Lec 4 15/02/22 (Tue)

$$\begin{cases} i\partial_t u - \Delta u = |u|^{p-1}u + \Phi\xi \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d), \quad \Phi \in HS(L^2; H^s) \end{cases}$$

<u>on $\mathbb{R}^d$:</u>

$$\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t')$$

(i) $\Psi \in C_t H^s_x$, a.s.

(ii) $\underline{\underline{\Psi \in L^q_T W^{s,r}_x,}}$ $\forall\ q<\infty$, $r \le \frac{2d}{d-2}$ ($r<\infty$, when $d=1,2$

$\forall$ finite $T>0$

<u>ex 1:</u> $d=1$, $p=3$, $s=0$.

$$\Gamma_{u_0,\Phi}(u) = S(t)u_0 - i\underbrace{\int_0^t S(t-t')\overset{\mathbb{1}_{[0,T]}(t')}{\swarrow}|u|^{p-1}u(t')\,dt'}_{\text{Apply the nonhomog Strichartz}} - i\Psi.$$

<u>Recall:</u> We say (q,r) is <u>admissible</u> if

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2} \qquad (q,r,2)\neq(2,\infty,2)$$

$$2\le q,\ r\le\infty$$

- <u>$d=1$</u>: $(q,r)=(8,4)$, $(\infty,2)$, admissible

Set

$$X(T)=C_T L^2_x \cap L^8_T L^4_x.$$

$$\|\Gamma(u)\|_{X(T)} \overset{\text{Strichartz}}{\lesssim} \underline{\|u_0\|_{L^2}} + \underbrace{\||u|^2u\|_{L^{8/7}_T L^{4/3}_x}} \qquad \frac{7}{8}=\frac{3}{8}+\frac{4}{8}$$

$$+\underbrace{\|\Psi\|_{X(T)}}_{\underline{<\infty,\ \text{a.s.}}}$$

$$\underset{\uparrow\ \text{Hölder int}}{\le} T^{1/2}\|u^3\|_{L^{8/3}_T L^{4/3}_x}$$

$$= T^{1/2}\|u\|^3_{L^8_T L^4_x}$$

$$\le \underline{T^{1/2}\|u\|^3_{X(T)}}$$

$$\underset{T\le 1}{\Rightarrow} \quad \|\Gamma(u)\|_{X(T)} \le \left(C_0\|u_0\|_{L^2} + \|\Psi\|_{X(1)}\right) + C_1 \underline{\underline{T}}^{1/2} \|u\|^3_{X(T)}$$

Also,

$$\|\Gamma(u) - \Gamma(v)\|_{X(T)} \lesssim \underline{\underline{T}}^{1/2}\left(\|u\|^2_{X(T)} + \|v\|^2_{X(T)}\right)\|u-v\|_{X(T)}$$

Let $R = 2\left(C_0\|u_0\|_{L^2} + \|\Psi\|_{X(1)}\right)$. ← random

→ Proceeding as before, we conclude that Γ is a contraction on $\overline{B_R} \subset X(T)$ by choosing $T = T(R) > 0$ suff small.

$\Rightarrow$ LWP in $L^2(\mathbb{R})$ with $\Phi \in HS(L^2; L^2)$

Rmk: If $u_0 \in H^s$ and $\Phi \in HS(L^2; H^s)$ for some s, then we can use the fractional Leibniz rule:

$$\|fg\|_{\dot{W}^{s,r}} \lesssim \|f\|_{\dot{W}^{s,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|g\|_{\dot{W}^{s,r_2}}$$

$$1 < r, p_j, q_j < \infty \qquad 0 < s < 1,$$

$$\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}, \quad j = 1, 2$$

$$\text{“}\ D^s(fg) \approx D^s f \cdot g + f \cdot D^s g\ \text{”}$$

to show $u \in C_T H^s_x$ $\Leftarrow$ persistence of regularity

- **Scaling symmetry on NLS:** $i\partial_t u - \Delta u = \pm |u|^{p-1} u$

u is a soln to NLS with $u|_{t=0} = u_0$

$\Longleftrightarrow$ $\underline{u_\lambda(t,x) = \frac{1}{\lambda^{2/p-1}}\, u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)}$ is a soln to (NLS) with $\underline{\underline{u_\lambda(0,x) = \frac{1}{\lambda^{\frac{2}{p-1}}}\, u_0\left(\frac{x}{\lambda}\right)}} =: u_{0,\lambda}(x)$

Set $\boxed{s_{crit} = \frac{d}{2} - \frac{2}{p-1}.}$ Then, we have

critical Sobolev regularity index

$$\|u_{0,\lambda}\|_{\dot H^{s_{crit}}} = \|u_0\|_{\dot H^{s_{crit}}}$$

NLS has several conservation laws:

mass $\underline{M(u) = \int |u|^2 dx}$, $P(u) = \mathrm{Im}\int \overline{\nabla u}\cdot u \, dx$ momentum

energy $\underline{E(u) = \frac{1}{2}\int |\nabla u|^2 dx \mp \frac{1}{p+1}\int |u|^{p+1} dx.}$

We say that (NLS) is

- mass-critical (= L^2-critical) if $s_{crit} = 0$
- energy-critical (= H^1-critical) if $s_{crit} = 1$.

