi\Theta_{m_2} e_k\big)}(x) |u(t',x)|^2 dx \Big)^2 dt' \bigg|^{q/2} \bigg] \\ &\le \mathbb{E}\bigg[\bigg| \int_0^T \eta_R(u)^2(t') \Big\| \big\| \Phi\Theta_{m_2} e_k(x) \big\|_{\ell^2_k} |u(t',x)|^2 \Big\|^2_{L^1_x} dt' \bigg|^{q/2} \bigg] \\ &\lesssim \mathbb{E}\bigg[\bigg| \int_0^T \eta_R(u)^2(t') M(u)^2(t') \bigg|^{q/2} \bigg] \\ &\le T^{\frac{q}{2}-1} \mathbb{E}\bigg[\int_0^T \eta_R(u)^q(t') M(u)^q(t') dt' \bigg]. \end{aligned}$$

Thus, for $M^*_t = \sup_{0\le r\le t} M(u)(r)$, we deduce that

$$\mathbb{E}\big[|M^*_T|^q\big] \lesssim |M(u_0)|^q + T^{q-1} \int_0^T \eta_R(u)^q(t) \mathbb{E}\big[|M^*_t|^q\big] dt + T^{\frac{q}{2}-1} \int_0^T \eta_R(u)^q(t) \mathbb{E}\big[|M^*_t|^q\big] dt,$$

where $\eta_R(u)$ ensures the finiteness of integrals. Thus, by Grönwall's inequality, we obtain

$$\mathbb{E}\big[|M^*_T|^q\big] \le C_1 M(u_0)^q \exp\big(C_2 (T^{q-1} + T^{\frac{q}{2}-1}) \big),$$

where C_1 and C_2 are constants that do not depend on R, m_1, m_2, or N. Finally, by sending $m_1 \to +\infty$, $m_2 \to +\infty$, $N \to \infty$, and $R \to \infty$, we achieve our goal for establishing the bound (0.3).

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LECTURE 13

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So far, we studied SPDEs with multiplicative noises in the Itô sense (i.e. we interpreted the stochastic convolution in $L^2(\Omega)$), also known as random field theory. We now turn our attention to the *pathwise well-posedness theory.* First thing we stumble upon, however, is the following issue:

Issue. Consider the following differential equation

$$\mathrm{d}Y_t = Y_t \,\mathrm{d}B_t,$$

where $B = (B_t)_{t\in[0,1]}$ is a Brownian motion. A first guess could be

$$Y_t = \int_0^t B_s \,\mathrm{d}B_s.$$

We can interpret the right hand side as a Wiener-Itô integral.

Even worse, one has the following result by Lyons in [2]:

Theorem 0.1 (Non-existence of path integral)**.** *There is no separable Banach space $X \subset C([0,1])$ such that*

(i) $B \in X$*, a.s. ,*
(ii) *The map*

$$(f,g) \mapsto \int_0^{\cdot} f(t)\partial_t g(t)\,\mathrm{d}t,$$

defined for smooth functions $(f,g) \in C^\infty([0,1]) \times C^\infty([0,1])$ extends continuously to a map: $X \times X \to C([0,1])$.

This theorem tells us that we can *not* construct $\int_0^t B_s \,\mathrm{d}B_s$ pathwise, i.e. we have to rely on probabilistic methods to construct the integral in $L^2(\Omega)$. The theory of rough paths is a framework for which integration with respect to Brownian motion can be made sense of pathwise.

Main idea: augment the data $B = (B_t)_{t\in[0,1]}$ by

$$\mathbb{B}(s,t) =: \int_s^t (B_r - B_s)\,\mathrm{d}B_r,$$

where the right-hand side is defined by the left-hand side. In other words, we are prescribing what integration of the Brownian motion with respect to itself should be. This is the main idea of rough paths theory developed by Lyons in [1]. For more about the theory look at the lecture notes from Spring 2020.

1. Pathwise integration

We consider the map

$$\begin{aligned}(f,g) \mapsto I(f,g) &= \int_0^{\cdot} f(t)\partial_t g(t)\,\mathrm{d}t,\\ C^\alpha \times C^\beta &\to C^\gamma,\end{aligned} \tag{1.1}$$

where C^α is the well-known α-Hölder space. Obviously, the expression given for I seems to require g to be differentiable. However, there are ways to interpret the integral through different point of views without requiring differentiability of g.

1.1. Differential calculus point of view.

We can say that $I(f,g)$ is the unique solution to

$$\partial_t I(f,g) = f\partial_t g, \quad I(f,g)(0) = 0.$$

Note that for functions $g \in C^\beta$ that are not classically differentiable we have that $\partial_t g$ is a distribution and the differential equation above is understood in distributional sense. We do not focus on this point of view in the sequel.

1.2. First increment point of view.

The first increment point of view is decreeing $I = I(f,g)$ to satisfy

$$\begin{cases} I(t) - I(s) &= f(s)(g(t)-g(s)) + o(|t-s|), \quad 0 \le s \le t \le 1,\\ \qquad I(0) &= 0, \end{cases} \tag{1.2}$$

uniformly over all $0 \le s \le t \le 1$, where $o(\cdot)$ is the little-o notation as $|t-s| \to 0$.

Remark 1.1. (i) *Note that* (1.2) *is clearly satisfied if $g \in C^1([0,1])$ and $f \in C([0,1])$, and in such case*

$$I(t) = \int_0^t f(s)\partial_s g(s)\,\mathrm{d}s,$$

is well-defined and satisfies by the fundamental theorem of calculus

$$I(t) - I(s) - f(t)(g(t)-g(s)) = \int_s^t (f(r)-f(s))\partial_r g(r)\,\mathrm{d}r.$$

By assumption f is continuous on the compact interval $[0,1]$ and is therefore uniformly continuous, in particular the function admits a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ such that

$$|f(r) - f(s)| \le \omega(|r-s|), \quad r,s \in [0,1].$$

The modulus of continuity ω is increasing and is continuous at 0 and $\omega(0) = 0$. In particular

$$\begin{aligned}\left|\int_s^t (f(r)-f(s))\partial_r g(r)\,\mathrm{d}r\right| &\le \int_s^t |f(r)-f(s)||\partial_r g(r)|\,\mathrm{d}r\\ &\le \|\partial_r g\|_{L^\infty} \int_s^t \omega(|r-s|)\,\mathrm{d}r\\ &\le \|\partial_r g\|_{L^\infty}\omega(|t-s|)|t-s|,\end{aligned}$$

which shows that

$$\int_s^t (f(r) - f(s))\partial_r g(r)\,\mathrm{d}r = o(|t-s|),$$

yielding (1.2).

(ii) *Furthermore,* (1.2) *determines I. Suppose that J also satisfies* (1.2). *Set $D = I - J$ and note that*

$$D(t) - D(s) = o(|t-s|),$$

and this shows that

$$\left|\frac{D(t) - D(s)}{t-s}\right| = o(1),$$

from which follows

$$\lim_{s\to t} \frac{D(t) - D(s)}{t-s} = 0.$$

Hence $D'(t) = 0$ for all $t \in [0,1]$ and $D(0) = 0$ which implies $D \equiv 0$.

