Lec 6 : 01/03/22 (Tue)

$$\Psi(t) = \Psi[u](t) = \int_0^t S(t-t')\,\sigma(u)(t')\,\Phi\, dW(t')$$

$$\sigma(u) = |u|^{\gamma-1}u, \quad \gamma \in 2\mathbb{N}+1$$

Prop: $s > d/2$. $\Phi \in HS(L^2; H^s)$

Then, for $u \in L^{2\gamma}_{ad}(\Omega; C_T H^s_x)$, then we have

$$\Psi = \Psi[u] \in C_T H^s_x, \text{ a.s.}$$

Moreover, if $u \in L^{q\gamma}(\Omega; C_T H^s_x)$ for some finite $q \geq 2$,

$$\Rightarrow \boxed{\mathbb{E}\left[\|\Psi\|^q_{C_T H^s_x}\right] \leq \underbrace{C(s,q,T)}_{\leq C(s,q)T^\theta \text{ for some } \theta>0} \mathbb{E}\left[\|u\|^{q\gamma}_{C_T H^s_x}\right] \times \|\Phi\|^q_{HS(L^2;H^s)}}$$

1 linear stoch. Schrödinger equation

$$\begin{cases} i\partial_t u = \Delta u + u \Phi \xi \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

· <u>Duhamel formula</u>:

$$u(t) = S(t) u_0 - i \overbrace{\int_0^t \underbrace{S(t-t')\big(u(t')\Phi\, dW(t')\big)}_{\big(S(t-t')\circ M_{u(t')}\circ\Phi\big)dW}}^{=\Psi[u](t)}$$

$$=: \Gamma_{u_0,\Phi}(u)(t)$$

<u>WTS</u>: $\Gamma = \Gamma_{u_0,\Phi}$ is a contraction on a ball in <u>$L^2(\Omega; C_T H^s_x)$</u>.

$$\boxed{s > d/2}$$

$$\|\Gamma(u)\|_{L^2_\omega C_T H^s_x} \underset{\substack{\uparrow \\ \text{Prop (via BDG ineq) for } \underline{s > d/2}}}{\leq} \|u_0\|_{H^s} + C T^\theta \|u\|_{L^2_\omega C_T H^s_x}$$

$$\|\Gamma(u) - \Gamma(v)\|_{L^2_\omega C_T H^s_x} \leq C T^\theta \|u - v\|_{L^2_\omega C_T H^s_x}.$$

Let $R = 2\|u_0\|_{H^s}$

Then, by choosing $T = T(R) > 0$ suff. small,

we have, for any $u, v \in \overline{B_R} \subset L^2_\omega C_T H^s_x$,

$$\|\Gamma(u)\|_{L^2_\omega C_T H^s_x} \leq R$$

$$\|\Gamma(u) - \Gamma(v)\|_{L^2_\omega C_T H^s_x} \leq \frac{1}{2}\|u - v\|_{L^2_\omega C_T H^s_x}$$

$\Rightarrow$ Γ is a contraction on $\overline{B_R}$.

② SNLS with multip. noise:

$$\begin{cases} i\partial_t u = \Delta u + |u|^{k-1} u + |u|^{\gamma-1} u \Phi \xi \\ u|_{t=0} = u_0 \in H^s, \quad s > d/2 \end{cases}$$

$k, \gamma \in 2\mathbb{N}+1$

· Duhamel formulation:

(SNLS)

$$u(t) = S(t) u_0 - i \int_0^t S(t-t') |u|^{k-1} u(t') \, dt' - i \underbrace{\int_0^t S(t-t') |u|^{\gamma-1} u(t') \Phi \, dW(t')}_{= \Psi[u](t)}$$

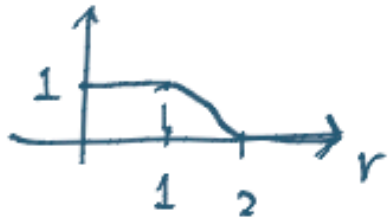
Want to construct $u \in L^2(\Omega; C_T H^s_x)$

$\Rightarrow$ Need to bound $\Phi[u]$ in $L^2(\Omega; C_T H^s_x)$

<u>But</u> by prop, we would need $u \in L^{2\sigma}(\Omega; C_T H^s_x)$

- <u>Truncation method</u>: de Bouard - Debussche '99.

