Lec 3 08/02/22 (Tue):

· Linear Schrödinger eqn:
$$\begin{cases} i\partial_t u = \Delta u \\ u|_{t=0} = u_0 \end{cases}$$

$$\Rightarrow \quad u(t) = S(t)\, u_0$$

$$S(t) = e^{-it\Delta} \qquad \widehat{S(t) f}(\xi) = e^{it|\xi|^2} \hat{f}(\xi)$$

↑ smoothing (in a sense different from the heat semigp $P(t) = e^{t\Delta}$

① Unitary in H^s: $\| S(t) f \|_{H^s} = \| f \|_{H^s}$

② dispersive estimate:

Only on $\mathbb{R}^d$

$$\| S(t) f \|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}} \| f \|_{L^1_x}, \quad t \neq 0$$

$\Leftarrow$ "smoothing" (removes high peaks.)

3 Strichartz estimate:

We say that (q,r) is admissible if

$$\boxed{\frac{2}{q} + \frac{d}{r} = \frac{d}{2}}, \quad (q,r,d) \neq (2,\infty,2)$$

$$2 \leq q, r \leq \infty$$

$$\boxed{\begin{array}{l} d=1: \ r \leq \infty \\ d=2: \ r < \infty \\ d \geq 3: \ r \leq \frac{2d}{d-2} \end{array}}$$

Only on $\mathbb{R}^d$

(i) (homogeneous), (q,r) admis

$$\| S(t) f \|_{L^q_t L^r_x} \lesssim \| f \|_{L^2_x}$$

(ii) (dual). (q,r), admis

$$\left\| \int_{\mathbb{R}} S(-t) F(t)\, dt \right\|_{L^2_x} \lesssim \| F \|_{L^{q'}_t L^{r'}_x} \qquad \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1$$

(iii) (nonhomogeneous / retarded)

$$\left\| \int_0^t S(t-t') F(t')\, dt' \right\|_{L^q_t L^r_x} \lesssim \| F \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

(q,r), $(\tilde{q},\tilde{r})$, admis

- For the proof (non endpt case), see the NLS course note
  Tao's book
- $f \in L^2_x \implies S(t) f \in L^q_t L^r_x$

  In particular, $S(t) f \in L^r_x$, a.s. $t \in \mathbb{R}$
- Only on $\mathbb{R}^d \Leftarrow$ Unitarity & dispersive estimate
  & H-L-S inequality (in time)

  On $\mathbb{T}^d$, some Strichartz estimates hold.

  Bourgain '93, Bourgain-Demeter '15.

- Heat eqn: Schauder estimate. $P(t) = e^{t\Delta}$

$$\| P(t) f \|_{W^{s,q}} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{s}{2}} \|f\|_{L^p_x}$$

$t > 0$

- wave eqn: Strichartz estimates.

Pf of (ii):
$$\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t')$$
$$\Phi \in HS(L^2; H^s),$$
$$\Phi W = \sum_{n\in\mathbb{N}} \beta_n(t)\Phi(e_n), \quad \{e_n\}_{n\in\mathbb{N}} = \text{O.N.B of } L^2(\mathbb{R}^d)$$

$$\|\langle\nabla\rangle^s \Psi(t,x)\|_{L^2(\Omega)}$$
$$\overset{\text{indep}}{\underset{\text{Ito isometry}}{=}} \Big\| \Big(\int_0^t |S(t-t')\langle\nabla\rangle^s \Phi(e_n)(x)|^2 dt'\Big)^{1/2} \Big\|_{\ell^2_n}$$
$$t' \to \tau = t - t'$$
$$\overset{\text{ch. of var}}{=} \|S(\tau)\langle\nabla\rangle^s\Phi(e_n)(x)\|_{\ell^2_n L^2_\tau([0,t])}$$

