Lec 8 : 15/03/22 (Tue)

Stochastic nonlinear heat eqn with multiplicative space-time white noise

$$(\mathrm{SNLH}) \quad \begin{cases} \partial_t u - \Delta u = \mathcal{N}(u) + \sigma(u)\underline{\underline{\xi}} & \text{on } \mathbb{T}^d \\ u|_{t=0} = u_0 \end{cases}$$

space-time white noise i.e. $\Phi = Id$.

· <u>Duhamel formulation</u> (= mild formulation)

$$u(t) = P(t)u_0 + \int_0^t P(t-t')\mathcal{N}(u)(t')dt' \qquad P(t) = e^{t\Delta}$$

$$\underbrace{+ \int_0^t P(t-t')\sigma(u)(t')\,dW(t')}_{=:\,\Psi[u](t)}$$

- **Schauder estimate** : $1 \le p \le q \le \infty$, $s \ge 0$

(*) $$\| P(t) f \|_{W^{s,q}} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{s}{2}} \| f \|_{L^p}$$

for any $t > 0$ on $\mathbb{R}^d$

$0 < t \le 1$ on $\mathbb{T}^d$

(For $\tilde{P}(t) = e^{t(\Delta - 1)}$, then

(*) holds for any $t > 0$ even on $\mathbb{T}^d$.

- **Besov spaces & Hölder-Besov spaces**

Hölder norm : $\| u \|_{C^s} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^s}$, $0 < s < 1$

↑ a semi-norm

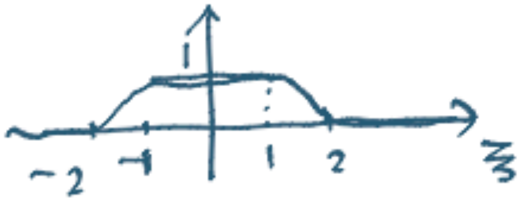
· Grafakos
· Bahouri-Chemin-Danchin ③

· **Littlewood-Paley freq projector:**

· P_j = "smooth" freq projector onto freq. $\{|n| \sim 2^j\}$, $j \geq 1$
$\{|n| \lesssim 1\}$, $j = 0$

· $\varphi \in C_c^\infty(\mathbb{R}; [0,1])$

$$\varphi(\xi) \equiv 1 \text{ for } |\xi| \leq 1$$
$$\varphi(\xi) \equiv 0 \text{ for } |\xi| \geq 2$$



Now, set $\varphi_j(\xi) = \begin{cases} \varphi(|\xi|), & j = 0 \\ \underbrace{\varphi\left(\frac{|\xi|}{2^j}\right) - \varphi\left(\frac{|\xi|}{2^{j-1}}\right)}_{\substack{\equiv 1 \text{ for } |\xi| \sim 2^j \\ \equiv 0 \text{ for } |\xi| \ll 2^j \\ \text{or } |\xi| \gg 2^j}}, & j \geq 1. \end{cases}$

$$\Rightarrow \widehat{P_j u}(n) = \varphi_j(n)\, \hat{u}(n)$$

- **Littlewood-Paley thm:** $1 < p < \infty$.

$$\underbrace{\Big\| \Big(\sum_{j=0}^{\infty} |P_j(u)|^2 \Big)^{1/2}}_{\text{square function}} \Big\|_{L^p} \sim \|f\|_{L^p}$$

- LP characterization of H^s:

$$\|u\|_{H^s} \sim \Big\| 2^{js} \|P_j(u)\|_{L^2_x} \Big\|_{\ell^2_{j\geq 0}}, \quad s \in \mathbb{R}$$

- **Besov space $B^s_{p,q}$** (or $B^{s,p}_q$)

$$\|u\|_{B^s_{p,q}} = \Big\| 2^{js} \|P_j(u)\|_{L^p_x} \Big\|_{\ell^q_{j\geq 0}}$$

$$1 \leq p, q \leq \infty, \qquad s \in \mathbb{R}$$

On $\mathbb{R}^d$, by setting $\psi_j(\xi) = \varphi\left(\frac{|\xi|}{2^j}\right) - \varphi\left(\frac{|\xi|}{2^{j-1}}\right)$, $j \in \mathbb{Z}$ ⑤

and $\widehat{Q_j u}(\xi) = \psi_j(\xi)\hat{u}(\xi)$, we can define

the homogeneous Besov space $\dot{B}^s_{p,q}$ by

$$\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\| 2^{js} \|Q_j(u)\|_{L^p_x} \right\|_{l^q_{j\in\mathbb{Z}}}$$

