Lec 5 22/02/22 (Tue)

$\Phi : H \to B$ is γ-radonifying (i.e. $\Phi \in \gamma(H;B)$)

iff $\Phi_{\#}\mu_H = \mu_H \circ \Phi^{-1}$ $\qquad d\mu_H \sim e^{-\frac{1}{2}\|u\|_H^2} du$

= image measure of μ_H under Φ
(push-forward)

has an extension to a countably additive (Gaussian probability) measure μ_Φ on B. (i.e. (H, B, μ_Φ) is an A.W.S.)

Rmk: · Fernique Thm: $\int_B e^{c\|u\|_B^2} \mu_\Phi(du) < \infty$ (for some $c>0$)

· $\|\Phi\|_{\gamma(H;B)} = \left(\int_B \|u\|_B^2 \, \mu_\Phi(du) \right)^{1/2}$

$\Updownarrow$

$\mu_\Phi(\|u\|_B > \lambda) \le C e^{-c\lambda^2}, \forall \lambda > 0$

- Burkholder - Davis - Gundy inequality:

- We say that a Banach space B is of martingale type p (M-type p) for some $1 \leq p \leq 2$ if

$$\boxed{\|f_N\|_{L^p(\Omega;B)} \leq C\left(\sum_{n=0}^{N}\|\underbrace{f_n - f_{n-1}}_{df_n}\|^p_{L^p(\Omega;B)}\right)^{1/p}}$$

for any B-valued L^p-martingales $\{f_n\}_{n=0}^{N}$ ($f_{-1}=0$).

filtration $\{\mathcal{F}_n\}_{n=0}^{\infty} \subset \mathcal{F}$

martingale: $\mathbb{E}[f_n \mid \mathcal{F}_m] = f_m$, $m \leq n$.

$\Rightarrow \mathbb{E}[df_n \mid \mathcal{F}_m] = 0$, $\forall\, m < n$.

$$\left(\begin{array}{l} \text{M-type } p \Longrightarrow \text{type } p \\ \text{M-type } p \Longleftarrow \text{type } p \text{, UMD} \end{array} \right.$$

UMD: unconditional martingale diff.

$\Updownarrow$ Burkholder, Bourgain '83.

HT space B

i.e. Hilbert transform on B-valued functions are bdd in $L^2(\mathbb{S};B)$

- If B has M-type p (for some $1 \le p \le 2$), then $\underline{L^r(A;B)}$ has M-type $\underline{p \wedge r}$ $(1<r<\infty)$

  A = measure space

- Hilbert space is of M-type 2.

$\Longrightarrow$ $L^r(\mathbb{R}^d;\mathbb{C})$ is of M-type 2 for $\underline{2 \le r < \infty}$

$$\boxed{\begin{array}{l} L^q(\mathbb{R}, L^r(\mathbb{R}^d; \mathbb{C})) = L^q_t L^r_x \\ \text{is of M-type 2 for } 2 \le q, r < \infty \end{array}}$$

- We say that a stopping time τ is accessible if $\exists$ increasing seq $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping times s.t.

$$\tau_n < \tau \qquad \text{a.s.}$$

$$\lim_{n\to\infty} \tau_n = \tau$$

predictable ?

BDG inequality (for stoch. integrals)

$1 < p < \infty$, $\quad B$ = Banach space of M-type 2.
also called "2-smooth".

Then, $\exists$ $C(p,B) > 0$ s.t.

$$\boxed{\mathbb{E}\left[\sup_{0<t<\tau} \left\| \int_0^t F(t)\, dW(t) \right\|_B^p\right] \le C(p,B)\, \mathbb{E}\left[\left(\int_0^\tau \|F(t)\|^2_{\gamma(K;B)}\, dt\right)^{p/2}\right]}$$

for any accessible stopping time $\tau > 0$

and $\gamma(K;B)$-valued progressively measurable F

- W = K-cylindrical Wiener process
  (for us, $K = L^2(\mathbb{R}^d)$ or $L^2(\mathbb{T}^d)$.)

- Pisier '76, Ondreját '04, Seidler '10 (optimal const)
- Brzeźniak '97 (stronger assumption: UMD & type 2).

- Proof of BDG inequality: later by Guangqu.

- Back to SNLS with multip noise

stochastic convolution

$$\Psi(t) = \Psi[u](t) = \int_0^t S(t-t')\big(\sigma(u)\,\Phi\,dW(t')\big)$$

$$\boxed{\sigma(u) = |u|^{\gamma-1}u}\quad , \quad \gamma \geq 1.$$

Assume $\gamma \in 2\mathbb{N}+1$
s.t. $\sigma(u)$ is algebraic

Prop: Let $s > \frac{d}{2}$ and $\Phi \in HS(L^2; H^s)$.