$\Rightarrow$ $\forall p \geq 2$
$$\|\langle\nabla\rangle^s\Psi(t,x)\|_{L^p(\Omega)} \lesssim p^{1/2}\|\cdots\|_{L^2(\Omega)} \sim p^{1/2}\|S(\tau)\langle\nabla\rangle^s\Phi(e_n)(x)\|_{\ell^2_n L^2_\tau([0,t])}$$

$$\Big\| \, \|\Phi\|_{L^q_T W^{s,r}_x} \Big\|_{L^p(\Omega)}$$

$q, r < \infty$

$r \le \frac{2d}{d-2}$

finite $T > 0$

Minkowski integral ineq

$$\le \Big\| \, \| \langle\nabla\rangle^s \Phi(t,x) \|_{L^p(\Omega)} \Big\|_{L^q_T L^r_x} \quad \text{for } p \ge q \vee r = \max(q,r)$$

$$\le C_p \Big\| \, \| S(\tau) \langle\nabla\rangle^s \Phi(e_n) \|_{\ell^2_n L^2_\tau([0,t])} \Big\|_{L^q_T L^r_x}$$

replace by $[0,T]$ in t over $[0,T]$

Mink

$$\le C_p T^{1/q} \Big\| \, \| S(\tau) \langle\nabla\rangle^s \Phi(e_n) \|_{L^2_\tau([0,T]; L^r_x)} \Big\|_{\ell^2_n}.$$

Given $2 \le r < \infty$, let $(\tilde{q}, r)$ be admissible for some $\tilde{q} \ge 2$

Hölder

$$\le C_p T^\theta \Big\| \, \| S(\tau) \langle\nabla\rangle^s \Phi(e_n) \|_{L^{\tilde{q}}_\tau([0,T]; L^r_x)} \Big\|_{\ell^2_n}$$

$$\underset{\text{Strichatz}}{\lesssim} C_p T^\theta \underbrace{\Big\| \, \| \Phi(e_n) \|_{H^s} \Big\|_{\ell^2_n}}_{= \|\Phi\|_{HS(L^2; H^s)}} < \infty$$

$$\Rightarrow \quad \Phi \in L^q_T W^{s,r}_x, \quad a.s.$$

$T < \infty$

$\boxtimes$

- Local well-posedness of SNLS with additive noise

existence, unique of solns, stability under perturbation (in initial data & noise

for "short" time

(SNLS) $\begin{cases} i\partial_t u - \Delta u = |u|^{2k}u + \underbrace{\Phi\xi} \\ u|_{t=0} = u_0 \in H^s \end{cases}$ additive noise

$\Phi \in HS(L^2; H^s)$,

We say that u is a soln to (SNLS) if u satisfies the mild formulation (= Duhamel formulation):

$$u(t) = \boxed{S(t)u_0 - i\int_0^t S(t-t')|u|^{2k}u(t')dt' - i\underbrace{\int_0^t S(t-t')\Phi\, dW(t')}_{=\Psi(t)}} =: \Gamma_{u_0,\Phi}(u)$$

, $S(t) = e^{-it\Delta}$

① $\boxed{\underline{\text{Simple case}}: \ s > d/2}$ H^s is an algebra.

Want to show $\exists!\ u \in C_T H^s$ s.t. $\underline{u = \Gamma(u)}$. fixed pt.

(i) $\| S(t) u_0 \|_{H^s} \overset{\text{unitarity}}{=} \| u_0 \|_{H^s}, \quad \forall t$ ⑧

$\Rightarrow \| S(t) u_0 \|_{C_T H^s_x} = \| u_0 \|_{H^s}$

(ii) $\left\| \int_0^t S(t-t') |u|^{2k} u(t')\, dt' \right\|_{H^s_x}$

$$\overset{\text{Mink}}{\leq} \int_0^t \| \underbrace{\cancel{S(t-t')}}_{\text{unitarity}} \underbrace{|u|^{2k} u}_{= u^{k+1} \bar{u}^k} (t') \|_{H^s_x} \, dt'$$

$$\overset{\text{product esti}}{\lesssim}_{\boxed{s > \frac{d}{2}}} \int_0^t \| u(t') \|_{H^s}^{2k+1} dt' \leq \int_0^t \| u \|_{C_T H^s}^{2k+1} dt' \quad \forall t \in [0, T]$$

$$\Rightarrow \left\| \int_0^t S(t-t') |u|^{2k} u(t')\, dt' \right\|_{C_T H^s} \lesssim T \| u \|_{C_T H^s}^{2k+1}.$$

