Lec 9 22/03/22 (Tue)

· **Back to SNLS with multiplicative noise** ($s \le d/2$)

$$\text{(SNLS)} \begin{cases} i\partial_t u - \Delta u = \mathcal{N}(u) + \sigma(u)\,\Phi\,\xi \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d) \end{cases}$$

↳ we use the Strichartz estimates.

On $\mathbb{R}^d$

· **Stochastic Strichartz estimates** (Brzeźniak - Millet '14, Hornung '18)

$$I_{[0,\tau)}F(t) = \int_0^t \mathbb{1}_{[0,\tau)}(t')\, S(t-t')\, F(t')\, dW(t')$$

$F : \mathbb{R}_+ \to HS(L^2; H^s)$

τ, accessible stopping time.

(q, r), admissible : $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $(q, r, d) \neq (2, \infty, 2)$

Then, we have (except for $r=\infty \Rightarrow d=1$.) ②

$$\Big\| \, \| I_{[0,\tau)} F \|_{L^q_T W^{s,r}_x(\mathbb{R}^d)} \Big\|_{L^p(\Omega)}$$

for finite times

$$\leq C(p,q,r,T) \, \Big\| \Big(\int_0^\tau \| F(t) \|^2_{HS(L^2; H^s)} \, dt \Big)^{1/2} \Big\|_{L^p(\Omega)}$$

Rmk: Usual deterministic Strichartz estimate:

$$\Big\| \int_0^t S(t-t') N(t') \, dt' \Big\|_{L^q_t W^{s,r}_x(\mathbb{R} \times \mathbb{R}^d)}$$

$$\lesssim \| N \|_{L^{\tilde{q}'}_t W^{s,\tilde{r}'}_x(\mathbb{R} \times \mathbb{R}^d)}$$

(q,r), $(\tilde{q},\tilde{r})$ admis.

Pf: Let $0 \le t \le T$.

$$I_{[0,\tau)} F(t) = \int_0^T \mathbb{1}_{[0,\tau)}(t') \, G_{t'}(t) \, dW(t')$$

$$= \left(\int_0^T \mathbb{1}_{[0,\tau)}(t') \, G_{t'} \, dW(t') \right)(t),$$

where $G_{t'} : [0,T] \ni t \longmapsto \mathbb{1}_{[t',T]}(t) \, S(t-t') F(t')$

$t' \in [0,T]$

$$\Rightarrow \| I_{[0,\tau)} F \|_{L^q_T W^{s,r}_x}$$

$$= \left\| \int_0^T \mathbb{1}_{[0,\tau)}(t') \, G_{t'} \, dW(t') \right\|_{L^q_t([0,T]; W^{s,r}_x)}$$

Let $q < \infty$, $r < \infty$

When $q = \infty$, we have $r = 2$.
When $r = \infty$, we have $d = 1$.

$L^q_T W^{s,r}_x$, $2 \le q, r < \infty$
is of M-type 2.

$$\Big\| \, \| I_{[0,\tau)} F \|_{L^q_T W^{s,r}_x} \Big\|_{L^p(\Omega)}$$

BDG ineq

$$\lesssim \Big\| \Big(\int_0^T \mathbb{1}_{[0,\tau)}(t') \boxed{\| G_{t'} \|^2_{\gamma(L^2; \, L^q_T W^{s,r}_x)}} \, dt' \Big)^{1/2} \Big\|_{L^p(\Omega)}$$

$$\underset{t}{\overset{T}{\psi}}$$

$$\lesssim \| F(t') \|^2_{HS(L^2; H^s)}$$

$$\| G_{t'} \|^q_{\gamma(L^2; \, \underbrace{L^q_T W^{s,r}_x}_{//B})}$$

$$\sim \mathbb{E} \int_0^T \| \sum_n g_n G_{t'}(e_n)(t) \|^q_{W^{s,r}_x} \, dt \qquad \{e_n\} = \text{O.N.B of } L^2(\mathbb{R}^d)$$

$$= \mathbb{E} \int_{t'}^T \| \underbrace{\sum_n g_n \big(S(t-t') \circ F(t') \big)(e_n)}_{= S(t-t') \left(\sum_n g_n F(t')(e_n) \right)} \|^q_{W^{s,r}_x} \, dt$$

(5)

$$= \mathbb{E}\left[\int_{t'}^{T} \| S(t) \circ S(t-t')\Big(\sum_n g_n F(t')(e_n)\Big) \|_{W_x^{s,r}}^{q} \, dt. \right]$$

$$\leq \mathbb{E}\left[\| S(t)\, (\cdots) \|_{L_t^q(\mathbb{R}; W_x^{s,r})}^{q} \right]$$