Given $u_0 \in H^s(\mathbb{R}^d)$, we say that the Cauchy problem is

- subcritical if $s > s_{crit}$ ⇐ expect well-posedness

  smaller data, longer time

  $$\|u_{0,\lambda}\|_{\dot{H}^s} = \lambda^{s_{crit} - s}\|u_0\|_{\dot{H}^s}$$

- critical if $s = s_{crit}$

  scaling does not change the $\dot{H}^s$-norm of initial data

- supercritical if $s < s_{crit}$.

  larger data, longer time ⇐ too good to be true

  ⇐ expect ill-posedness

$$s_{crit} = \frac{d}{2} - \frac{2}{p-1}$$

- $\underline{p=3}$: $d=1 \Rightarrow s_{crit} = -1/2$

  $d=2 \Rightarrow s_{crit} = 0.$

$\underline{ex2}$: $d=2$, $p=3$, $\underline{s=0}$ $\longrightarrow$ $u_0 \in L^2(\mathbb{R}^2)$, $\Phi \in HS(L^2;L^2)$

critical problem

$(q,r) = (4,4), (\infty, 2)$ admissible.

$$\underset{\underset{L^4_T L^4_x}{\parallel}}{\|\Gamma(u)\|_{L^4_{T,x}}} \le \underline{\|S(t)u_0\|_{L^4_{T,x}}} + \|\Psi\|_{L^4_{T,x}}$$

$$+ C \underbrace{\||u|^2 u\|_{L^{4/3}_{T,x}}}_{= \|u\|^3_{L^4_{T,x}}}$$

$\uparrow$ $\underline{\text{NO } T^\theta\,!!}$

$$\boxed{\text{Set } R = 2\left(\|S(t)u_0\|_{L^4_{T,x}} + \|\Psi\|_{L^4_{T,x}} \right) \quad ⊛}$$

$\Rightarrow$ For $u \in \overline{B_R} \subset L^4_{T,x}$, we have

$$\cdot \ \|\Gamma(u)\|_{L^4_{T,x}} \le \frac{1}{2}R + CR^3 \underset{\substack{\uparrow \\ \text{for } R \ll 1}}{\le} R$$

Also

$$\cdot \ \|\Gamma(u) - \Gamma(v)\|_{L^4_{T,x}} \le \underbrace{C' R^2}_{\le 1/2 \ \text{ for } R \ll 1} \|u - v\|_{L^4_{T,x}}$$

$$\cdot \ \|S(t)u_0\|_{L^4(\mathbb{R}; L^4_x)} \lesssim \|u_0\|_{L^2_x} < \infty$$

$$\cdot \ \|\Psi\|_{L^4([0,1]; L^4_x)} < \infty, \text{ a.s.}$$

$\Rightarrow$ We can choose $T = T(\omega) > 0$ small s.t. $R \ll 1$

⇒ Γ is a contraction on $\overline{B_R} \subset L^4_{T,x}$

⇒ $\exists!$ soln $u \in \overline{B_R} \subset L^4_{T,x}$

$$u = \underset{\substack{\Uparrow \\ C_t L^2_x}}{S(t)u_0} - i \underbrace{\int_0^t S(t-t')|u|^2 u(t')\,dt'} - i \underset{\substack{\cap \\ C_t L^2_x,\ a.s}}{\Psi}$$

$$\| \cdots \|_{C_T L^2_x} \overset{\text{Str}}{\lesssim} \| |u|^2 u \|_{L^{4/3}_{T,x}} = \| u \|^3_{L^4_{T,x}} < \infty$$

⇒ $u \in C_T L^2_x$, a.s

<u>Note</u>: uniqueness holds only in $\overset{\text{the ball in}}{C_T L^2_x \cap L^4_{T,x}}$

⇐ conditional uniqueness

$|u|^2 u \Leftarrow u$ must belong to L^3_{loc} (in x).

Now, we turn our attention to

SNLS with multiplicative noise.

$$i\partial_t u - \Delta u = \mathcal{N}(u) + \sigma\underset{\overset{\uparrow}{|u|^{\gamma-1}u}}{(u)}\Phi\xi.$$

- Stochastic convolution:

$$\Psi(t) = \Psi[u](t) = \int_0^t S(t-t')\,\sigma(u)(t')\,\Phi\,dW(t')$$

- $\Phi \in HS(L^2; H^s)$ $\Leftarrow$ NOT sufficient.
- the argument in the additive case does not apply here to the lack of indep.

- γ - radonifying operator

$\Leftarrow$ Banach generalization of HS operators.

H = separable Hilbert space

$$d\mu \sim e^{-\frac{1}{2}\|x\|_H^2} dx$$

$\Leftarrow$ NOT countably additive if $\dim H = \infty$

$\Rightarrow$ Need to enlarge H to make sense of μ.

$$i : H \hookrightarrow B$$

(H, B, μ) is called an abstract Wiener space

if μ makes sense as a Gaussian prob meas on B.