Then, we have

$$\|u\|_{C^s} = \|u\|_{\dot{B}^s_{\infty,\infty}}, \quad 0 < s < 1$$

· If we set $\Lambda^s = C^s \cap L^\infty$, then we have

on $\mathbb{T}^d$, $\mathbb{R}^d$.

$$\|u\|_{\Lambda^s} = \|u\|_{B^s_{\infty,\infty}}, \quad 0 < s < 1$$

Lipschitz space

$(C^s = \dot{\Lambda}^s)$

make sense for any $s \in \mathbb{R}$.

- Hölder – Besov space: $\mathcal{C}^s = B^s_{\infty,\infty}$, $s \in \mathbb{R}$

  - $\|u\|_{\mathcal{C}^s} = \|u\|_{B^s_{\infty,\infty}} = \sup_{j\geq 0} 2^{js} \|P_j(u)\|_{L^\infty_x}$.

  - $\mathcal{C}^s \supset W^{s,\infty}$

  - $\underline{s>0}$: $\mathcal{C}^s$ is an algebra.

  - Schauder: $\|P(t)f\|_{\mathcal{C}^{s_2}} \lesssim t^{-\frac{s_2-s_1}{2}} \|f\|_{\mathcal{C}^{s_1}}$

    $s_2 \geq s_1$

- We want to study the stochastic convolution

$$\Psi[u](t) = \int_0^t P(t-t')\,\sigma(u)(t')\,dW(t')$$

via BDG ineq. (in $C_T \mathcal{C}^s_x$).

· Ideal property of γ-radonifying operator:

$$B_1 \xrightarrow{S} B_2 \xrightarrow{\Phi} B_3 \xrightarrow{T} B_4$$

$$\Phi \in \gamma(B_2, B_3)$$

$$S \in \underset{\text{lin, bdd}}{\underset{\uparrow}{L}}(B_1, B_2), \quad T \in L(B_3, B_4)$$

Then,

$$T \circ \Phi \circ S \in \gamma(B_1, B_4) \quad \text{with}$$

$$\| T \circ \Phi \circ S \|_{\gamma(B_1, B_4)} \lesssim \| T \|_{L(B_3, B_4)} \| \Phi \|_{\gamma(B_2, B_3)} \| S \|_{L(B_1, B_2)}$$

$$\cdot \; \mathcal{C}^s(\mathbb{T}^d) \underset{\text{Young}}{\supset} W^{s,\infty}(\mathbb{T}^d) \underset{\text{Sobolev}}{\supset} W^{s+\varepsilon, r}(\mathbb{T}^d), \quad \varepsilon r > d.$$

$r \gg 1$ s.t. $\varepsilon = 0+$.

Fix $t \geq 0$. Then, by BDG ineq,

$$\mathbb{E}\left[\|\Psi[u](t)\|^p_{\mathcal{C}^s_x}\right] \leq \mathbb{E}\left[\sup_{\underline{0\leq t_0\leq t}}\left\|\int_0^{t_0} P(t-t')\sigma(u)(t')dW(t')\right\|^p_{W_x^{s+\varepsilon,r}}\right]$$

$$\overset{BDG}{\lesssim} \mathbb{E}\left[\underbrace{\left(\int_0^t \|P(t-t')\circ M_{\sigma(u)(t')}\|^2_{\gamma(L^2;W^{s+\varepsilon,r})}dt'\right)}_{\text{When is this finite?}}{}^{p/2}\right]$$

Note: $\mathrm{Id} \in \gamma(L^2; W^{\alpha,r})$ iff $\underline{\alpha < -d/2}$, $1 \leq r \leq \infty$