This shows that I is the only function whose increment matches the "germ" $f(s)(g(t) - g(s))$ modulo a negligible error.

1.3. **Several notations and the Sewing Lemma.** Let V be a vector space, e.g. $V = \mathbb{R}$. For $n \geq 1$ and $s \leq t$ define

$$\Delta_n(s,t) := \{(s_1, ..., s_n) \in \mathbb{R}^n \ : \ s \leq s_1 \leq \cdots \leq s_n \leq t\}.$$

Furthermore, set $C_1(V) := C([0,1], V)$ and for $n \geq 2$, define $C_n(V) \subset C(\Delta_n, V)$ as the space of all functions $f : \Delta_n \to V$ such that $f(s_1, ..., s_n) = 0$ whenever $s_1 = \cdots = s_n$. An element in $C_n(V)$ is *n-cochain.* We also define the *coboundary operator* $\delta : C_n(V) \to C_{n+1}(V)$ for any $f \in C_n(V)$ by

$$\delta f(s_1, ..., s_{n+1}) := \sum_{k=1}^{n+1} (-1)^{n-k} f(s_1, ..., s_{k-1}, s_{k+1}, ..., s_{n+1}).$$

We may use the notation δ_n to specify the dependence on n in $C_n(V)$.

Example 1. *For $f \in C_1$ we have*

$$\delta f(s,t) = f(t) - f(s).$$

For $f \in C_2$ we have

$$\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u).$$

There are several **facts** that we state without proof:

- We have
$$\delta \circ \delta = 0,$$
in particular $\operatorname{Im} \delta_{n-1} \subset \operatorname{Ker} \delta_n$.
- We get the following cochain complex:
$$0 \to \mathbb{R} \to C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \xrightarrow{\delta} \cdots \tag{1.3}$$
This complex is *exact*, i.e. $\operatorname{Im} \delta_{n-1} = \operatorname{Ker} \delta_n$ and the cohomology $H^n = \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n-1} = \{0\}$. Hence if $\delta f = 0$, then $f = \delta g$ for some g.

These newly introduced notation will be used in our context as follows. Let $A(s,t) = f(s)\delta g(s,t) = f(s)(g(t)-g(s))$ and note that (1.2) is equivalent to

$$A = \delta I + R,$$

for some R such that $R(s,t) = o(|t-s|)$. Taking δ on both sides yields

$$\delta A = \delta R,$$

since $\delta \circ \delta I = 0$. The so-called *sewing map* allows us to recover R from $\delta A \in C_3$.

Before formulating the main result, let us define a topology on C_n. We say $f \in C_n^\alpha([s,t])$ if

$$\|f\|_{C_n^\alpha([s,t])} := \sup_{(s_1,\ldots,s_n)\in\Delta_n(s,t)} \frac{|f(s_1,\ldots,s_n)|}{|s_n - s_1|^\alpha} < \infty.$$

See the paper by Gubinelli and Tindel [3] for a variant norm and more on the cochains.

We set

$$C_n^{\alpha^+} := \bigcup_{\beta>\alpha} C_n^\beta.$$

Remark 1.2. *We have $\delta C_1 \cap C_2^{1^+} = \{0\}$ which follows from the fact that any function satisfying $|f(x)-f(y)| \lesssim |x-y|^\alpha$ for $\alpha > 1$ is constant. Indeed any $f \in \delta C_1$ is of the form $f = \delta g$ and $f \in C_2^{1^+}$ implies*

$$|g(t)-g(s)| = |\delta g(s,t)| = |f(s,t)| \lesssim |t-s|^\alpha,$$

for some $\alpha > 1$. In particular g is constant and therefore $f \equiv 0$. This shows the claim.

Theorem 1.3 (Sewing Lemma)**.** *There exists a unique map $\Lambda : C_3^{1^+} \cap \delta C_2 \to C_2^{1^+}$ such that*

$$\delta\Lambda = \mathrm{Id}_{C_3\cap\delta C_2},$$

and for any closed interval $I \subset \mathbb{R}_+$ and $\alpha > 1$, there exists a constant $C = C(\alpha)$ such that

$$\|\Lambda h\|_{C_2^\alpha(I)} \le C\|h\|_{C_3^\alpha(I)}, \qquad \text{for all } h \in C_3^\alpha \cap \delta C_2. \tag{1.4}$$

Proof. See the course notes from Spring 2020 or [3]. The proof can also be found in [4]. □

The way we apply the Sewing Lemma is along the following lines. Recall that $A = \delta I + R$ which gave us $\delta A = \delta R$. Assuming $\delta A \in C_3^{1^+} \cap \delta C_2$, we have $\delta\Lambda\delta A = \delta A$ which gives

$$\delta(R - A) = \delta(R - \Lambda\delta A) = 0.$$

Therefore by the fact the cochain in (1.3) is exact we get

$$R - \Lambda\delta A = \delta f,$$

for some $f \in C_1$. Note that $R_{s,t} = o(|t-s|)$ and $\Lambda\delta A \in C_2^{1^+}$ which gives that $\delta f = o(|t-s|)$ yielding $\delta f \equiv 0$. Hence

$$R = \Lambda\delta A.$$

We conclude that

$$\delta I = A - R = (\mathrm{Id} - \Lambda\delta)A.$$

1.4. **Young integral.** Let $f \in C^\alpha$ and $g \in C^\beta$ with $\alpha + \beta > 1$. With the first increment point of view, it is natural to write

$$\delta I(f,g)(s,t) = \int_s^t f(u)\,\mathrm{d}g(u) = \int_s^t f(s)\,\mathrm{d}g(u) - R(s,t) = f(s)\delta g(s,t) - R(s,t),$$

where $R(s,t) = o(|t-s|)$. We set $A(s,t) = f(s)\delta g(s,t)$ to obtain $\delta I = A - R$. Now note that

$$\begin{aligned}\delta A(s,u,t) &= A(s,t) - A(u,t) - A(s,u)\\ &= f(s)\delta g(s,t) - f(u)\delta g(u,t) - f(s)\delta g(s,u)\\ &= f(s)(\delta g(s,t) - \delta g(s,u)) - f(u)\delta g(u,t).\end{aligned}$$

We have

$$\delta g(s,t) - \delta g(s,u) = g(t) - g(s) - (g(u) - g(s)) = g(t) - g(u) = \delta g(u,t),$$

so that

$$\delta A(s,u,t) = f(s)\delta g(u,t) - f(u)\delta g(u,t) = (f(s) - f(u))\delta g(u,t) = -\delta f(s,u)\delta g(u,t).$$

By assumption $f \in C^\alpha$ and $g \in C^\beta$ which gives

$$|\delta A(s,u,t)| = |\delta f(s,u)\delta g(u,t)| \lesssim |u-s|^\alpha |t-u|^\beta \lesssim |t-s|^{\alpha+\beta},$$

yielding $\delta A \in C_3^{\alpha+\beta}$. Since $\alpha + \beta > 1$ we get by the Sewing Lemma given in 1.3 (and in particular the discussion after the statement), that

$$\delta I(f,g) = (\mathrm{Id} - \Lambda\delta)A = (\mathrm{Id} - \Lambda\delta)(f\delta g).$$