(iii) Recall that

$$\big\| \|\Psi\|_{C_T H^s_x} \big\|_{L^2(\Omega)} \le C_T \|\Phi\|_{HS(L^2; H^s)}$$

$$\Rightarrow \|\Psi\|_{C_T H^s_x} \le C(T, \|\Phi\|_{HS(L^2;H^s)}, \omega) < \infty, \text{ a.s.}$$

From (i), (ii), and (iii), we obtain, for $0 \le T \le 1$

$$\|\Gamma(u)\|_{C_T H^s_x} \le \underbrace{\boxed{\Big(\|u_0\|_{H^s} + \|\Psi\|_{C([0,1]; H^s_x)} \Big)}}_{=: \frac{1}{2} R_\omega} + \underbrace{\boxed{C_1 T \|u\|^{2k+1}_{C_T H^s}}}_{\le \frac{1}{2} R}$$

Similarly,

$$\|\Gamma(u) - \Gamma(v)\|_{C_T H^s_x} \le \overbrace{C_2 T \Big(\|u\|^{2k}_{C_T H^s} + \|v\|^{2k}_{C_T H^s} \Big)}^{\le 1/2} \|u - v\|_{C_T H^s_x}.$$

$$|u|^{2k} u - |v|^{2k} v = P_{2k}(u, \bar{u}, v, \bar{v})(u - v) + Q_{2k}(u, \bar{u}, v, \bar{v})(\bar{u} - \bar{v})$$

poly of degree $2k$

Let $u, v \in \overline{B_R} \subset C_T H^s_x$.

Ⓐ $C_1 T \|u\|^{2k+1}_{C_T H^s} \le C_1 T R^{2k+1} \le \frac{1}{2} R$

Ⓑ $C_2 T \left(\|u\|^{2k}_{C_T H^s} + \|v\|^{2k}_{C_T H^s} \right) \le 2 C_2 T R^{2k} \le \frac{1}{2}$

- Banach fixed pt thm: X = complete metric space.

T = contraction on X : $T : X \circlearrowleft$

$d(Tx, Ty) \le \theta\, d(x, y)$

for some $0 < \theta < 1$, $\forall x, y \in X$.

Then, $\exists!\, x \in X$ s.t. $Tx = x$ $\Leftarrow$ fixed pt.

Now, set $R = R(\omega) = 2 \left(\|u_0\|_{H^s} + \|\Psi\|_{C([0,1]; H^s_x)} \right)$.

$\Rightarrow$ choose $T = T(R_\omega) > 0$ small s.t. Ⓐ & Ⓑ hold.

$\Rightarrow$ By the Banach fixed pt thm, $\exists!\, u = u^\omega \in \overline{B}_{R^\omega}$ 11

s.t. $\quad u = \Gamma(u) \quad$ on $[0, T_\omega]$.

· <u>Stability?</u>

Take $u_{0,1}, u_{0,2} \in H^s$, $\quad \Phi_1, \Phi_2 \in HS(L^2; H^s)$

$$\Downarrow$$

$$\Psi_j = \int_0^t S(t-t')\Phi_j \, dW(t')$$

$\Rightarrow$ Let u_j, $j=1,2$, be the solns to (SNLS):

$$\begin{cases} i\partial_t u_j - \Delta u_j = |u_j|^{2k} u_j + \Phi_j \xi \\ u_j|_{t=0} = u_{0,j}. \end{cases}$$

A slight modification gives

$$\underbrace{\| u_1 - u_2 \|_{C_T H^s_x}} = \| \Gamma_{u_{0,1}, \Phi_1}(u_1) - \Gamma_{u_{0,2}, \Phi_2}(u_2) \|_{C_T H^s_x}$$

$$\leq \| u_{0,1} - u_{0,2} \|_{H^s} + \| \Psi_1 - \Psi_2 \|_{C_T H^s_x}$$

$$+ \underbrace{\frac{1}{2} \| u_1 - u_2 \|_{C_T H^s_x}}_{\text{hide in LHS.}} \quad \text{by choosing } T = T_\omega \text{ small.}$$

$$\Rightarrow \boxed{\| u_1 - u_2 \|_{C_T H^s_x} \lesssim \| u_{0,1} - u_{0,2} \|_{H^s} + \| \Psi_1 - \Psi_2 \|_{C_T H^s_x}}$$