Then, for any $u \in L^{2\gamma}_{ad}(\Omega; C_T H^s_x)$, we have

$$\Psi = \Psi[u] \in C_T H^s_x, \text{ a.s.}$$

adapted

Pf: We use the factorization method

(Da Prato Book '04 "Kolmogorov eqn for SPDEs" Lemma 2.7)

· **Lemma**: Let $0<\alpha<1$ and $q>\frac{1}{\alpha}$

Suppose $f \in L^q_T H^s_x$ for some $T>0$.

Then,

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')\,dt', \quad 0 \le t \le T$$

belongs to $C_T H^s_x$. Moreover, we have

$$\Longrightarrow \quad \sup_{0\le t\le T} \|F(t)\|_{H^s_x} \lesssim \|f\|_{L^q_T H^s_x}$$

· FACT: $\boxed{\int_\mu^t (t-t')^{\alpha-1}(t'-\mu)^{-\alpha}\,dt' = \frac{\pi}{\sin(\pi\alpha)}}$

$\forall\ 0<\alpha<1, \quad 0\le\mu\le t'\le t$ (Beta function)

$$\Psi(t) = \int_0^t S(t-\mu)\ \sigma(u)(\mu)\ \Phi\, dW(\mu)$$

$$= \frac{\sin(\pi\alpha)}{\pi} \int_0^t \left[\int_\mu^t (t-t')^{\alpha-1} (t'-\mu)^{-\alpha}\, dt'\right] \times S(t-\mu)\, \sigma(u)(\mu)\, \Phi\, dW(\mu)$$

$$= \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(t-t')(t-t')^{\alpha-1} \times \underbrace{\left[\int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\, \sigma(u)(\mu)\, \Phi\, dW(\mu)\right]}_{= f(t')} dt'.$$

In view of Lemma, it suffices to show

$$f \in L^q_T H^s_x, \quad \text{for some} \quad \frac{1}{\alpha} < q < \infty.$$

$$f(t') = \int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\sigma(u)(\mu)\Phi\, dW(\mu)$$

$\circledast$ WTS:

$$\mathbb{E}\left[\int_0^T \|f(t')\|^q_{H^s_x}\, dt'\right] \le C(T, q, \Phi) < \infty$$

By BDG inequality, for finite $q \ge 1$,

$$\mathbb{E}\left[\|f(t')\|^q_{H^s_x}\right] \lesssim \mathbb{E}\left[\Bigg(\underbrace{\int_0^{t'} \|\cancel{S(t'-}\mu)\underline{(t'-\mu)}^{-\alpha}\sigma(u)(\mu)\Phi\|^2_{HS(L^2;H^s)}\, d\mu}_{\displaystyle = \int_0^{t'} (t'-\mu)^{-2\alpha}\|\sigma(u)(\mu)\Phi(e_n)\|^2_{l^2_n H^s_x}\, d\mu}\Bigg)^{q/2}\right]$$

$$\int_0^{t'} \underline{(t'-\mu)^{-2\alpha}}\, \|\sigma(u)(\mu)\Phi(e_n)\|^2_{l^2_n H^s_x}\, d\mu$$

$0<\alpha<\frac{1}{2}$ $\quad \lesssim \quad$ $s > d/2$

$$\|\sigma(u)\|^2_{C_T H^s_x}\|\Phi\|^2_{HS(L^2;H^s)} \qquad \{e_n\} = \text{O.N.B. of } L^2.$$

$$\lesssim \mathbb{E}\left[\|u\|^{\gamma q}_{C_T H^s_x}\right] \times \|\Phi\|^q_{HS(L^2;H^s)} < \infty.$$

Now, integrate from $t'=0$ to T

$\Rightarrow$ (*) follows. for any finite $q \geq 1$ (in particular $q > \frac{1}{\alpha}$)

$\Rightarrow$ By Lemma, $\Psi \in C_T H^s_x$, a.s.

We need $\alpha > \frac{1}{2}$.

$\Rightarrow$ We can take any $2 \leq q < \infty$

Note: ① On p. 9, we viewed

$$S(t'-\mu)\big(\sigma(u)(\mu)\,\Phi\, dW(t')\big)$$

as

$$S(t'-\mu) \circ M_{\sigma(u)(\mu)} \circ \Phi \text{ applied to } dW(t')$$

where M_F = multiplication by a function F.

② With $\sigma(u) = |u|^{\gamma-1} u$, we used

$\| \sigma(u) \|_{H^s} \lesssim \| u \|_{H^s}^{\gamma}$ for $s > d/2$.

Here, we used the fact that $\gamma \in 2\mathbb{N}+1$

$\Rightarrow$ $\sigma(u)$ is algebraic. (i.e. a product).

- When $\gamma \notin 2\mathbb{N}+1$, we can not consider $s \gg 1$ due to the lack of smoothness of $\sigma(\cdot)$

In general, given $s > d/2$ (s.t. $H^s \hookrightarrow L^\infty$), we need $\sigma \in C^k(\mathbb{C} \cong \mathbb{R}^2; \mathbb{C})$ with $k \geq [s]+1$

For example, see

- Lemma A.9 in Tao's dispersive PDE book.
- Lemma 4.10.2 in Cazenave's book.

(Also, see the fractional chain rule.)