Strichartz

$$\lesssim \mathbb{E}\left[\| \sum_n g_n F(t')(e_n) \|_{H^s}^{q} \right]$$

$$\sim \| \sum_n g_n F(t')(e_n) \|_{L^2(\Omega; H^s)}^{q}$$

$$= \| F(t') \|_{\gamma(L^2; H^s)}^{q} = \| F(t') \|_{HS(L^2; H^s)}^{2}$$

Next, we consider the case $q = \infty \Rightarrow r = 2$

$$\Big\| \, \| \mathcal{I}_{[0,\tau)} F \|_{L^\infty_T H^s_x} \Big\|_{L^p(\Omega)}$$

$$\mathcal{I}_{[0,\tau)} F(t) = \int_0^t \mathbb{1}_{[0,\tau)}(t') \underbrace{S(t-t')}_{= S(t)\circ S(-t')} F(t')\,dt'$$

Unitarity of $S(t)$ on H^s_x

$$= \Big\| \sup_{0 \le t \le T} \Big\| \int_0^t \mathbb{1}_{[0,\tau)}(t') S(-t') F(t')\, dW(t') \Big\|_{H^s} \Big\|_{L^p(\Omega)}$$

M-type 2.

BDG

$$\lesssim \Big\| \Big(\int_0^\tau \| \cancel{S(-t')} F(t') \|^2_{HS(L^2; H^s)} \, dt' \Big)^{1/2} \Big\|_{L^p(\Omega)}$$

$$\| \cancel{S(-t')} F(t')(e_n) \|_{\ell^2_n H^s_x}$$

$\square$

Rmk: · de Bouard – Debussche '99, '03

$\Leftarrow$ They used the BDG ineq & the dispersive estimate:

$$\| S(t) f \|_{L^r_x} \lesssim \frac{1}{|t|^{\frac{d}{2}-\frac{d}{r}}} \| f \|_{L^2_x}.$$

· Stochastic convolution:

$$\Psi[u](t) = \int_0^t S(t-t') \underbrace{\sigma(u)(t')\,\Phi}\, dW(t')$$

$\sigma(u) \sim u^\gamma$, $\gamma \geq 1$.

F(t') in the stoch Stri.

Assume

$\Phi \in \gamma(L^2; L^\infty)$.

$\Rightarrow$ In view of the stoch. Strichartz estimate, we need to bound

$$\| \sigma(u)\,\Phi \|_{L^2_T HS(L^2; L^2)}$$

$$\| \sigma(u) \Phi \|_{L^2_T HS(L^2; L^2)}$$

$$= \Big\| \Big(\sum_n \underbrace{\| \sigma(u)(t) \Phi(e_n) \|^2_{L^2_x}}_{\overset{\text{Hölder}}{\leq} \| \sigma(u)(t) \|^2_{L^2_x} \| \Phi(e_n) \|^2_{L^\infty_x}} \Big)^{1/2} \Big\|_{L^2_T}$$

$$\leq \| \sigma(u) \|_{L^2_T L^2_x} \underbrace{\| \Phi \|_{\gamma(L^2; L^\infty)}}_{< \infty.}$$

$$\| \sigma(u) \|_{L^2_T L^2_x} = \| u^\gamma \|_{L^2_T L^2_x} = \| u \|^\gamma_{L^{2\gamma}_T L^{2\gamma}_x} \overset{\text{WANT}}{\lesssim} C(T) \| u \|^\gamma_{L^q_T L^r_x}, \quad \text{some admis } (q, r)$$

$$q \geq 2\gamma, \quad r = 2\gamma$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad q \geq 2\gamma, \quad r = 2\gamma$$

$$\frac{1}{2\gamma}(d+2) \; \mathbin{/\!/}$$

$$\Rightarrow \boxed{\gamma \leq \frac{d+2}{d} = 1 + \frac{2}{d}}$$

Rmk: As for the nonlinear part $\mathcal{N}(u) = |u|^{k-1}u$, we need the L^2-(sub)criticality: $s_{crit} \geq 0$ in order to study SNLS with $u_0 \in L^2_x$

$$\Downarrow$$

$$k \leq 1 + \frac{4}{d}$$

· By the stoch. Strichartz estimate and the computation on pp 8–9, 10

$$\Big\| \, \| \Psi[u](t) \|_{L_T^{\tilde q} L_x^{\tilde r}} \Big\|_{L^p(\Omega)}$$