Note that for any partition $P = \{0 = t_0 \le t_1 \le t_2 \le .... \le t_n = t\}$, we can write through telescoping

$$\begin{aligned}I(f,g)(t) &= I(f,g)(t) - I(f,g)(0)\\ &= \sum_P \delta I(f,g)(t_i,t_{i+1})\\ &= \sum_P (f(t_i)(g(t_{i+1}) - g(t_i)) + \Lambda\delta(f\delta g)(t_i,t_{i+1}))\\ &= \sum_P f(t_i)(g(t_{i+1}) - g(t_i) + \sum_P \Lambda\delta(f\delta g)(t_i,t_{i+1}).\end{aligned}$$

Note that since the partition P is arbitrary, we get an equality when we take the mesh size $|P| \to 0$, in other words we get

$$I(f,g)(t) = \lim_{|P|\to 0} \sum_P f(t_i)(g(t_{i+1}) - g(t_i) + \sum_P \Lambda\delta(f\delta g)(t_i,t_{i+1}).$$

We note that the map Λ maps continuously into $C_2^{\alpha+\beta}$ which yields that

$$\sum_P |\Lambda\delta(f\delta g)(t_i,t_{i+1})| \lesssim \sum_P |t_{i+1} - t_i|^{\alpha+\beta} \lesssim |P|^{\alpha+\beta-1} \underbrace{\sum_P |t_{i+1} - t_i|}_{=t},$$

where we have used the fact that $\alpha + \beta > 1$ in the last inequality. We obtain

$$\lim_{|P|\to 0} \sum_P \Lambda\delta(f\delta g)(t_i,t_{i+1}) = 0,$$

and consequently

$$I(f,g)(t) = \lim_{|P|\to 0} \sum_P f(t_i)(g(t_{i+1}) - g(t_i)). \tag{1.5}$$

1.5. **Young differential equations.** Now consider an equation of the form

$$\mathrm{d}Y_t = Y_t \,\mathrm{d}X_t, \quad Y|_{t=0} = Y_0 \tag{1.6}$$

with some $\alpha > 1/2$, $X \in C^\alpha$ and $y \in \mathbb{R}$ (for the initial condition). An example that falls into this category is when X is a fractional Brownian motion with Hurst parameter $H > 1/2$.

Now assuming $Y \in C^\alpha$ satisfies the DE, we get

$$Y_t = Y_r + \int_r^t Y_u \,\mathrm{d}Y_u \tag{1.7}$$

$$= Y_r + \int_r^t Y_r \,\mathrm{d}X_u + \underbrace{\int_r^t \delta Y_{ur} \,\mathrm{d}X_u}_{=:R_{tr}} \tag{1.8}$$

In here and in the sequel, we use the notation $A_{tr} = A(r,t)$ for any $A \in C_2$, and also if $f \in C_1$, we may write f_{tr} to mean $\delta f_t r = \delta f(r,t)$. We have

$$\int_r^t Y_u \,\mathrm{d}X_u = \underbrace{\int_r^t Y_r \,\mathrm{d}X_u}_{Y_r \delta X_{tr}} + R_{tr}.$$

Taking δ of both sides together with the fact that $\delta\delta = 0$ (the left hand side is basically δI), yields

$$\delta R_{t_1t_2t_3} = -\delta(Y\delta X)_{t_1t_2t_3} := \delta X_{t_1t_2}\delta Y_{t_2t_3} = O(|t_1 - t_3|^{2\alpha}).$$

Note that $2\alpha > 1$ by the assumption that $\alpha > 1/2$, so we can apply the Sewing Lemma and get that

$$R = -\Lambda\delta(Y\delta X).$$

Hence

$$\int_r^t Y_u \,\mathrm{d}X_u = (\mathrm{Id} - \Lambda\delta)(Y\delta X)_{tr}.$$

Recalling (1.7) our Young differential equation (1.6) is boils down to solving the following fixed point problem:

$$Y_t - Y_r = (\mathrm{Id} - \Lambda\delta)(Y\delta X)_{tr} =: (\delta\Gamma Y)_{tr}. \tag{1.9}$$

In the above we have set

$$(\Gamma Y)(t) := (\mathrm{Id} - \Lambda\delta)(Y\delta X)_{t0} + Y_0.$$

Recall on the space of Hölder functions C^α the norm can be given by

$$\|Z\|_{C^\alpha} := |Z_0| + |Z|_{C^\alpha},$$

where

$$|Z|_{C^\alpha} := \sup_{s\neq t} \frac{|Z_t - Z_s|}{|t-s|^\alpha}.$$

Note that we can write

$$|(\delta\Gamma Y)_{tr}| \leq \|Y\|_{L^\infty}\|X\|_{C^\alpha}|t-r|^\alpha + |\Lambda\delta(Y\delta X)_{tr}| \leq \|Y\|_{L^\infty}\|X\|_{C^\alpha}|t-r|^\alpha + \|\delta(Y\delta X)\|_{C_2^\alpha}|t-r|^\alpha.$$

We note that for any $t \le T$, we have

$$|Z_t| \le |Z_0| + |Z_t - Z_0| \le |Z_0| + |Z|_{C^\alpha} T^\alpha.$$

This yields

$$\|Z\|_{L^\infty_T} \le |Z_0| + T^\alpha |Z|_{C^\alpha}. \tag{1.10}$$

We have

$$\|\delta(Y\delta X)\|_{C_2^\alpha} \lesssim T^\alpha \|\Lambda(\delta Y \delta X)\|_{C_2^{2\alpha}} \overset{(1.4)}{\lesssim} T^\alpha \|\delta Y \delta X\|_{C_3^{2\alpha}} \lesssim T^\alpha \|Y\|_{C^\alpha} \|X\|_{C^\alpha}.$$

Therefore

$$|\Gamma Y|_{C^\alpha} \lesssim \|Y\|_{L^\infty_T} \|X\|_{C^\alpha} + T^\alpha \|Y\|_{C^\alpha} \|X\|_{C^\alpha}.$$

In particular

$$|\Gamma Y|_{C^\alpha} \lesssim (|Y_0| + T^\alpha \|Y\|_{C^\alpha}) \|X\|_{C^\alpha}.$$

Therefore, since $(\Gamma Y)(0) = Y_0$ we get

$$\|\Gamma Y\|_{C^\alpha} \le c_0(1 + |Y_0|)\|X\|_{C^\alpha} + CT^\alpha \|Y\|_{C^\alpha} \|X\|_{C^\alpha},$$

for some constants $c_0, C > 1$ only depending on α. We can take $R := c_0(1 + |Y_0|)\|X\|_{C^\alpha}$ and define

$$B_R := \{Y \in C^\alpha \ : \ \|Y\|_{C^\alpha} \le R, \ Y|_{t=0} = Y_0\}.$$

Now take $T \ll 1$ small enough so that $CT^\alpha \|Y\|_{C^\alpha} \le \frac{1}{2} R$ for all $Y \in B_R$. This establishes $\Gamma Y : B_R \to B_R$.