η = smooth cutoff function

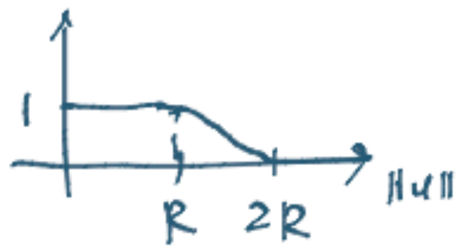


$\eta \equiv 1$ on $[0, 1]$

$\eta \equiv 0$ on $[2, \infty)$.

For $R > 0$, set

$$\eta_R(u)(t) = \eta\left(\|u\|_{C([0,t); H^s)} \Big/ R \right)$$



$\uparrow$
random

Modify the Duhamel formulation and consider the following fixed pt problem: ⑥

$$(SNLS_R) \quad u(t) = S(t)u_0 - i\int_0^t S(t-t')\,\eta_R(u)(t')\,|u|^{k-1}u(t')\,dt' - i\int_0^t S(t-t')\,\eta_R(u)(t')\,|u|^{\gamma-1}u(t')\,\Phi\,dW(t')$$

$$=: S(t)u_0 + \mathrm{I}[u] + \mathrm{II}[u]$$

$$=: \Gamma(u)$$

$$\|\mathrm{I}[u]\|_{C_T H^s_x} \le \left\| \int_0^t \eta_R(u)(t')\,\|u(t')\|^k_{H^s_x}\,dt' \right\|_{C_T}$$

$$\lesssim T R^k$$

Want to estimate $I[u_1] - I[u_2]$.

- Define stopping times $t_{j,R}$, $j = 1, 2$, by

$$t_{j,R} = \sup \{ t \in [0,T] : \| u_j \|_{C([0,t];H^s_x)} \le 2R \}$$

- WLOG, assume $t_{1,R} \le t_{2,R}$.

$$\| I[u_1] - I[u_2] \|_{C_T H^s_x} \qquad N(u) = |u|^{k-1} u.$$

$$\le \left\| \int_0^t S(t-t') \, \eta_R(u_1)(t') \big(N(u_1)(t') - N(u_2)(t') \big) dt' \right\|_{C_T H^s_x}$$

$$+ \left\| \int_0^t S(t-t') \big(\eta_R(u_1)(t') - \eta_R(u_2)(t') \big) N(u_2)(t') \, dt' \right\|_{C_T H^s_x}$$

$$=: I_1 + I_2.$$

- $\underline{\underline{I_2}}$: $\left| \eta_R(u_1)(t') - \eta_R(u_2)(t') \right|$

$$(*) \quad \le \frac{\|\eta'\|_{L^\infty}}{R} \underline{\|u_1 - u_2\|_{C_T H^s_x}} \qquad \text{for } t' \le T.$$

MVT & triangle ineq

$$\|I_2\|_{C_T H^s_x} \le T \, \| (\eta_R(u_1) - \eta_R(u_2)) \, N(u_2) \|_{C_T H^s_x}$$

$$= T \, \| (\eta_R(u_1) - \eta_R(u_2)) \, \underline{\underline{N(u_2)}} \|_{C([0, \underline{t_{2,R}}]; H^s_x)}$$

$$\overset{\text{by } (*)}{\lesssim} \frac{T}{R} \, \underline{\underline{R^k}} \, \underline{\|u_1 - u_2\|_{C_T H^s_x}}$$

$$= T \, R^{k-1} \, \|u_1 - u_2\|_{C_T H^s_x}$$

- $\underline{I_1}$:

$$\| I_1 \|_{C_T H^s_x} \leq T \| \eta_R(u_1) (\mathcal{N}(u_1) - \mathcal{N}(u_2)) \|_{C_T H^s_x}$$

$$= T \| \eta_R(u_1) (\mathcal{N}(u_1) - \mathcal{N}(u_2)) \|_{C([0, t_{1,R}]; H^s_x)}$$

$$\lesssim T R^{k-1} \| u_1 - u_2 \|_{C_T H^s_x}.$$

$$\| u_j \|_{C([0, t_{1,R}]; H^s_x)} \leq 2R$$

$$\Rightarrow \boxed{\| I[u_1] - I[u_2] \|_{C_T H^s_x} \lesssim T R^{k-1} \| u_1 - u_2 \|_{C_T H^s_x}.}$$

- We now consider $\mathbb{II}[u]$.

