Lec 7 : 08/03/22 (Tue)

- Last time, we constructed global solns u_R to the truncated eqn (SNLS$_R$).

(*) Recall $\theta_R(u_R)(t) = \theta\left(\dfrac{\|u_R\|_{C([0,t];H^s_x)}}{R}\right)$

$\theta \equiv 1$ on $[0,1]$, 0 on $[2,\infty)$

- Let $t_R = \inf\{t \geq 0 : \|u_R\|_{C([0,t];H^s)} \geq R\}$.

(stopping time)

$\Rightarrow \quad u = u_R$ on $[0, t_R]$

soln to (SNLS)

- t_R is non-decreasing in R.

($\Leftarrow$ Given $R < R'$, we have $u_R = u_{R'} = u$ on $[0, t_R]$

(**) $\Rightarrow$ Set $t_* = \lim_{R \to \infty} t_R$

random

and define u on $[0, t_*)$ by setting $u = u_R$ on $[0, t_R]$

$\Rightarrow$ u is a soln to (SNLS) on $[0, t_*)$.

- Blowup alternative: If $t_* < \infty$, then by (*) and (**), we get $\lim_{t \uparrow t_*} \|u(t)\|_{H^s} = \infty$

Digression • On the algebra property of H^s & smoothness of a nonlinearity.

$s > d/2$

$N(u)$ = nonlinearity in u and $\bar{u}$.

• homogeneous of degree p.

By FTC,

$$N(u(x)) - N(u(y))$$

$$= \int_0^1 \partial_z N\big(u(y) + \theta(u(x) - u(y))\big)\,(u(x) - u(y))$$

$$+ \partial_{\bar z} N\big(u(y) + \theta(u(x) - u(y))\big)\,\overline{(u(x) - u(y))}\,d\theta$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{with } z = x + iy.$$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \quad |N(u(x)) - N(u(y))|$$

$$(\dagger) \qquad \lesssim \underbrace{\|N'(u)\|_{L^\infty_x}}_{\overset{\lesssim}{\text{assume}} \|u\|^{p-1}_{L^\infty_x}} |u(x) - u(y)|.$$

$$\Rightarrow \|N(u)\|_{\dot{H}^s} = \left(\int_M \int_M \frac{|N(u(x)) - N(u(y))|^2}{|x-y|^{d+2s}} \, dx\, dy \right)^{1/2}$$

$0 < s < 1$

$M = \mathbb{R}^d$ or $\mathbb{T}^d$

Stein   Bényi-Oh

$$\overset{\text{by } (\dagger)}{\lesssim} \|u\|^{p-1}_{L^\infty_x} \underbrace{\left(\iint \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx\, dy \right)^{1/2}}_{= \|u\|_{\dot{H}^s}}$$

$s > \frac{d}{2}$

$$\underset{\text{Sobolev}}{\lesssim} \|u\|^p_{H^s}$$

$$\|N(u)\|_{L^2} \underset{\text{assume}}{\lesssim} \|u\|_{L^{2p}}^p \underset{\text{Sobolev}}{\lesssim} \|u\|_{H^s}^p \qquad s \geq d\left(\frac{1}{2}-\frac{1}{p}\right)$$

$\Rightarrow$ Since $H^s = L^2 \cap \dot{H}^s$, we conclude

$$\underbrace{\|N(u)\|_{H^s}}_{C^1} \lesssim \|u\|_{H^s}^p .$$

$$\left.\begin{array}{l} 0<s<1 \\ s>\frac{d}{2} \end{array}\right\} \Rightarrow d=1$$

- In order to study a nonlinear PDE, say SNLS, we need to estimate the difference $N(u)-N(v)$.

Once again by F.T.C., we write

product

$$N(u)-N(v) = \int_0^1 \underbrace{\partial_z N(v+\theta(u-v))}\,\underbrace{(u-v)} + \underbrace{\partial_{\bar z} N(v+\theta(u-v))}\,\underbrace{\overline{(u-v)}}\, d\theta$$

For $s > \frac{d}{2}$,

$$\|N(u)-N(v)\|_{H^s} \underset{\text{alge.}}{\overset{\text{Mink}}{\le}} \int_0^1 \underbrace{\|N'(v+\theta(u-v))\|_{H^s}}_{④-⑤} \|u-v\|_{H^s}\, d\theta$$

$$\lesssim \|v+\theta(u-v)\|_{H^s}^{p-1}$$

$$\lesssim \|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1}$$

$0 < s < 1$ and $N \in C^2$.

· What if $s > 1$?

Write $s = \underbrace{[s]}_{\text{integer part}} + \overbrace{\{s\}}^{\text{fractional part}}$

First, compute $\partial_x^{[s]} N(u)$

· Then, repeat the computation on ④ – ⑥ to compute the $H^{\{s\}}$-norm of $\partial_x^{[s]} N(u)$. $\Leftarrow$ Need $N \in C^{[s]+1}$

· For the difference estimate, need $N \in C^{[s]+2}$.