$(\tilde q, \tilde r)$, any admin pair
$\tilde r < \infty$

$$\lesssim \Big\| \, \| \sigma(u) \Phi \|_{L_T^2 HS(L^2; L^2)} \Big\|_{L^p(\Omega)}$$

$$\lesssim \| \Phi \|_{\gamma(L^2; L^\infty)} \Big\| \, \| u \|_{L_T^q L_x^r} \Big\|_{L^p(\Omega)}$$

$r = 2\gamma$, $1 \le \gamma \le 1 + \frac{2}{d}$

(q, r), admissible

$\Rightarrow$ By using the truncation method, we can prove LWP of SNLS in $L^2(\mathbb{R}^d)$

when $1 \leq \gamma \leq 1 + \frac{2}{d}$ and $1 < k \leq 1 + \frac{4}{d}$

<u>Note</u>: Unlike the "algebra" case ($s > \frac{d}{2}$), we need the following cutoff:

$$\eta_R(u)(t) = \eta\left(\frac{\|u\|_{C([0,t];L^2_x)} + \|u\|_{L^q([0,t];L^r_x)}}{R}\right)$$

$\Leftarrow$ compare this with the cutoff on p.5 of Lec 6
p.1 of Lec 7.

<u>ex</u>: $d=1$, $k=3$, $\gamma=2$. (1-d cubic SNLS)

$\Rightarrow$ $(q, r) = (8, \underset{\uparrow\, 2\gamma}{4})$ $\Leftarrow$ same pair as in the additive case in Lec 4.

Rmk: When $\gamma < 1 + \frac{2}{d}$, we can relax the condition $\Phi \in \gamma(L^2; L^\infty)$. On p. 8, suppose we put $\sigma(u)$ in the $L^2_T L^\alpha_x$-norm. $(\alpha > 2)$

$$\| \sigma(u) \|_{L^2_T L^\alpha_x} = \| u \|^{\gamma}_{L^{2\gamma}_T L^{\alpha\gamma}_x} \qquad \begin{matrix} r = \alpha\gamma \\ q \geq 2\gamma \end{matrix}$$

Then, in the 2nd step on p. 8, we put Φ in $\gamma(L^2; L^{\frac{2\alpha}{\alpha-2}})$ $\quad (<\infty)$
(instead of $\gamma(L^2; L^\infty)$.)

$$\frac{1}{2} = \frac{1}{\alpha} + \boxed{\frac{\alpha - 2}{2\alpha}} = 1/\beta$$

$$\begin{cases} r = \alpha\gamma \\ q \geq 2\gamma \\ (q, r), \text{ admis} \end{cases} \Rightarrow \frac{d}{2} = \frac{2}{q} + \frac{d}{r} \leq \frac{1}{\gamma} + \frac{d}{r} \Rightarrow \frac{1}{\alpha} = \frac{\gamma}{r} \geq \frac{\gamma}{2} - \frac{1}{d}$$

$$\Rightarrow \frac{1}{\beta} = \frac{1}{2} - \frac{1}{\alpha} \leq \frac{1}{2} - \frac{\gamma}{2} + \frac{1}{d} = \frac{1}{2}\left(1 + \frac{2}{d} - \gamma\right)$$

Enough to assume $\Phi \in \gamma(L^2; L^\beta)$, $\beta \geq \dfrac{2}{1 + \frac{2}{d} - \gamma} \vee 2$.

· **Blowup alternative**: If the maximal time of existence $T_{max} = T_{max}(\omega) < \infty$, then

$$(\star) \quad \lim_{t \nearrow T_{max}} \|u\|_{C([0,t]; H^s_x)} + \|u\|_{L^q([0,t], L^r_x)} = \infty$$

· In some cases, we can reduce this to

$$(\star\star) \quad \lim_{t \nearrow T_{max}} \|u\|_{C([0,t]; H^s_x)} = \infty.$$

NOT automatic but one needs to prove this!!

ex: $d=1$, $k=3$, $\gamma=1$

We construct soln in $C_T L^2_x \cap L^8_T L^4_x$ admis.

$$\cdot \left\| \int_0^t S(t-t') |u|^2 u(t') dt' \right\|_{L^8_T L^4_x} \overset{\text{Str.}}{\lesssim} \| |u|^2 u \|_{L^{\frac{4}{3}}_T L^1_x}$$

$(4, \infty)$ is admis

$$\leq T^{1/2} \|u\|_{L^\infty_T L^2_x} \|u\|^2_{L^8_T L^4_x}$$

$\Rightarrow$ Using the Duhamel formulation, we have

$$\|u\|_{L^8_T L^4_x} \leq C\|u_0\|_{L^2} + C T^{1/2} \|u\|_{L^\infty_T L^2_x} \|u\|^2_{L^8_T L^4_x} + \|\Psi[u]\|_{L^8_T L^4_x}$$