We still need to estimate the difference of ΓY and $\Gamma \tilde{Y}$ for $Y, \tilde{Y} \in B_R$ to complete the fixed point argument through Banach's fixed point theorem. We have by a similar argument

$$\|\Gamma Y - \Gamma \tilde{Y}\|_{C^\alpha} \lesssim \|Y - \tilde{Y}\|_{L^\infty} \|X\|_{C^\alpha} + T^\alpha \|Y\|_{C^\alpha} \|X\|_{C^\alpha}.$$

We have $Y_0 = \tilde{Y}_0$ so that

$$\|Y - \tilde{Y}\|_{L^\infty_T} \le T^\alpha \|Y - \tilde{Y}\|_{C^\alpha},$$

which gives

$$\|\Gamma Y - \Gamma \tilde{Y}\|_{C^\alpha} \le C_1 T^\alpha \|Y - \tilde{Y}\|_{C^\alpha} \|X\|_{C^\alpha}.$$

By taking T small enough to ensure $CT^\alpha \|X\|_{C^\alpha} \le \frac{1}{2}$, we get

$$\|\Gamma Y - \Gamma \tilde{Y}\|_{C^\alpha} \le \frac{1}{2} \|Y - \tilde{Y}\|_{C^\alpha}.$$

This shows existence of a unique solution in B_R. One can show that this solution is the unique local solution in C^α through the standard argument.

1.6. **Towards rough paths.** After the theory for Young differential equations for C^α with $\alpha > 1/2$, we may ask:

Question: What about Brownian motion?

The theory in the subsection above does not cover whenever X is a Brownian motion. In such case $X \in C^\alpha \setminus C^{1/2}$, a.s. for any $\alpha < 1/2$.

Let $X \in C^\alpha(\mathbb{R}_+, V)$ for some $1/3 \le \alpha < 1/2$, and $\mathbb{X} \in C_2^{2\alpha}(\mathbb{R}_+, V \otimes V)$, satisfying Chen's relation, i.e.

$$\mathbb{X}_{t_1 t_3} - \mathbb{X}_{t_1 t_2} - \mathbb{X}_{t_2 t_3} = \delta X_{t_1 t_2} \otimes \delta X_{t_2 t_3}. \tag{1.11}$$

We call $(X, \mathbb{X})$ a *rough path*.

1.7. **Controlled rough paths.** Controlled rough path is a path Y taking values that somehow locally behaves like X. This notion was introduced by Gubinelli in [4]. Let W be some Banach space and denote by $\mathcal{L}(V,W)$ the space of linear bounded operators from V to W. The precise definition of controlled rough path can be stated as follows:

Definition 1.4 (Controlled rough path)**.** *A path $Y \in C^\alpha([0,T];W)$ is controlled rough path if*

$$\delta Y_{tr} = Y'_r X_{tr} + R^Y_{tr}, \tag{1.12}$$

with $Y' \in C^\alpha([0,T];\mathcal{L}(V,W))$ and $R^Y \in C_2^{2\alpha}(V)$. We call Y' the Gubinelli derivative and we use the notation $(Y,Y') \in \mathcal{D}_X^{2\alpha}$ for a controlled rough path.

Example 2. *Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. An example of controlled rough path is $Y = F(X)$ for some function $F : \mathbb{R}^n \to \mathbb{R}^m$. In this case $Y' = DF(X)$ which can verified by Taylor's theorem.*

From now on $V = W = \mathbb{R}$. We let Y be a controlled rough path. Note that for the case $V = W = \mathbb{R}$, we have $\mathcal{L}(V,W) \simeq \mathbb{R}$ and consequently $Y' : [0,T] \to \mathbb{R}$. Consider

$$I_{tr} = \int_r^t Y_u \, \mathrm{d}X_u.$$

By (1.12), we formally have

$$\begin{aligned} I_{tr} &= \int_r^t Y_r \, \mathrm{d}X_u + \int_r^t \delta Y_{ur} \, \mathrm{d}X_u \\ &= Y_r \delta X_{tr} + \int_r^t Y'_r X_{tr} + R^Y_{tr} \, \mathrm{d}X_u \\ &= Y_r \delta X_{tr} + Y'_r \underbrace{\int_r^t X_{ur} \, \mathrm{d}X_u}_{\text{undefined}} + \underbrace{\int_r^t R^Y_{ur} \, \mathrm{d}X_u}_{\text{undefined}}. \end{aligned}$$

The last two terms are not well-defined, but we impose that

$$\int_r^t X_{ur} \, \mathrm{d}X_u = \mathbb{X}_{tr},$$

which leads us to the definition of the integral through the following. We define I by $I(0) = 0$ and

$$I_{tr} = Y_r \delta X_{tr} + Y'_r \mathbb{X}_{tr} + o(|t-r|). \tag{1.13}$$

As in (1.2) on page 2, it turns out that (1.13) characterizes I_{tr} through the Sewing Lemma.

Remark 1.5. *When Y and X are smooth then $\int_r^t Y_u \, \mathrm{d}X_u$ satisfies (1.13) with $\mathbb{X}_{tr} = \int_r^t X_{ur} \, \mathrm{d}X_u$ which for smooth X is well-defined.*

We have

$$I_{tr} = \delta X_{tr} Y_r + \mathbb{X}_{tr} Y'_r + K_{tr}$$

and note that

$$\begin{aligned} \delta K_{t_1 t_2 t_3} &= -\delta(\delta X_{tr} Y_r + \mathbb{X}_{tr} Y'_r)_{t_1 t_2 t_3} \\ &= \delta X_{t_1 t_2} \delta Y_{t_2 t_3} - \delta(\mathbb{X}_{tr} Y'_r)_{t_1 t_2 t_3} \\ &=: \mathrm{I} - \mathrm{II}. \end{aligned}$$

In here $\delta(\mathbb{X}_{tr}Y'_r)_{t_1t_2t_3}$ means δ applied to $(r,t)\mapsto \mathbb{X}_{tr}Y'_r$ and evaluated at (t_1,t_2,t_3). For I we write

$$\mathrm{I} \overset{(1.12)}{=} \delta X_{t_1t_2}\delta X_{t_2t_3}Y'_{t_3} + \underbrace{\delta X_{t_1t_2}R^Y_{t_2t_3}}_{\in C_3^{2\alpha}}. \tag{1.14}$$

For II we write

$$\begin{aligned}\mathrm{II} &= \mathbb{X}_{t_1t_3}Y'_{t_3} - \mathbb{X}_{t_1t_2}Y'_{t_2} - \mathbb{X}_{t_2t_3}Y'_{t_3}\\ &= (\mathbb{X}_{t_1t_3} - \mathbb{X}_{t_2t_3})Y'_{t_3} - \mathbb{X}_{t_1t_2}Y'_{t_2}.\end{aligned}$$