Also, we have

(**) $$\Big\| \, \| \Psi_1 - \Psi_2 \|_{C([0,T]; H^s_x)} \Big\|_{L^p(\Omega)} \lesssim p^{1/2} \| \Phi_1 - \Phi_2 \|_{HS(L^2; H^s)}$$

$\forall$ finite $p \geq 1$

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$$\left(\Leftarrow \quad \Psi_1 - \Psi_2 = \int_0^t S(t-t')\,(\Phi_1 - \Phi_2)\,dW(t') \right.$$

By Chebyshev's inequality,

$$\| \Psi_1 - \Psi_2 \|_{C([0,1];H^s_x)} \leq K \, \| \Phi_1 - \Phi_2 \|_{HS(L^2;H^s)}$$

outside a set of probability $< C e^{-cK^2}$.

$\Longrightarrow$ LWP when $\boxed{s > \frac{d}{2}}$

$u_0 \in H^s \quad , \quad \Phi \in HS(L^2; H^s)$

Rmk: • same proof on $\mathbb{T}^d$.

• At this point, we proved uniqueness of the soln only in $\overline{B_R} \subset C_T H^s_x$, but the uniqueness in fact holds in the entire $C_T H^s_x$. (unconditional uniqueness).

Method 1: Observe that if $u \in C_T H^s_x$, then

$$\lim_{t \to 0+} \|u(t)\|_{H^s} = \|u_0\|_{H^s} \leq \frac{1}{2} R.$$

Thus, by shrinking the time, we can invoke the uniqueness in $\overline{B_R}$. But, we must ensure that we can basically keep the same local existence time (up to a constant factor)

$\Leftarrow$ use a continuity argument / bootstrap argument

Method 2: Apply the Gronwall ineq to the difference of two given solns.

Appendix: $\circledast$ on page 9 and $\circledast\circledast$ on page 12.

$\cdot$ $\circledast$ on page 9:

Claim: $\big\| \|\Phi\|_{C_T H^s_x} \big\|_{L^2(\Omega)} \leq C_T \|\Phi\|_{HS(L^2; H^s)}$

Write $\Phi(t) = \Phi(t) - \underbrace{\Phi(0)}_{=0}$

and let $F(t) = \Phi(t) \big/ \|\Phi\|_{HS(L^2; H^s)}$.

Then, using the computation on page 14 of Lec 2, we have

$$\| F(t_1) - F(t_2) \|_{L^p(\Omega; H^s)} \lesssim p^{1/2} \| F(t_1) - F(t_2) \|_{L^2(\Omega; H^s)}$$

$$\lesssim p^{1/2} |t_1 - t_2|^{1/2}.$$

Then, by Kolmogorov's conti criterion, we have

$$P\left(\sup_{0<t\le T} \frac{\|F(t)-F(0)\|_{H^s}}{|t|^{\alpha/p-\varepsilon}} \ge \lambda\right) \le \frac{C_1}{\lambda^p}$$

where $p \gg 1$ and $\alpha = \frac{p}{2} - 1$

Fubini (a.k.a. Layer-cake thm)

$$\Rightarrow \|F\|_{L^2(\Omega;\, C_T H^s_x)} = 2\int_0^\infty \lambda^1\, P\left(\|F\|_{C_T H^s_x} \ge \lambda\right) d\lambda$$

$$\lesssim 1 + \int_1^\infty \lambda^{1-p}\, d\lambda$$

for some $p \gg 1$

$$< \infty.$$

$$= P\left(\frac{\|F\|_{C_T H^s_x}}{T^{\alpha/p-\varepsilon}} \ge C_T \lambda\right)$$

$$\le P\left(\frac{\|F\|_{C_T H^s_x}}{|t|^{\alpha/p-\varepsilon}} \ge C_T \lambda\right)$$

$$\le C_T / \lambda^p$$

$$\Rightarrow \left\| \|\Phi\|_{C_T H^s_x} \right\|_{L^2(\Omega)} \le C_T \|\Phi\|_{HS(L^2;\, H^s)}$$

i.e. (*) on page 9.