① $\underline{\sigma(u) \equiv 1}$ (i.e. additive case)

$$\|P(t-t')\circ \mathrm{Id}\|_{\gamma(L^2;W^{s+\varepsilon,r})}$$

$$\lesssim \|P(t-t')\|_{L(W^{-\frac{d}{2}-\varepsilon,r},\,W^{s+\varepsilon,r})}\underbrace{\|\mathrm{Id}\|_{\gamma(L^2;W^{-\frac{d}{2}-\varepsilon,r})}}_{\lesssim 1}$$

$$\overset{\text{Schauder}}{\lesssim} (t-t')^{-(s+\frac{d}{2}+2\varepsilon)/2}$$

$$\Rightarrow \quad \int_0^t \| P(t-t') \circ I d \|^2_{\gamma(L^2;\, W^{s+\varepsilon, r})} \, dt'$$

$$\lesssim \int_0^t (t-t')^{-(s+\frac{d}{2}+2\varepsilon)} \, dt'$$

$$< \infty \iff s + \frac{d}{2} + 2\varepsilon < 1$$

$$\iff \underline{s < 1 - \frac{d}{2} - 2\varepsilon}$$

On the other hand, we need $s > 0$ (s.t. C^s is an algebra) to handle the nonlinearity $N(u) = u^k$

$\Rightarrow$ Hence, $\quad 0 < s < 1 - \frac{d}{2} - 2\varepsilon$

$$\Rightarrow \boxed{d = 1}$$

<u>Rmk:</u> • In the additive case ($\sigma(u) \equiv 1$), we do <u>not</u> need to work in $L^2(\Omega; C_T \mathcal{C}^s_x)$ with the BDG ineq and the truncation method. Instead, we can directly prove <u>pathwise</u> local well-posedness

• We imposed the condition $s>0$ s.t. u is a function (in x). In the additive case, the solution theory can be built for higher dimensions ($d=2,3$) even when $u(t)$ is only a distribution (in x). In this case, we need to introduce a <u>renormalization</u> to give a proper meaning to the nonlinearity $N(u)$. See my course note from Spring 2021.

② general case $\sigma(u) = u^\gamma$

$\boxed{\text{Correction}}$ In this case, we can NOT close the argument in $C_T C_x^s$, simply using the BDG ineq. We instead work in $C_T W_x^{s,r}$ for some $s \geq 0$, $2 \leq r < \infty$.

$\Longrightarrow$ By BDG ineq, we need to study

$$\| P(t-t') \circ M_{\sigma(u)(t')} \|_{\gamma(L^2; W^{s,r})}$$

$$\sim \Big(\mathbb{E}' \Big[\big\| \sum_k g_k \langle \nabla \rangle^s P(t-t') \circ M_{\sigma(u)(t')} (e_k) \big\|_{L^r_x}^2 \Big] \Big)^{1/2}$$

$\uparrow$ for $\{g_k\}$ $\qquad$ $\uparrow$ indep of u

$e_k = e^{2\pi i k x}$

$$\sim \big\| \| \cdots \|_{L^r_x} \big\|_{L^r(\Omega')} = \big\| \| \cdots \|_{L^r(\Omega')} \big\|_{L^r_x}$$

u on Ω $\leftarrow$ indep

$\{g_k\}$ on Ω' $\swarrow$

$$\sim \big\| \| \cdots \|_{L^2(\Omega')} \big\|_{L^r_x}$$

$$\sim \Big\| \Big(\sum_k \big| \langle \nabla \rangle^s P(t-t') \circ M_{\sigma(u)(t)} (e_k) \big|^2 \Big)^{1/2} \Big\|_{L^r_x}$$

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(In general, $\|\Phi\|_{\gamma(L^2;L^r)} \sim \left\| \left(\sum_k |\Phi(e_k)|^2 \right)^{1/2} \right\|_{L^r_x}$.

$$\left\| \left(\sum_k \Big| \underbrace{\langle\nabla\rangle^s P(t-t') \circ M_{\sigma(u)(t')}(e_k)}_{= \sum_{n\in\mathbb{Z}^d} e^{-(t-t')|n|^2} \langle n\rangle^s \widehat{\sigma(u)(t')}(n-k)\, e_n(x)} \Big|^2 \right)^{1/2} \right\|_{L^r_x}$$

$$\underset{r\geq 2}{\overset{\text{Minkowski}}{\leq}} \Big\| \, \| \cdots \|_{L^r_x} \Big\|_{\ell^2_k}$$

$$\overset{\text{Hausdorff-Young}}{\leq} \Big\| \, \| e^{-(t-t')|n|^2} \langle n\rangle^s \widehat{\sigma(u)(t')}(n-k) \|_{\ell^{r'}_n} \Big\|_{\ell^2_k} \qquad \underline{r' \leq 2}$$

$$\overset{\text{Minkowski}}{\leq} \Big\| e^{-(t-t')|n|^2} \langle n\rangle^s \underbrace{\| \widehat{\sigma(u)(t')}(n-k) \|_{\ell^2_k}}_{= \|\sigma(u)(t')\|_{L^2_x}} \Big\|_{\ell^{r'}_n}$$