$\{e_n\}$ = O.N.B of H.

$${}_{B}\langle \underset{\overset{\Uparrow}{B}}{x}, e_n \rangle_{B^*}, \quad n = 1, \cdots, N$$

- Gross '60's. Books by Kuo, Nualart

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ex: $$d\mu_s \sim \underline{e^{-\frac{1}{2}\|u\|^2_{H^s}} du} \quad \text{"on } H^s(\mathbb{T}^d)\text{"}.$$

$$\Leftrightarrow \boxed{u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x}}$$

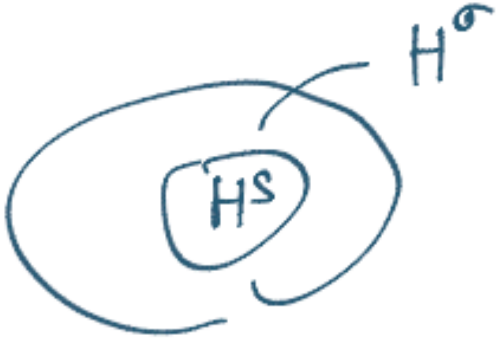
$\{g_n\}_{n \in \mathbb{Z}^d}$ = indep standard $\mathbb{C}$-valued Gaussians

$$e^{-\frac{1}{2}\|u\|_{H^s}^2}\,du = e^{-\frac{1}{2}\sum_n \langle n\rangle^{2s}|\hat{u}(n)|^2}\,du$$

$$= \prod_{n\in\mathbb{Z}^d} e^{-\frac{1}{2}\underbrace{\langle n\rangle^{2s}|\hat{u}(n)|^2}_{|g_n|^2}}\;\underbrace{d\hat{u}(n)}_{\text{Lebesgue meas on } \mathbb{C}\cong\mathbb{R}^2}$$

$$\mathbb{E}\left[\|u\|_{H^\sigma}^2\right] = \sum_{n\in\mathbb{Z}^d}\frac{\mathbb{E}[|g_n|^2]}{\langle n\rangle^{2s-2\sigma}} < \infty$$

iff $\boxed{\sigma < s - \frac{d}{2}}$



$$u_N = \sum_{|n|\le N}\frac{g_n}{\langle n\rangle^s}e^{in\cdot x} \longrightarrow u \text{ in } H^\sigma(\mathbb{T}^d),\ \text{a.s.}$$

if $\sigma < s - d/2$

On $\mathbb{T}^d$

- (H^s, H^σ, μ_s) is an abstract Wiener space. if $\sigma < s - \frac{d}{2}$
- $(H^s, W^{\sigma,p}, \mu_s)$ " $p \leq \infty$

Back to $W(t) = \sum \beta_n(t) e_n$

↑ Gaussian for fixed t.

- $\Phi \in HS(L^2; H^s)$

$$\Rightarrow \Phi W(t) \in H^s_x$$

$$\boxed{\begin{array}{l} \cdot\ \Phi \in \underline{\underline{\gamma(L^2; B)}}\ \left(\text{or } M(L^2; B) \text{ or } R(L^2; B)\right) \\ \qquad \Rightarrow \Phi W(t) \in B \end{array}}$$

$(L^2, B, \text{Law}(\Phi W(t)))$ is an abstract Wiener space.

- $\Phi \in \gamma(H ; B)$ ← Banach space

  ↑ sep. Hilbert space

  indep std Gaussian ↙

  if $$\|\Phi\|_{\gamma(H;B)} = \left(\mathbb{E}\left\|\sum_n g_n \Phi(e_n)\right\|_B^2\right)^{1/2} < \infty$$

- If B is a Hilbert space,

  then $\gamma(H;B) = HS(H;B)$.

- $$\|\Phi\|_{\gamma(H;B)} \underset{\uparrow}{\sim} \left(\mathbb{E}\left\|\sum_n g_n \Phi(e_n)\right\|_B^p\right)^{1/p}, \quad 1 < p < \infty$$

  Kahane-Khintchin ineq.

$\sum_n g_n \Phi(e_n)$ ← Gaussian series.

different modes of convergence are equivalent
(Ito - Nisio thm)

- Hytönen, van Neerven, Veraar, Weis.
  Analysis in Banach spaces. vol 1–2.

<u>Aside:</u> Def of an abstract Wiener space, starting from μ on H.

We say that a semi-norm $|||\cdot|||$ is <u>measurable</u>

if $\forall \varepsilon > 0$, $\exists$ a finite-dim'l ortho projection P_ε of H

s.t. $$\mu\left(|||Px||| > \varepsilon\right) < \varepsilon$$

for any finite-dim'l ortho projection $P \perp P_\varepsilon$.

$\Rightarrow$ $B =$ completion of H under $|||\cdot|||$.

$\Rightarrow$ (H, B, μ) is an abstract Wiener space.

(Think of "P_ε = low freq projection"
"P = high freq projection $\perp P_\varepsilon$"