· (**) on page 17:

A similar computation on pp. 15-16 gives

$$\Big\| \, \|\Psi_1 - \Psi_2\|_{C([0,1]; H^s_x)} \Big\|_{L^p(\Omega)} \leq C_p \, \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}$$

for any finite $p \geq 1$

$\Rightarrow$ By Chebyshev's ineq,

$$\|\Psi_1 - \Psi_2\|_{C([0,1]; H^s_x)} \leq K \, \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}$$

outside a set of probability $\lesssim 1/K^p$ (a bit weaker than p. 13 but OK.

Q: $C_p \sim p^{1/2}$: Yes, but need to work harder.

In the following, let $\Psi = \Psi_1 - \Psi_2$

$\Phi = \Phi_1 - \Phi_2$

- Given $k \in \mathbb{Z}_{\geq 0}$, let $\{t_{l,k} : l = 0, 1, \cdots, 2^k\}$ be 2^k+1 equally spaced pts in $[0,1]$, i.e. $t_{0,k} = 0$

$$t_{l,k} - t_{l-1,k} = 2^{-k}$$

Then, we have (with $\Psi(0) = 0$)

$$\boxed{\Psi(t) = \sum_{k=1}^{\infty} \left(\Psi(t_{l_k,k}) - \Psi(t_{l_{k-1},k-1}) \right)}$$

for some $l_k = l_k(t) \in \{0, 1, \cdots, 2^k\}$.

(Binary expansion of $t \in [0,1]$: $t = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$, $b_k \in \{0,1\}$

$\Rightarrow t_{l_k,k} = \sum_{j=1}^{k} \frac{b_j}{2^j}$ (binary exp of t up to order k

$$\Rightarrow \sup_{0 \leq t \leq 1} \|\Psi(t)\|_{H^s} \leq \sum_{k=1}^{\infty} \underline{\max_{0 \leq l_k \leq 2^k}} \|\Psi(t_{l_k,k}) - \Psi(t_{\underline{l'_{k-1},k-1}})\|_{H^s}$$

where $|t_{l_k,k} - t_{l'_{k-1},k-1}| \leq 2^{-k}$

Fix $p \gg 1$. Then, we have

$$\Big\| \, \|\Psi\|_{C([0,1];H^s)} \Big\|_{L^p(\Omega)}$$

$$\leq \sum_{k=1}^{\infty} \underbrace{\Big\| \max_{0 \leq l_k \leq 2^k} \|\Psi(t_{l_k,k}) - \Psi(t_{l'_{k-1},k-1})\|_{H^s} \Big\|_{L^p(\Omega)}}$$

replace max by ℓ^q-sum ($q \geq p$)

$$\leq \Big(\int \sum_{l_k=0}^{2^k} \| \cdots \|_{H^s}^q \, dP(\omega) \Big)^{1/q}$$

$$\lesssim \underline{\underline{2^{k/q}}} \max_{0 \leq l_k \leq 2^k} \|\Psi(t_{l_k,k}) - \Psi(t_{l'_{k-1},k-1})\|_{L^q(\Omega; H^s)}$$

$\lesssim 1$ for $q \gtrsim k$. $\Longrightarrow$ choose $q(k) = p + k$.

$$\lesssim \sum_{k=1}^{\infty} \cancel{\max_{0 \leq l_k \leq 2^k}} \underline{2^{-\frac{k}{2}}} (p+k)^{\frac{1}{2}} \|\Phi\|_{HS(L^2; H^s)}$$

from p.14 of Lec 2.

$|t_{l_k,k} - t_{l'_{k-1},k-1}|^{1/2}$

Use $(p+k)^{1/2} \lesssim p^{1/2} \cdot k^{1/2}$.

$$\Rightarrow \Big\| \|\Phi\|_{C_T H^s_x} \Big\|_{L^p(\Omega)} \lesssim p^{1/2} \|\Phi\|_{HS(L^2; H^s)} \underbrace{\sum_{k=1}^{\infty} 2^{-k/2} \cdot k^{1/2}}_{\lesssim 1.}$$

This shows (**) on p. 12.

($\Rightarrow$ Then, apply (exponential) Chebyshev to obtain the bound on p. 13.