  We need to go back to the proof of Prop

- <u>On page 9 of Lec 5</u>:

  Instead of $\|\sigma(u)\|_{C_T H^s_x}$, we have

- $\|\gamma_R(u)\,\sigma(u)\|_{C_T H^s_x}$ for bounding $\mathbb{II}[u]$

- $\|\gamma_R(u_1)\,\sigma(u_1) - \gamma_R(u_2)\,\sigma(u_2)\|_{C_T H^s_x}$

  for bounding the difference $\mathbb{II}[u_1] - \mathbb{II}[u_2]$

$\Leftarrow$ these terms can be estimated as on pp. 6-9.

As a result (of modify the proof of Prop via the BDG ineq)

we obtain

$$\| \mathrm{II}[u] \|_{L^2_\omega C_T H^s_x} \lesssim T^\theta \| \Phi \|_{HS(L^2; H^s)} R^\gamma,$$

$$\| \mathrm{II}[u_1] - \mathrm{II}[u_2] \|_{L^2_\omega C_T H^s_x}$$

$$\lesssim T^\theta \| \Phi \|_{HS(L^2; H^s)} R^{\gamma-1} \| u_1 - u_2 \|_{L^2_\omega C_T H^s_x}$$

· Putting everything together, we have

$$\text{(A)} \quad \| \Gamma(u) \|_{L^2_\omega C_T H^s_x} \le \| u_0 \|_{H^s_x} + C_1 T^\theta \left(R^k + \| \Phi \|_{HS(L^2; H^s)} R^\gamma \right),$$

$$\text{(B)} \quad \| \Gamma(u_1) - \Gamma(u_2) \|_{L^2_\omega C_T H^s_x} \lesssim T R^{k-1} \| u_1 - u_2 \|_{L^2_\omega C_T H^s_x}$$

$$+ T^\theta \| \Phi \|_{HS(L^2; H^s)} R^{\gamma-1} \| u_1 - u_2 \|_{L^2_\omega C_T H^s_x}$$

In particular, the difference estimate

B$'$ $$\|\Gamma(u_1)-\Gamma(u_2)\|_{L^2_\omega C_T H^s_x} \le \underline{C_2 T^\theta \|\Phi\|_{HS(L^2;H^s)}} \|u-v\|_{L^2_\omega C_T H^s_x}$$

holds for <u>ANY</u> $u_1, u_2 \in L^2_{ad}(\Omega; C_T H^s_x)$,

<u>regardless of their sizes.</u>

$\Leftarrow$ thanks to the cutoff function γ_R.

- Fix $R>0$. Then, by taking $T = T(\|\Phi\|_{HS(L^2;H^s)}) > 0$ suff. small, Γ is a contraction on a ball $\overline{B_{R_0}} \subset L^2_\omega C_T H^s_x$, where $R_0 = \|u_0\|_{H^s} + C_1 T^\theta (R^k + \|\Phi\|_{HS(L^2;H^s)} R^{\sigma-1})$.

$= $ (RHS) of A on the previous page

$\Rightarrow$ LWP of $(SNLS_R)$ in H^s, $s > d/2$, for <u>any finite $R>0$</u>

(with $\Phi \in HS(L^2; H^s)$).