Now, we estimate

$$\| e^{-(t-t')|n|^2} \langle n \rangle^s \|_{\ell^{r'}_n} = \left(\sum_{n\in\mathbb{Z}^d} e^{-r'(t-t')|n|^2} \langle n \rangle^{sr'} \right)^{1/r'}$$

$$\lesssim \left(\sum_{n\in\mathbb{Z}^d} \frac{1}{(t-t')^\alpha \langle n \rangle^{2\alpha - sr'}} \right)^{1/r'}$$

$$e^{-r'(t-t')|n|^2} \lesssim \min\left(\frac{1}{(t-t')|n|^2}, 1 \right)^\alpha$$

for any $\alpha \geq 0$

$$\lesssim \frac{1}{(t-t')^{\alpha/r'}}, \text{ provided that } 2\alpha - sr' > d$$

$$\iff \boxed{s < \frac{2}{r'}\alpha - \frac{d}{r'}}$$

$\Rightarrow$ Under $s < \frac{2}{r'}\alpha - \frac{d}{r'}$ we then have

$$\int_0^t \| P(t-t') \circ M_{\sigma(u)(t')} \|^2_{\gamma(L^2; W^{s,r})} dt'$$

$$\lesssim \underbrace{\int_0^t \frac{1}{(t-t')^{2\alpha/r'}} dt'}_{\lesssim 1 \text{ iff } 2\alpha < r'} \times \| \sigma(u) \|^2_{C_T L^2_x}, \quad 0 \le t \le T$$

$$\lesssim \| \sigma(u) \|^2_{C_T L^2_x}$$

$$\begin{cases} s < \frac{2}{r'}\alpha - \frac{d}{r'} \\ 2\alpha < r' \end{cases} \Rightarrow \underbrace{0 \le s}_{} < 1 - \frac{d}{r'}, \quad 1 < r' \le 2$$

$$\Rightarrow \boxed{d = 1}$$

$\Updownarrow$

$0 \le s < \frac{1}{r}$ when $d = 1$.

With $\sigma(u) = u^\gamma$, we have

$$\|\sigma(u)\|_{C_T L^2_x} = \|u\|^\gamma_{C_T L^{2\gamma}_x}$$

$$\lesssim \|u\|^\gamma_{C_T W^{s,r}_x},$$

- $s \geq 0$ if $r \geq 2\gamma$
- Otherwise, by Sobolev $(d=1)$, $s \geq \frac{1}{r} - \frac{1}{2\gamma} \left(< \frac{1}{r} \right)$

- Note: The BDG ineq with the computations on pp. 11–15 shows

$$\mathbb{E}\left[\|\Psi[u](t)\|^p_{W^{s,r}} \right] \lesssim \mathbb{E}\left[\|u\|^{\gamma p}_{C_T W^{s,r}_x} \right]$$

for any fixed $0 \leq t \leq T$. BUT, we need to insert $\sup_{0<t<T}$ under the expectation on LHS. See pp. 18–23 below.

· As for the nonlinear part, we have

$$\left\| \int_0^t P(t-t')N(u)(t')\,dt' \right\|_{C_T W_x^{s,r}} \le \left\| \int_0^t \|P(t-t')N(u)(t')\|_{W^{s,r}} dt' \right\|_{C_T}$$

Schauder

$$\lesssim \left\| \int_0^t (t-t')^{\boxed{-\frac{s}{2}-\frac{1}{2}(1-\frac{1}{r})}} \underbrace{\|N(u)(t')\|_{L^1_x}}_{} dt' \right\|_{C_T}$$

Note that

$\frac{s}{2} + \frac{1}{2}(1-\frac{1}{r}) < 1$

since $0 < s < \frac{1}{r} < 1$.