We now use Chen's relations (1.11) to write

$$\mathbb{X}_{t_1t_3} - \mathbb{X}_{t_2t_3} = \mathbb{X}_{t_1t_2} + \delta X_{t_1t_2}\delta X_{t_2t_3},$$

which yields

$$\mathrm{II} = \underbrace{-\mathbb{X}_{t_1t_2}\delta Y'_{t_2t_3}}_{\in C_2^{3\alpha}} + \delta X_{t_1t_2}\delta X_{t_2t_3}Y'_{t_3}. \tag{1.15}$$

Note that the last term in the previous expression is the same term as the first term in the expression for I in (1.14). Therefore these term cancel each other when we subtract (1.15) from (1.14), and we will be left with

$$\delta K_{t_1t_2t_3} = \mathrm{I} - \mathrm{II} = \delta X_{t_1t_2}R^Y_{t_2t_3} - \mathbb{X}_{t_1t_2}\delta Y'_{t_2t_3} \in C_2^{3\alpha}\cap C_2.$$

Note that $\alpha \geq 1/3$ which implies that $K \in C_3^{1^+}\cap \delta C_2$ allowing us to apply the Sewing Lemma 1.3 yielding

$$K = -\Lambda\delta(\delta X_{tr}Y_r + \mathbb{X}_{tr}Y'_r),$$

and hence

$$I_{tr} = \int_r^t Y_u \,\mathrm{d}X_u = \left[(\mathrm{Id} - \Lambda\delta)(\delta X_{t_1t_2}Y_{t_2} + \mathbb{X}_{t_1t_2}Y'_{t_2})\right]_{tr}.$$

This defines I_{tr}, but for a more pleasing formula concerning Riemann-type sums, can obtained by a similar calculations as was done to obtain (1.5) for the Young integral, namely

$$\int_0^t Y_r \,\mathrm{d}X_r = \lim_{|P|\to 0}\sum_P Y_{t_i}X_{t_{i+1}t_i} + Y'_{t_i}\mathbb{X}_{t_{i+1}t_i}.$$

1.8. **Rough differential equations.** Now we consider rough differential equations (RDE) which are of the following type

$$\mathrm{d}Y_t = Y_t\,\mathrm{d}X_t, \quad Y|_{t=0} = Y_0,$$

where $X \in C^\alpha$ with $1/3 < \alpha \leq 1/2$ is a rough path, i.e. enhanced with some $\mathbb{X} \in C_2^{2\alpha}$ satisfying Chen's relation (1.11). We have that a solution Y would satisfy

$$Y_t - Y_r = \int_r^t Y_u\,\mathrm{d}X_u = \left[(\mathrm{Id} - \Lambda\delta)(\delta X\cdot Y + \mathbb{X}\cdot Y')\right]_{tr}. \tag{1.16}$$

Note that (1.16) for a controlled path $(Y,Y')\in \mathcal{D}_X^{2\alpha}$, we have $\delta Y_{tr} = Y_r\delta X_{tr}$ + "error", which implies that $Y' = Y$. We can endow $\mathcal{D}_X^{2\alpha}$ with the seminorm

$$\|(Y,Y')\|_{X,2\alpha} = \|Y'\|_{C^\alpha} + \|R^Y\|_{C_2^{2\alpha}},$$

which makes $\mathcal{D}_X^{2\alpha}$ a Banach space under the norm

$$|||(Y,Y')||| := |Y_0| + |Y_0'| + \|(Y,Y')\|_{X,2\alpha}. \tag{1.17}$$

We solve the RDE through a fixed point argument. To that end, we set

$$\Gamma(Y,Y')(t) := \left(\int_0^t Y_u \,\mathrm{d}X_u, Y_t\right) + (Y_0, 0).$$

Let us also set

$$Z_t := \int_0^t Y_u \,\mathrm{d}X_u, \quad Z_t' = Y_t.$$

In order to estimate the seminorm $\|\cdot\|_{X,2\alpha}$ of $\Gamma(Y,Y')$ we need to write Z_t as a controlled path

$$\delta Z_{tr} = Z_r' \delta X_{tr} + R_{tr}^Z.$$

Since $Z_t = \int_0^t Y_u \,\mathrm{d}X_u$, we have (from (1.13) on page 8)

$$Z' = Y.$$

However, what is R^Z? We have

$$\begin{aligned} R_{tr}^Z &= \delta Z_{tr} - Z_r' \delta X_{tr} \\ &= \int_r^t Y_u \,\mathrm{d}X_u - Y_r \delta X_{tr} \\ &\overset{(1.16)}{=} \mathbb{X}_{tr} Y_r - \left[\Lambda \delta(\delta X \cdot Y + \mathbb{X} \cdot Y')\right]_{tr}. \end{aligned} \tag{1.18}$$

By writing $\Gamma(Y,Y') = (\Gamma_1(Y,Y'), \Gamma_2(Y,Y'))$ with $\Gamma_1(Y,Y')$ the Gubinelli derivative of $\Gamma(Y,Y')$ and $\Gamma_2(Y,Y')$ the remainder. We have for the Gubinelli derivative $\Gamma_1(Y,Y')$

$$\begin{aligned} \|\Gamma_1(Y,Y')\|_{C^\alpha} &= \|Z'\|_{C^\alpha} \\ &= \|Y\|_{C^\alpha} \\ &\overset{(1.12)}{\leq} \|Y'\|_{L_T^\infty} \|X\|_{C^\alpha} + T^\alpha \|R^Y\|_{C_2^{2\alpha}} \\ &\leq (|Y_0'| + T^\alpha \|Y'\|_{C^\alpha}) \|X\|_{C^\alpha} + T^\alpha \|R^Y\|_{C_2^{2\alpha}}, \end{aligned} \tag{1.19}$$

where we have used the bound (1.10). Moreover for the remainder $\Gamma_2(Y,Y')$ we have using (1.18) together with the Sewing Lemma (in particular (1.4)), that

$$\begin{aligned} \|\Gamma_2(Y,Y')\|_{C_2^{2\alpha}} &= \|R^Z\|_{C_2^{2\alpha}} \\ &\overset{(1.18)}{=} \|\mathbb{X}_{tr} Y_r - \left[\Lambda \delta(\delta X \cdot Y + \mathbb{X} \cdot Y')\right]_{tr} \|_{C_2^{2\alpha}} \\ &\overset{(1.4)}{\leq} \|\mathbb{X}\|_{C_2^{2\alpha}} \|Y\|_{L_T^\infty} + C \|\delta X_{t_1 t_2} R_{t_1 t_2}^Y - \mathbb{X}_{t_1 t_2} \delta Y_{t_2 t_3}'\|_{C^{2\alpha}} \\ &\leq \|\mathbb{X}\|_{C_2^{2\alpha}} \|Y\|_{L_T^\infty} + C T^\alpha (\|X\|_{C^\alpha} \|R^Y\|_{C^2 \alpha_2} + \|\mathbb{X}\|_{C_2^{2\alpha}} \|Y'\|_{C^\alpha}), \end{aligned} \tag{1.20}$$

where the constant C is coming from (1.4) and is only depending on α. For the first term in the previous inequality we use

$$\|Y\|_{L_T^\infty} \leq |Y_0| + T^\alpha \|Y\|_{C^\alpha} \overset{(1.19)}{\leq} |Y_0| + T^\alpha (|Y_0'| + T^\alpha \|Y'\|_{C^\alpha}) \|X\|_{C^\alpha} + T^{2\alpha} \|R^Y\|_{C_2^{2\alpha}}. \tag{1.21}$$