· <u>Claim</u>: ($SNLS_R$) is globally well-posed

<u>Pf</u>: Consider $SNLS_R$ on $[T, 2T]$ with the initial cond $u_R(T)$

soln on $[0,T]$

$$u(t) = S(t-T)\, u_R(T) - i\int_T^t S(t-t')\,|u|^{p-1}u(t')\,dt'$$

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truncation $\chi_R(u)$

$$- i \underbrace{\int_T^t S(t-t')\,|u|^{\gamma-1}u(t')\,\Phi\, dW(t')}_{\Psi^T[u]} =: \Gamma^T(u)$$

for $T \le t' \le 2T$.

· Change of variables : $\tau = t - T$
$\tau' = t' - T$

$$\Rightarrow \quad \Psi^T[u](\tau) = \int_0^\tau S(\tau-\tau')\,|u|^{\gamma-1}u(\tau'+T)\,\Phi\, dW^T(\tau')$$

where $W^T(\tau) = W(t+\tau)$. $\Leftarrow$ shifted in time but <u>same in law</u>.

$\Psi^T[u]$ satisfies the same bound as $\Psi[u]$ on $[T, 2T]$.

$$\Rightarrow \|\Gamma^T(u)\|_{L^2_\omega C_{I_2} H^s_x} \le \underline{\|u_R(T)\|_{L^2_\omega H^s_x}} + \underbrace{C_1(R, \|\Phi\|_{HS(L^2; H^s)})}_{\text{same as in A on p. 11}}$$

$$\le \|u_0\|_{H^s} + 2C_1(R, \|\Phi\|_{HS(L^2; H^s)})$$

$\uparrow$ from LWP on $[0, T]$

where $I_j = [(j-1)T, jT]$.

and

$$\|\Gamma^T(u_1) - \Gamma^T(u_2)\|_{L^2_\omega C_{I_2} H^s_x} \le \overbrace{\underline{\underline{C_2 T^\theta \|\Phi\|_{HS(L^2; H^s)}}}}^{\text{want } \le 1/2\text{. does NOT depend on the size of } u\,!!} \|u_1 - u_2\|_{L^2_\omega C_{I_2} H^s_x}$$

$\uparrow$ same factor as in B′ on p. 12.

By a contraction argument, we can prove LWP on $[T, 2T]$ (i.e. time of existence does NOT shrink, while the soln may grows.

- By iterating this argument on $I_j = [(j-1)T, jT]$,

we have

$$\|\Gamma^{jT}(u)\|_{L^2_\omega C_{I_j} H^s_x} \leq \|u_0\|_{H^s} + \underbrace{j}_{\text{soln grows}} C_1(R, \|\Phi\|_{HS(L^2;H^s)})$$

and

$$\|\Gamma^{jT}(u_1) - \Gamma^{jT}(u_2)\|_{L^2_\omega C_{I_j} H^s_x} \leq \underbrace{C_2 T^\theta \|\Phi\|_{HS(L^2;H^s)}}_{\substack{\text{same const and hence} \\ \text{the time of local existence at the } j^{th} \text{ step} \\ \text{does NOT shrink.}}} \|u_1 - u_2\|_{L^2_\omega C_{I_j} H^s_x}$$

$\Rightarrow$ GWP of the truncated eqn $(SNLS_R)$ for any $R > 0$.

<u>Next step</u>: "Construction" of soln u to (SNLS).

Rmk: We did not check if the soln u to the lin. stoch. Schrödinger eqn or ($SNLS_R$) is indeed adapted.

Consider the lin. stoch Schrödinger eqn on p. 4.

$$u = \Gamma(u)(t) = S(t)u_0 - i\Psi[u](t).$$

· We proceeded with a contraction argument, namely by a Picard iteration.

$$\begin{cases} P_1 = S(t)u_0 \\ P_j = S(t)u_0 - i\Psi[P_{j-1}], \quad j \geq 2 \end{cases}$$

· Clearly, P_1 is adapted (since it is deterministic)

· If P_{j-1} is adapted, then so is $\Psi[P_{j-1}]$ as an Ito integral and so is P_j.

By induction, P_j is adapted for any $j \in \mathbb{N}$.

Since $u = \lim_{j \to \infty} P_j$, u is also adapted.

<u>Moral</u>: When we construct a soln via a Picard iteration, it is automatically adapted (if basic objects are adapted)