$N(u) = u^k$

$$\le \|u\|^k_{C_T L^k_x}$$

$$\lesssim \|u\|^k_{C_T W_x^{s,r}} \quad \text{for} \begin{cases} s \ge 0 & \text{if } r \ge k \\ s \ge \frac{1}{r} - \frac{1}{k} \ (< \frac{1}{r}) \end{cases}$$

$$\lesssim T^\theta \|u\|^k_{C_T W_x^{s,r}}.$$

With a truncation, we can perform a contraction argument in $L^p_{ad}(\Omega; C_T W^{s,r}_x)$ to construct u_R by choosing

$p \gg 1$. see p.20

① $0 \le s \le 1/r$

② $r \ge 2\gamma$ or $s \ge \frac{1}{r} - \frac{1}{2\gamma}$

③ $r \ge k$ or $s \ge \frac{1}{r} - \frac{1}{k}$

$\Rightarrow$ Then, LWP of (SNLH) in $W^{s,r}_x(\mathbb{T})$.

· Back to $\mathbb{E}\left[\sup_{0<t<T} \|\Psi[u](t)\|^p_{W^{s,r}}\right]$ set $T=1$

We repeat the argument on pp. 18-20 in Lec 3.

Let $t_{\ell,k} = \frac{\ell}{2^k}$, $\ell = 0, 1, 2, \cdots, 2^k$.

and write

$$(*) \quad \Psi[u](t) = \sum_{k=1}^{\infty} \left(\Psi[u](t_{\ell_k,k}) - \Psi[u](t_{\ell_{k-1},k-1}) \right)$$

for some $\ell_k = \ell_k(t) \in \{0, 1, \cdots, 2^k\}$

Note: In $(*)$, we used continuity (in t) of $\Psi[u]$ (which we want to show). Strictly speaking, we perform the following argument to $\pi_N u$ in place of u and take $N \to \infty$ (π_N = smooth freq. projection onto freq $\{|n| \lesssim N\}$

$$\Rightarrow \sup_{0\le t\le 1} \|\Psi[u](t)\|_{W^{s,r}_x}$$

$$\le \sum_{k=1}^{\infty} \max_{0\le l_k\le 2^k} \|\Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1})\|_{W^{s,r}_x}$$

where $|t_{l_k,k} - t_{l'_{k-1},k-1}| \le 2^{-k}$.

$$\Rightarrow \Big\| \sup_{0\le t\le 1} \|\Psi[u](t)\|_{W^{s,r}} \Big\|_{L^p(\Omega)}$$

$$\le \sum_{k=1}^{\infty} \|\Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1})\|_{L^p(\Omega;\, \ell^p_k W^{s,r}_x)}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{k/p} \underline{\max_{0\le l_k\le 2^k}} \|\Psi[u](t_{l_k,k}) - \Psi[u](t_{l'_{k-1},k-1})\|_{\underline{L^p(\Omega;\, W^{s,r}_x)}}$$

Claim:

$$\sup_{0 \le t_1 < t_2 \le T} \| \Psi[u](t_2) - \Psi[u](t_1) \|_{L^p(\Omega; W^{s,r}_x)}$$

$$\lesssim (t_2 - t_1)^{\theta} \| \sigma(u) \|_{L^p(\Omega; C_T L^2_x)}$$

for some $\theta > 0$, indep of p.

$\Rightarrow$ By assuming Claim, we have

$$\Big\| \sup_{0 \le t \le 1} \| \Psi[u](t) \|_{W^{s,r}} \Big\|_{L^p(\Omega)}$$

$$\lesssim \underbrace{\sum_{k=1}^{\infty} 2^{k/p} \, 2^{-k\theta}}_{\lesssim 1 \text{ by choosing } p \gg 1.} \| \sigma(u) \|_{L^p(\Omega; C([0,1]; L^2_x))}$$

Pf of Claim: For $0 \le t_1 < t_2 \le T$, we have

$$
\begin{aligned}
&\Psi[u](t_2) - \Psi[u](t_1) \\
&= \int_{t_1}^{t_2} P(t_2 - t')\, \sigma(u)(t')\, dW(t') \\
&\quad + \int_0^{t_1} [P(t_2 - t') - P(t_1 - t')]\, \sigma(u)(t')\, dW(t') \\
&=: \mathrm{I}(t_1, t_2) + \mathrm{II}(t_1, t_2).
\end{aligned}
$$