Cauchy's ineq

$$\leq C_0 (1 + \|u\|_{L^\infty_T L^2_x})^2 + \|\Psi[u]\|_{L^8_T L^4_x} \qquad ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$$+ C_1 T \|u\|^4_{L^8_T L^4_x}$$

· Fix a stopping time $\tau < T_{max} = T_{max}(\omega)$. We have

$$(\ast) \qquad \underbrace{\|u\|_{L^8_I L^4_x}}_{\text{over an interval } I} \leq C_0 (1 + \|u\|_{L^\infty_\tau L^2_x})^2 + \|\Psi[u]\|_{L^8_\tau L^4_x}$$

$$+ C_1 \underbrace{|I|}_{\text{the Length}} \underline{\|u\|^4_{L^8_I L^4_x}}$$

for any interval $I \subset [0, \tau)$

By a continuity argument, by choosing $|I| \ll 1$,
we have

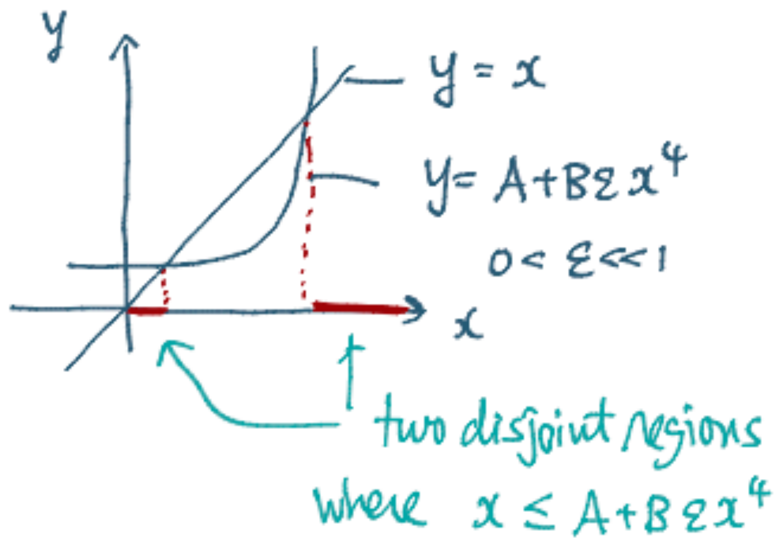
$$(**) \qquad \|u\|_{L^8_I L^4_x} \le \underbrace{2C_0\left(1+\|u\|_{L^\infty_\tau L^2_x}\right)^2}_{=:\,K(\omega)} + 2\|\Psi[u]\|_{L^8_\tau L^4_x}.$$

- continuity argument : Suppose that a conti func $X(t)$ satisfies

$$X(t) \le A + B\varepsilon X^4(t)$$
for any $t \in [t_0, t_1]$

and
$$X(t_0) \le A.$$

Then, by choosing $\varepsilon < 0$, we have
$$X(t) \le 2A \quad \text{for any } t \in [t_0, t_1]$$



OR The same conclusion follows from a bootstrap argument

$$X(t_0) \le A$$

$$X(t) \le A + B\varepsilon X^4(t), \quad \forall t \in I = [t_0, t_1]$$

WTS $\Longrightarrow$ $X(t) \le 2A$, $\forall t \in I$, by choosing $\varepsilon \ll 1$.

"Pf": $X(t_0) \le A \xrightarrow{\text{by conti of } X} X(t) \le 10A$, $\forall t \in [t_0, t_0 + \delta_1]$

$\Longrightarrow X(t) \le A + B\varepsilon (10A)^4 \le 2A$, $\forall t \in [t_0, t_0 + \delta_1]$

In particular, $X(t_0 + \delta_1) \le 2A$.

$\xrightarrow{\text{by conti}}$ $X(t) \le 10A$, $\forall t \in [t_0 + \delta_1, t_0 + \delta_1 + \delta_2]$

$\Longrightarrow$ $X(t) \le A + B\varepsilon(10A)^4 \le 2A$, $\forall t \in [t_0 + \delta_1, t_0 + \delta_1 + \delta_2]$

$\Longrightarrow$ repeat until the entire interval $I = [t_0, t_1]$ is covered.