Note that $\Gamma(Y,Y')|_{t=0} = (Y_0, Y_0)$ yields

$$|||\Gamma(Y,Y')||| \leq 2|Y_0| + \|\Gamma(Y,Y')\|_{X,2\alpha}.$$

For the second term we use (1.19), (1.20) and (1.21) to get

$$|||\Gamma(Y,Y')||| \leq \underbrace{c_0(1+\|\mathbb{X}\|_{C_2^{2\alpha}})+\|X\|_{C^\alpha}\|\mathbb{X}\|_{C_2^{2\alpha}}|Y_0'|}_{=:\frac{1}{2}R} + c(\|X\|_{C^\alpha},\|\mathbb{X}\|_{C_2^{2\alpha}})T^\alpha(\|R^Y\|_{C_2^{2\alpha}}+\|Y'\|_{C^\alpha}),$$

for any $0 \leq T \leq 1$. As already underbraced above, we set

$$R := 2\left(c_0(1+\|\mathbb{X}\|_{C_2^{2\alpha}})+\|X\|_{C^\alpha}\|\mathbb{X}\|_{C_2^{2\alpha}}|Y_0'|\right).$$

Then by choosing $T = T(R, \|X\|_{C^\alpha}, \|\mathbb{X}\|_{C_2^{2\alpha}}) > 0$ sufficiently small, we have

$$\begin{aligned}|||\Gamma(Y,Y')||| &\leq \frac{1}{2}R + C(\|X\|_{C^\alpha},\|\mathbb{X}\|_{C_2^{2\alpha}})T^\alpha \underbrace{\|(Y,Y')\|_{X,2\alpha}}_{\leq |||(Y,Y')|||} \\ &\leq R,\end{aligned} \tag{1.22}$$

for any $(Y,Y') \in B_R \subset \mathcal{D}_X^{2\alpha} \cap \{(Y,Y')|_{t=0} = (Y_0,Y_0)\}$. One has the B_R is a closed subset in $\mathcal{D}_X^{2\alpha}$ and the intersection is to make sure the initial condition is satisfied. This all shows that $\Gamma : B_R \to B_R$. We need to show that Γ is also a contraction.

To that end, we need to estimate the difference between two elements. Through an exact similar approach as done above, but now done for $Y - \tilde{Y}$ yields the following for the Gubinelli derivative

$$\|\Gamma_1(Y,Y') - \Gamma_1(\tilde{Y},\tilde{Y}')\|_{C^\alpha} = \|Y - \tilde{Y}\|_{C^\alpha} \tag{1.23}$$

$$\leq T^\alpha \|Y' - \tilde{Y}'\|_{C^\alpha}\|X\|_{C^\alpha} + T^\alpha \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\alpha}} \tag{1.24}$$

for any $(Y,Y'),(\tilde{Y},\tilde{Y}') \in B_R$, where we have to remark that B_R include the initial condition ensuring $(Y - \tilde{Y})_0 = 0$.

For the remainder term we get

$$\|\Gamma_2(Y,Y') - \Gamma_2(\tilde{Y},\tilde{Y}')\|_{C_2^{2\alpha}} \leq \|\mathbb{X}\|_{C_2^{2\alpha}}\|Y-\tilde{Y}\|_{L_T^\infty} + CT^\alpha(\|X\|_{C^\alpha}\|R^Y - R^{\tilde{Y}}\|_{C_2^{2\alpha}} + \|\mathbb{X}\|_{C_2^{2\alpha}}\|Y'-\tilde{Y}'\|_{C^\alpha}), \tag{1.25}$$

where the constant C is by the Sewing Lemma (1.4). We also have

$$\|Y-\tilde{Y}\|_{L_T^\infty} \leq T^\alpha\|Y-\tilde{Y}\|_{C^\alpha} \overset{(1.23)}{\leq} T^{2\alpha}\|Y'-\tilde{Y}'\|_{C^\alpha}\|X\|_{C^\alpha} + T^{2\alpha}\|R^Y - R^{\tilde{Y}}\|_{C_2^{2\alpha}}. \tag{1.26}$$

By gathering the inequalities (1.23), (1.25) and (1.26), we obtain

$$|||\Gamma(Y,Y') - \Gamma(\tilde{Y},\tilde{Y}')||| \leq C(\|X\|_{C^\alpha}\|\mathbb{X}\|_{C_2^{2\alpha}})T^\alpha |||(Y,Y') - (\tilde{Y},\tilde{Y}')||| \tag{1.27}$$

We can make $C(\|X\|_{C^\alpha},\|\mathbb{X}\|_{C_2^{2\alpha}})T^\alpha \leq \frac{1}{2}$ by choosing $T \ll 1$. From (1.22) and (1.27) we conclude that Γ is a contraction on B_R, so a (local) solution exists by Banach's fixed point theorem.

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LECTURE 14 26/04/22 (TUE)

GUOPENG LI

1. Lecture 14

In this lecture, we start with some general background on the Rough path theory:

- Rough path theory was originally introduced in terms of V^p = functions of bounded p-variations. Moreover, we notice the following facts regarding to the V^p-space.
  - $C^\alpha \subset V^{1/\alpha}$, for any $0 < \alpha < 1$.
  - If a function $f \in V_c^p$, then $\exists$ reparametrization τ such that $f \circ \tau \in C^{1/p}$.
- The rough differential equation (RDE), RPDE:
  Given an SDE (or SPDE) with a noise X which is rough in time (C^α, $\alpha \le 1/2$, V^p for $p \ge 2$), we lift the noise X to a rough path $(X, \mathbb{X})$ and study the original equation, where an integral is interpreted as a rough integral.

In the dispersive PDEs, we introduce the *paracontrolled distributations* [3]. We consider

$$dY = Y dX,$$

then the paracontrolled ansatz:

$$Y = Y' \olessthan X + R. \tag{1.1}$$

Here, $X \in C^\alpha$ for $\alpha \le 1/2$, $Y' \in C^\alpha$ denote as the "Gubinelli derivative", and the remainder term $R \in C^{2\alpha}$. Moreover, the symbel $\olessthan$ denotes as paraproduce, we see the following for the precise definition.

Paraproduct decomposition [1]:

Let $\mathbf{P}_j$ be the Littlewood-Paley projection operators, and we can have the decomposition:

$$\begin{aligned} fg &= f \olessthan g + f \circledequal g + f \ogreaterthan g \\ &= \sum_{j<k-2} \mathbf{P}_j(f)\mathbf{P}_k(g) + \sum_{|j-k|\le 2} \mathbf{P}_j(f)\mathbf{P}_k(g) + \sum_{k<j-2} \mathbf{P}_j(f)\mathbf{P}_k(g) \end{aligned}$$

- $f \olessthan g$ = paraproduct of g by f. The function f is in the low-frequency and g is in the high-frequency, i.e.
  $$\text{frequency of } g \quad \gg \quad \text{frequency of } f.$$
  We notice that let $f \in C^{\alpha_1}$ and $g \in C^{\alpha_2}$, then $f \olessthan g$ is always well defined object. Moreover, the regularity $f \olessthan g \sim \min\{\alpha_2, \alpha_1 + \alpha_2\}$.
- $f \circledequal g$ = resonant product. It is well defined if $\alpha_1 + \alpha_2 > 0$. If $f \circledequal g$ is well defined, then the regularity $f \circledequal g \sim \alpha_1 + \alpha_2$.