By BDG ineq and repeating the computations on pp. 11-14

$$
\| \mathrm{I}(t_1,t_2) \|_{L^p(\Omega; W_x^{s,r})} \lesssim \left(\int_{t_1}^{t_2} \frac{1}{(t_2 - t')^{1 - \frac{2\theta}{p}}} dt' \right)^{p/2} \| \sigma(u) \|_{L^p(\Omega; C_T W_x^{s,r})}
$$

$$
\sim (t_2 - t_1)^{\theta} \quad \text{for some small } \theta > 0.
$$

As for II, first note that

$$\left| \mathcal{F}_x\left([P(t_2-t') - P(t_1-t')] f \right)(n) \right|$$

$$(*) \quad = \left| e^{-(t_2-t_1)|n|^2} - 1 \right| \cdot e^{-(t_1-t')|n|^2} |\hat{f}(n)|$$

$$\overset{\text{MVT}}{\lesssim} (t_2-t_1)^{\theta} |n|^{2\theta} \quad e^{-(t_1-t')|n|^2} |\hat{f}(n)|$$

Using $(*)$, we repeat the computation on p. 13.

$$\left\| \left(e^{-(t_2-t')|n|^2} - e^{-(t_1-t')|n|^2}\right) \langle n \rangle^s \right\|_{\ell^r_n}$$

$$\lesssim (t_2-t_1)^{\theta} \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(t_1-t')^{\alpha}} \frac{1}{\langle n \rangle^{2\alpha - sr' - 2\theta}} \right)$$

$$\lesssim (t_2-t_1)^{\theta} \frac{1}{(t_1-t')^{\frac{\alpha}{r'}}} \quad , \text{ if } \boxed{2\alpha - sr' - 2\theta > d}$$

(A)

Also, note
$$\int_0^{t_1} \frac{1}{(t_1-t')^{\frac{2\alpha}{r'}}} \cdot dt' \lesssim 1$$

if $\boxed{\frac{2\alpha}{r'} < 1}$ B

- By choosing $\theta > 0$ suff. small, the conditions A & B are satisfied in view of the conditions on pp 13-14.

$\boxtimes$

<u>Rmk:</u> · On p. 12, we put $\sigma(u)(t')$ in L^2_x when this term has more regularity. Thus, we can improve the argument a bit but it seems that we can <u>not</u> close the argument in $C_T C^s_x$ via the BDG ineq on p. 8 (since we would get two contradictory conditions $s > \frac{1}{2} - \frac{1}{r} > \frac{1}{2} - \varepsilon$ and $s < \frac{1}{2} - \varepsilon$.)

$$\varepsilon r > 1$$

· In the following, we directly show $\Psi[u]$ in $C_T \mathcal{C}^{s_0}_x$ for some $s_0 > 0$, where $u \in L^p_{ad} C_T H^{\frac{1}{2}-}_x$, $p \gg 1$.

In view of Kolmogorov's conti. criterion, it suffices to show

$$\mathbb{E}\left[\left| \Psi[u](t_1, x_1) - \Psi[u](t_2, x_2) \right|^p \right] \lesssim \left| (t_1, x_1) - (t_2, x_2) \right|^{1+\theta} \quad \text{for some } \theta > 0$$

and $p \gg 1$

- $\Psi[u](t_1, x_1) - \Psi[u](t_2, x_2)$

$$= \left(\Psi[u](\underline{\underline{t_1}}, x_1) - \Psi[u](\underline{\underline{t_2}}, x_1) \right) \Leftarrow$$

Use the ideas from the pf of Claim on p. 20

$$+ \left(\Psi[u](t_2, \underline{\underline{x_1}}) - \Psi[u](t_2, \underline{\underline{x_2}}) \right).$$

- We focus only on

$$\mathbb{E}\left[\left| \Psi[u](t, x_1) - \Psi[u](t, x_2) \right|^p \right]$$

$$\Psi[u](t,x_1) - \Psi[u](t,x_2)$$

$$= \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \int_0^t e^{-(t-t')|n|^2} \widehat{\sigma(u)(t')}(n-k)\, d\beta_k(t') \big(e_n(x_1) - e_n(x_2)\big)$$

$$= \sum_{k\in\mathbb{Z}^d} \int_0^t \boxed{\sum_{n\in\mathbb{Z}^d} e^{-(t-t')|n|^2} \widehat{\sigma(u)(t')}(n-k) \big(e_n(x_1) - e_n(x_2)\big)}\, d\beta_k(t')$$