- Stoch. nonlinear wave eqn (SNLW) with multiplicative space-time white noise

On $\mathbb{T}^d$ (on $\mathbb{R}^d$, $W(t)$ is "unbounded".)

spatial white noise

As we see later, $d=1$.

$$\text{(SNLW)} \begin{cases} \partial_t^2 u - \Delta u = \mathcal{N}(u) + \sigma(u)\xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

space time white noise

- Duhamel formulation (= mild formulation)

$$u(t) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1 + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}\mathcal{N}(u)(t')\,dt'$$

$$+ \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}\left(\sigma(u)(t')\,dW(t')\right)$$

With $S(t) = \frac{\sin(t|\nabla|)}{|\nabla|}$, $|\nabla| = \sqrt{-\Delta}$. $\widehat{S(t)f}(n) = \begin{cases} \frac{\sin(t|n|)}{|n|}\hat{f}(n), & n \neq 0 \\ t\hat{f}(0), & n = 0 \end{cases}$ (8)

$$u(t) = \partial_t S(t)u_0 + S(t)u_1 + \int_0^t S(t-t')\mathcal{N}(u)(t')\,dt' + \underbrace{\int_0^t S(t-t')\sigma(u)(t')\,dW(t')}_{=:\Psi[u]}$$

Write $\underbrace{S(t) = S_+(t) - S_-(t)}_{\text{No need to do this at the zeroth freq.}}$, where $S_\pm(t) = \frac{e^{\pm it|\nabla|}}{2i|\nabla|}$

Then, write $\Psi[u] = \Psi_+[u] - \Psi_-[u]$

$\Rightarrow$ By BDG ineq,

$$\mathbb{E}\left[\sup_{0\le t\le T}\|\Psi_\pm[u](t)\|_{H^s}^p\right] \lesssim \mathbb{E}\left[\left(\int_0^T \underbrace{\|S_\pm(-t')\sigma(u)(t')\|_{HS(L^2;H^s)}^2}_{\text{When is this finite?}}\,dt'\right)^{p/2}\right]$$

$$\| S_\pm(-t')\, \sigma(u)(t') \|_{HS(L^2;\, H^s)}$$

$$= \Big(\sum_k \| \underbrace{S_\pm \cancel{(-t')}}_{|\nabla|^{-1}} \circ \underbrace{M_{\sigma(u)(t')}}_{\text{multiplication by } \sigma(u)(t')} (e_k) \|_{H^s}^2 \Big)^{1/2} \qquad e_k = e^{2\pi i k\cdot x}$$

$$\sim \Big(\sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s-2} \underbrace{\Big| \sum_{n=n_1+n_2} \widehat{\sigma(u)}(n_1)\, \delta_{n_2 k} \Big|^2}_{|\widehat{\sigma(u)}(n-k)|^2} \Big)^{1/2}$$

Sum in k. Then in n.

$$\sim \|\sigma(u)\|_{L^2}$$

$$2s-2 < -d$$

$$s < -\frac{d}{2} + 1$$

Want $s > 0$ (or $s = \frac{1}{2} - \varepsilon$)

$$\Rightarrow \underline{d = 1}$$

$\underline{d=1}$: Take $s = \frac{1}{2} - \varepsilon$ for small $\varepsilon > 0$

- $\underset{\substack{\| \\ u^r}}{\|\sigma(u)\|}_{L^2} = \|u\|^r_{L^{2r}} \underset{\text{Sob}}{\lesssim} \|u\|^r_{H^s}$ $\boxed{s \geq \frac{1}{2} - \frac{1}{2r}}$

- $\Big\| \int_0^t S(t-t') \underset{\substack{\uparrow \\ u^k}}{\mathcal{N}(u)}(t')\, dt' \Big\|_{C_T H^s_x}$

$\lesssim T \|\mathcal{N}(u)\|_{C_T H_x^{\overbrace{s-1}^{<0}}} \overset{\text{Sobolev}}{\lesssim} T \|\mathcal{N}(u)\|_{C_T L^r_x}$ $\quad \underline{\underline{1-s \geq \frac{1}{r} - \frac{1}{2}}}$

$= T \|u\|^k_{C_T L^{kr}_x}$

$\overset{\text{Sobolev}}{\lesssim} T \|u\|^k_{C_T H^s_x}$, $\underline{\underline{s \geq \frac{1}{2} - \frac{1}{kr}}} \Rightarrow ks - \frac{k}{2} \geq -\frac{1}{r} \geq s - 1 - \frac{1}{2}$

$\Rightarrow \boxed{s \geq \frac{k-3}{2(k-1)}}$ $(< \frac{1}{2})$

- Use the truncation method and construct global solns $(u_R, \partial_t u_R) \in C(\mathbb{R}_+; \mathcal{H}^{\min(s, \frac{1}{2}-\varepsilon)}(\mathbb{T}))$ where $\mathcal{H}^s(\mathbb{T}) = H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$.

$s \geq \max\left(\frac{1}{2} - \frac{1}{2\delta}, \frac{k-3}{2(k-1)}\right)$

$\Rightarrow$ LWP of (SNLW) in $\mathcal{H}^s(\mathbb{T})$

i.e. given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T})$, $\exists!$ soln $(u, \partial_t u)$ to (SNLW) in $C([0, t_*]; \mathcal{H}^{s_0}(\mathbb{T}))$,

where $s_0 = \min(s, \frac{1}{2} - \varepsilon)$.

Set $N(u) \equiv 0$ and consider

$$\partial_t^2 u - \Delta u = \sigma(u)\xi \qquad \text{on } \mathbb{T}.$$

- GWP? : Yes, Mueller (Ann. Prob. '97)

  $\sigma(u)$, locally Lipschitz

  $|\sigma(u)| \lesssim \langle u \rangle \log(2+|u|)$.

Open question: $\sigma(u) \sim |u|^\gamma$ for $\gamma > 1$.

finite time blowup with positive probability?

- Similar question for Stoch. heat eqn:

$$\partial_t u - \Delta u = \sigma(u)\xi. \quad \text{on } \mathbb{T} \qquad \sigma(u) \sim u^\gamma.$$

  - finite time blowup for $\gamma \gg 1$: Mueller-Sowers, PTRF '93
    $\gamma > 3/2$: Mueller, Ann. Prob. '00
  - GWP for $\gamma < 3/2$: Mueller, PTRF '91