← "looks like a bootstrap

- $(*)$ on p. 14 $\Longrightarrow$ $(**)$ on p. 15
  by choosing $|I| \sim K(\omega)^{-3}$. $(***)$

- Fix $T_* \gg 1$. Then, by writing $[0, T_* \wedge \tau) = \bigcup I_j$
  where $|I_j| = I$ (except for the last interval), we have

$$
\begin{aligned}
\|u\|_{L^8([0,T_*\wedge\tau);\,L^4_x)} &= \Big(\sum_j \|u\|^8_{L^8_{I_j} L^4_x} \Big)^{1/8} \\
&\lesssim \Big(\frac{T_*}{|I|} \Big)^{1/8} K(\omega) \\
&\overset{(***)}{\lesssim} T_*^{1/8} K^{5/8}(\omega).
\end{aligned}
$$

$(+)$

- Our goal is to show ☆☆ on p.13

Suppose <u>not</u>. Then, we have

$$P\left(\sup_{t \le T_{max}} \|u(t)\|_{L^2} < \infty \text{ and } T_{max} < \infty \right) > 0$$

By taking $T_* \gg 1$, we have

$$P\left(\sup_{t \le T_{max}} \|u(t)\|_{L^2} < \infty \text{ and } T_{max} < T_* \right) > 0.$$

- Given $R \gg 1$, set a stopping time.

$$t_R = \inf \left\{ t_0 \in [0, T_*) : \|u(t_0)\|_{L^2} \ge R \text{ or } t_0 \ge T_{max} \right\}$$

Then,

A $$P(\underbrace{t_R = T_{max}}_{=: A_R}) > 0 \text{ by choosing } R \gg 1.$$

(i) From ⊕ on p. 17 with $\tau = t_R$, we have

$$\mathbb{E}\left[\|u\|^p_{L^8_{t_R} L^4_x}\right] \lesssim T_x^{1/8}\, \mathbb{E}\left[K(\omega)^{3p/8}\right].$$

(ii) From ⊛⊛ on p. 15 and the definition of t_R, we have

$$K(\omega) \le 2C_0\,(1+R)^2 + \|\Psi[u]\|_{L^8_{t_R} L^4_x}$$

(iii) By the corollary to the stoch. Strichartz estimate on p. 10,

$$\Big\| \|\Psi[u]\|_{L^8_{t_R} L^4_x} \Big\|_{L^p(\Omega)}$$

$$\lesssim \|\Phi\|_{\gamma(L^2;L^\infty)} \Big\| \underbrace{\|u\|_{L^\infty_{t_R} L^2_x}}_{\le R} \Big\|_{L^p(\Omega)}$$

From (i), (ii), and (iii), we conclude that

$$\text{(B)} \quad \mathbb{E}\left[\|u\|^p_{L^8_{t_R} L^4_x} \right] \le C(T_*, R)$$

• Finally, from (A) on p. 18 and (B), we have

$$\mathbb{E}\left[\mathbb{1}_{A_R} \|u\|^p_{L^8_{t_R} L^4_x} \right] \le C(T_*, R).$$

Namely, on A_R, we have

$$\|u\|_{L^\infty_{T_{max}} L^2_x} + \|u\|_{L^8_{T_{max}} L^4_x} < \infty$$

BUT

$$T_{max} < \infty \implies \text{contradiction to } (\star) \text{ on p. 13}$$

Conclusion: $d=1$, $k=3$, $r=1$.

If $T_{max} < \infty$, then $\lim_{t \uparrow T_{max}} \|u(t)\|_{L^2} = \infty$.

Hence, global well-posedness follows once we prove

$$\sup_{0 \le t \le T_{max}} \|u(t)\|_{L^2} < \infty$$

- We will prove this by applying Ito's lemma to the mass $M(u) = \int |u|^2 dx$ which is conserved under the deterministic NLS (i.e. $\Phi \equiv 0$)

$$i \partial_t u - \Delta u = |u|^{k-1} u.$$

$$\Rightarrow \partial_t M(u) = 2\mathrm{Re} \int u \overline{\partial_t u}\, dx \overset{\text{IBP}}{=} -2\mathrm{Re}\, i \int |\nabla u|^2 dx + 2\mathrm{Re}\, i \int |u|^{k+1} dx = 0$$

(at least for smooth solns.

- GWP of SNLS with multiplicative noise
  - de Bouard - Debussche '99, '03 ($\gamma = 1$)
  - Hornung '18 ($\underbrace{k < 1 + \frac{4}{d}}_{\text{mass-subcritical}}$, $1 \le \gamma \le \gamma_k$)
  - mass-critical case ($\gamma = 1$)
    Fan - Xu, APDE '21. Zhang, '18 ← preprint. ⇐ Bényi-Oh-Pocovnicu '15

- Well-posedness on $\mathbb{T}^d$.
  - Need to use the Fourier restriction norm method
  - Cheung - Mosincat, SPDE '19.