Let us come back to the paracontrolled ansatz (1.1):

$$\begin{aligned} Y &= Y' \olessthan X + R \\ &=: Z + R. \end{aligned} \tag{1.2}$$

We have the following computation by using paraproduct decomposition.

$$\partial_t Y = Y \partial_t X = Y \olessthan \partial_t X + Y \circledequal \partial_t X + Y \ogreaterthan \partial_t X.$$

Here, $\alpha_1 = 1/2+$ and $\alpha_2 = -1/2-$. Therefore, $Y\partial_t X \sim (1/2+) + (-1/2-) < 0$, $Y \olessthan \partial_t X \sim -1/2-$, $Y \circledequal \partial_t X$ is NOT well defined, and $Y \ogreaterthan \partial_t X \sim 0-$. Next, by using (1.2), we have

$$\partial_t Z = (Z + R) \olessthan \partial_t X, \tag{1.3}$$

which implies that $Z \sim 1/2-$, as $\partial_t X \sim -1/2-$. For the remainder term,

$$\partial_t R = (Z + R) \ogreaterthan \partial_t X + Z \circledequal \partial_t X + R \circledequal \partial_t X, \tag{1.4}$$

where $(Z + R) \ogreaterthan \partial_t X \sim 0-$. By ingoring the resonaut products, we would expect $R \sim 1-$. This implies that $R \circledequal \partial_t X \sim (1-) + (-1/2-) > 0$ is well defined. Moreover, we notice that from (1.3) and (1.4) that

$$\partial_t Y = (Z + R) \olessthan \partial_t X + \partial_t R.$$

Now, we see from (1.4) that the issue term is $Z \circledequal \partial_t X \sim (1/2-) + (-1/2-) < 0$. We use the structure of Z,

$$Z(t) = Z_0 + \int_0^t (Z + R) \olessthan \partial_t X(t')dt',$$

where Z_0 is constant. Therefore, $Z_0 \circledequal \partial_t X$ is well defined. Next, we consider the second term, and we obtain

$$\left(\int_0^t (Z + R) \olessthan \partial_t X(t')dt' \right) \circledequal \partial_t X = \left[(Z + R) \olessthan \delta X_{t_0} \right] \circledequal \partial_t X + \mathrm{Com}_1 \circledequal \partial_t X, \tag{1.5}$$

where $\partial_t X + \mathrm{Com}_1 \circledequal \partial_t X \sim (1/2 + 2\varepsilon) + (-1/2 - \varepsilon) > 0$, and which is a good and well defined. We continuite from RHS of (1.5) such that

$$\mathrm{RHS} = (Z + R) \olessthan (\delta X_{t_0} \circledequal \partial_t X) + \mathrm{Com}_1 \circledequal \partial_t X + \mathrm{Com}_2.$$

Here, we view $\delta X_{t_0} \circledequal \partial_t X$ as a part of given data, and $\mathrm{Com}_2 = [\olessthan, \circledequal](Z + R, \delta X_{t_0}, \partial_t X)$.

Now, all the terms of (1.3) and (1.4) make sense. Therefore, we can solve the system for Z and R by a standard contraction argument[1]. Finally, we summarise (i) paracontrolled and (ii) controlled path.

- paracontrolled:

$$\begin{aligned} X \xrightarrow{\star} (\partial_t X, \delta X_{t_0} \circledequal \partial_t X) X &\xrightarrow{\star\star} (Z, R) \\ &\to Y = Z + R \end{aligned}$$

- controlled path:

$$\begin{aligned} X \xrightarrow{\star} (X, \mathbb{X}) &\xrightarrow{\star\star} (Y', R)\big(\text{ or } (Y, Y') \big) \\ &\to Y \end{aligned}$$

[1]Strictly speaking, we need to multiply by a smooth cut-off function (in time) to prove LWP.

- Here, $\star$ is denoted as the process such that stochastic analysis to lift X to an enhanced data set/rough path.
- The second arrow, $\star\star$ is that deterministic analysis, and we notice there is NOT probability.

1.1. **Pathwise local well-posedness of SNLS with multiplicative noise on $\mathbb{T}^d$.** We study the Cauchy problem of the following SNLS with multiplicative noise on $\mathbb{T}^d$:

$$\begin{cases} i\partial_t u = \Delta u + \mathcal{N}(u) + u\Phi\xi \quad \text{ on } \mathbb{T}^d \\ u|_{t=0} = u_0, \end{cases} \tag{1.6}$$

where u is linear in u, but it is nonlinear in noise, $\mathcal{N}(u) = |u|^{p-1}u$ for $p \in 2\mathbb{N}+1$, and ξ is the space-time white noise. The interaction representation is defined such that,

$$v(t) := S(-t)u(t) = e^{it\Delta}u(t).$$

For simplicity, let the nonlinearity $\mathcal{N}(u) = 0$, we have

$$\begin{aligned} v(t) &= u_0 - i\int_0^t S(-t')\big(S(t')v(t')\Phi d\xi(t')\big) \\ &= u_0 - i\int_0^t S(-t')\big(S(t')v(t')dX_{t'}\big) \end{aligned}$$

We set $Y_t = v(t)$ and write

$$Y_t = Y_r - i\int_r^t S_{-t'}dX_{t'}S_{t'}Y_{t'}.$$

We observe that $S_{-t'}$ and $S_{t'}$ make things harder in a sense that we need to lose spatial regularity to compute C_t^α-norms (i.e. $D_t^\alpha e^{it|n|^2} \sim |n|^{2\alpha}e^{it|n|^2}$).

We then split into two cases: the Young case and the Rough case.

Young case:

Let $X \in C_t^\alpha H^s$ with $\alpha > 1/2$ and $\Phi \in \mathrm{HS}(L^2; H^s)$. Then, we have

$$\begin{aligned} \int_r^t S_{-t'}dX_{t'}S_{t'}Y_{t'} &= \int_r^t S_{-t'}dX_{t'}S_{t'}Y_r - i\int_r^t S_{-t'}dX_{t'}S_{t'}\int_r^{t'} S_{-t''}dX_{t''}S_{t''}Y_{t''} \\ &=: \mathrm{I} + \mathrm{II} \end{aligned}$$

From the RHS we have,

$$\delta\mathrm{I} = -\delta\bigg(\int_r^t S_{-t'}dX_{t'}S_{t'}Y_r\bigg) = X^1_{t_1t_2}\delta Y_{t_2t_3} = X^1\delta Y,$$

where we defined the operator $X^1_{tr} := \int_r^t S_{-t'}dX_{t'}S_{t'}$.