- By the BDG ineq for scalar martingales, we have

$$\mathbb{E}\Big[\big|\Psi[u](t,x_1) - \Psi[u](t,x_2)\big|^p\Big]$$

$$\lesssim \mathbb{E}\bigg[\Big(\sum_{k\in\mathbb{Z}^d} \int_0^t \Big| \boxed{\sum_{n\in\mathbb{Z}^d} \cdots} \Big|^2 dt'\Big)^{p/2}\bigg]$$

By MVT, we have $|e_n(x_1) - e_n(x_2)| \lesssim |n|^\delta |x_1 - x_2|^\delta$

for any $0 \le \delta \le 1$.

$$\sum_k \int_0^t \Big| \sum_n \qquad \Big|^2 dt'$$

$$\lesssim |x_1 - x_2|^{2\delta} \int_0^t \| a_n * b_n \|^2_{\ell^2_n} \, dt'$$

$$a_n = e^{-(t-t')|m|^2} |m|^\delta, \quad b_n = \widehat{\sigma(u)(t')}(n)$$

$$\lesssim \frac{1}{(t-t')^\theta \langle m \rangle^{2\theta - \delta}} \qquad \frac{1}{2} + 1 = \frac{1}{1+} + \frac{1}{2-}$$

Young

$$\lesssim \underline{|x_1 - x_2|^{2\delta}} \underbrace{\int_0^t \frac{1}{(t-t')^{2\theta}} dt'}_{\lesssim 1 \text{ for } \theta < 1/2} \underbrace{\Big\| \frac{1}{\langle m \rangle^{2\theta - \delta}} \Big\|^2_{\ell^{1+}_n}}_{\lesssim 1 \text{ by choosing } \theta = \frac{1}{2}-,\ \delta > 0 \text{ small.}} \underline{\| \widehat{\sigma(u)(t')}(n) \|^2_{\ell^{2-}_n}}$$

Hölder

$$\lesssim |x_1 - x_2|^{2\delta} \| \sigma(u) \|^2_{C_T H^\alpha_x}, \quad \alpha = 0+$$

Lastly, by the fractional Leibniz rule with $\sigma(u) = u^\gamma$

$$\|\sigma(u)\|_{C_T H^\alpha_x} \lesssim \|u\|^\gamma_{C_T W^{\alpha,2\gamma}_x}$$

$$\underset{\substack{\text{Sobolev} \\ \text{see p.15}}}{\lesssim} \|u\|^\gamma_{C_T H^s_x}, \qquad s \geq \underset{\substack{\| \\ 0+}}{\alpha} + \frac{1}{2} - \frac{1}{2\gamma} \;\left(\begin{array}{l} < 1/2 \\ \text{b/c } \alpha = 0+ \end{array} \right.$$

Hence, we proved

$$\mathbb{E}\left[\left| \Psi[u](t,x_1) - \Psi[u](t,x_2) \right|^p \right]$$

$$\lesssim |x_1 - x_2|^{\delta p} \, \mathbb{E}\left[\|u\|^{\gamma p}_{C_T H^s_x} \right]$$

$1+\theta$ by choosing $p \gg 1$.

As for the nonlinear part,

$$\left\| \int_0^t P(t-t')\,\mathcal{N}(u)(t') \right\|_{C_T \mathcal{C}_x^{s_0}}$$

$$\lesssim \left\| \int_0^t \| P(t-t')\,\mathcal{N}(u)(t') \|_{W_x^{s_0,\infty}}\, dt' \right\|_{C_T}$$

$$\underset{\text{Schauder}}{\lesssim} \underbrace{\left\| \int_0^t (t-t')^{-\frac{1}{2}-\frac{s_0}{2}}\, dt' \right\|_{C_T}}_{\lesssim T^\theta} \underbrace{\| \mathcal{N}(u) \|_{C_T L_x^1}}$$

$$= \| u \|_{C_T L_x^k}^k$$

$$\lesssim \| u \|_{C_T H^s}^k, \quad s \geq \frac{1}{2} - \frac{1}{k} \left(< \frac{1}{2} \right)$$