We now make the following claim:

Claim 1.1. *If X^1 maps H^s to H^s with $\|X^1_{tr}\|_{\mathcal{L}(H^s;H^s)} \lesssim |t-r|^\alpha$ (this is denote the class by $C_2^\alpha\mathcal{L}(H^s;H^s)$), then we have*

$$\delta I = X^1\delta Y \in C_3^{2\alpha} \quad \Longleftarrow \quad |t_1-t_2|^\alpha|t_2-t_3|^\alpha,$$

where $2\alpha > 1$.

Next, by the sewing lemma, we obtain

$$\int_r^t S_{-t'} dX_{t'} S_{-t'} Y_{t'} = (\mathrm{Id} - \Lambda\delta) X^1 Y,$$

where X^1 is the operator.

Now, the question left is: **How to verify claim 1.1?**

First of all, by taking the Fourier transform we obtain,

$$\begin{aligned}\widehat{X^1_{tr} f}(n) &= \sum_{n=n_1+n_2} \int_r^t e^{-it'(|n_1|^2-|n_2|^2)} \phi(n_1) d\beta^H_{n_1}(t') \widehat{f}(n_2) \\ &= \sum_{n=n_1+n_2} \int_r^t e^{-it'(|n_1|^2-|n_2|^2)} dX \widehat{f}(n_2),\end{aligned}$$

where we denote $\phi(n_1)d\beta^H_{n_1}(t') = dX$, when X is a fractional brownian motion in time with a Hurst parameter $H > 1/2$, which is slightly smoother in time than a brownian motion. Next, let us assume $\widehat{\Phi f}(n) = \phi(n)\widehat{f}(n)$ and $\Phi \in \mathrm{HS}(L^2; H^s)$ with $s > d/2$ (i.e., algebric property hold). Then, the main tool is the random matrix/tensor extimate from Deng-Nahmod-Yue [2] and [6]. Then, we can deduce $X^1 \in C^\alpha_2 \mathcal{L}(H^s; H^s)$ with $\alpha > 1/2$ ($\alpha = H-$). Finally, we perform a contraction argument as in Lecture 13, and we can conclude the Young case.

Rough case:

In the rough case, we take $X = \Phi W$, where W is the L^2-cylindrical Wiener process, and $\Phi \in \mathrm{HS}(L^2; H^s)$ with $s > d/2$. Then, we have

$$Y_t = Y_r - i \int_r^t S_{-t'} dX_{t'} S_{t'} (D^{-\varepsilon} Y_{t'}), \tag{1.7}$$

where the operator $X^1_{tr} = \int_r^t S_{-t'} dX_{t'} S_{t'}$. From controlled rough path,

$$\delta Y_{tr} = X^1_{tr} Y'_r + R^Y_{tr} = -i X^1_{tr} D^{-\varepsilon} Y_r + R^Y_{tr}. \tag{1.8}$$

By using (1.8) into (1.7), we get

$$\begin{aligned} i \int_r^t S_{-t'} dX_{t'} S_{t'} D^{-\varepsilon} Y_{t'} &= \int_r^t S_{-t'} dX_{t'} S_{t'} D^{-\varepsilon} Y_r - i \int_r^t S_{-t_1} dX_{t_1} S_{t_1} X^1_{t'r} D^{-2\varepsilon} Y_r + X^1_{tr} R_{\cdot r} \\ &= X^1_{tr} D^{-\varepsilon} Y_r - i \mathbb{X}^2_{tr} D^{-2\varepsilon} Y_r + X^1_{tr} R_{\cdot r}, \end{aligned}$$

where we remark that $(X^1, \mathbb{X}^2)$ is an operator-valved rough path adapted to the Schrödinger flow, satisfying

$$\mathbb{X}^2_{t_1 t_3} - \mathbb{X}^2_{t_1 t_2} - \mathbb{X}^2_{t_2 t_3} = \mathbb{X}_{t_1 t_2} \circ \mathbb{X}_{t_2 t_3}. \tag{1.9}$$

We then apply the controlled rough path, we have

$$\delta X^1_{tr} R_{\cdot r} = -\delta(X^1_{tr} D^{-\varepsilon} Y) + i\delta(\mathbb{X}^2 D^{-2\varepsilon} Y).$$

By using (1.8), the first term on the RHS:

$$\begin{aligned} -\delta(X^1_{tr} D^{-\varepsilon} Y)_{t_1 t_2 t_3} &= X^1_{t_1 t_2} \delta D^{-\varepsilon} Y_{t_2 t_3} \\ &= -i X^1_{t_1 t_2} \circ X^1_{t_2 t_3} D^{-2\varepsilon} Y_{t_3} + X^1_{t_1 t_2} D^{-\varepsilon} R_{t_2 t_3}. \end{aligned} \tag{1.10}$$

By using (1.9) on the second term of RHS:

$$\begin{aligned} i\delta(\mathbb{X}^2 D^{-2\varepsilon} Y) &= i\mathbb{X}^2_{t_1t_3} D^{-2\varepsilon} Y_{t_3} - i\mathbb{X}^2_{t_1t_2} D^{-2\varepsilon} Y_{t_2} - i\mathbb{X}^2_{t_2t_3} D^{-2\varepsilon} Y_{t_3} \\ &= -i\mathbb{X}^2_{t_1t_2} \delta D^{-2\varepsilon} Y_{t_2t_3} + iX^1_{t_1t_2} \circ X^1_{t_2t_3} D^{-2\varepsilon} Y_{t_3}. \end{aligned} \tag{1.11}$$

We observe that from (1.10) and (1.11) that $X^1_{t_1t_2} \circ X^1_{t_2t_3} D^{-2\varepsilon} Y_{t_3}$ get cancellation. Moreover, $2\alpha + \alpha = 3\alpha > 1$ of $\alpha > 1/3$. Therefore, we apply the sewing lemma such that

$$\int_r^t S_{-t'} dX_{t'} S_{t'} Y_{t'} = (\mathrm{Id} - \Lambda\delta)(X^1 Y - i\mathbb{X}^2 Y)$$

For this computation, we need

$$X^1 \in C_2^\alpha \mathcal{L}(H^2; H^{s-\varepsilon}), \qquad \mathbb{X}^2 \in C_2^\alpha \mathcal{L}(H^2; H^{s-2\varepsilon}).$$

The computation fails when $\varepsilon = 0$.

We, therefore, have the following theorem.

Theorem 1.2 (Oh-Zheng '22 [7]). *Let $\varepsilon \geq 0$, $\Phi \in (L^2; H^s)$ for $s > d/2$. Then, the SNLS* (1.6) *is* LWP *in $H^s(\mathbb{T}^d)$. In particular,* (i) *When W is Brownian in time, the LWP holds for $D^{-\varepsilon} u dW$ instead of udW;* (ii) *when W is fractional BM in time with Hurst index $H > 1/2$. Then, the LWP holds without $D^{-\varepsilon}$.*

Remark 1.3. *When $\varepsilon = 0$, this argument fails, and we need another idea. For the heat case, see Gubinelli-Tindel* [4]*, and also Hairer-Pardoux* [5] *for the regularity structure approach.*

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