Lec 2 : 01/02/22 (Tue)

Step 1: Step stoch process

does not peek in the future

$$f(t,\omega) = \sum_{j=1}^{n} \underline{a_{j-1}(\omega)}\, \mathbb{1}_{[t_{j-1}, t_j)}(t)$$

- a_j, $\mathcal{F}_{t_j}$ - meas.
- $\sum a_j^2 < \infty$

Define Ito integral by

$$I(f)(\omega) = \sum_{j=1}^{n} a_{j-1}(\omega)\left(B(t_j) - B(t_{j-1})\right)$$

(left endpt sum)

Then, ① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}\left[(I(f))^2\right] = \int_a^b \mathbb{E}[f^2]\, dt$.

Recall : conditional expectation

If $X \in L^1(\Omega; \mathcal{F})$ and $\mathcal{G} \subset \mathcal{F}$,

sub σ-field.

then define the conditional expectation of X given $\mathcal{G}$ by the unique r.v. Y s.t.

1. Y is $\mathcal{G}$-meas.
2. $\int_A X \, dP = \int_A Y \, dP, \quad \forall A \in \mathcal{G}$

Rmk: • Y is given by the Radon-Nikodym Thm.

$$\mu(A) = \int_A X \, dP, \quad A \in \mathcal{G}$$

$$\Rightarrow \mu \ll P|_{\mathcal{G}}$$

$$\overset{R\text{-}N}{\Rightarrow} d\mu = \underset{\in L^1}{Y} \, dP|_{\mathcal{G}}.$$

We denote Y by $\mathbb{E}[X \mid \mathcal{G}]$.

- If $X \in L^2(\Omega; \mathcal{F})$,

  then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{P}_{\underbrace{L^2(\Omega; \mathcal{G})}_{\text{closed.}}}(X)$

- Basic properties:

(i) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$

(often used in computing an expectation by conditioning

(ii) If X is $\mathcal{G}$-meas, $\mathbb{E}[X \mid \mathcal{G}] = X$.

(iii) If X and $\mathcal{G}$ are indep (i.e. $\{X \in U\}$ and $A \in \mathcal{G}$ are indep $\forall U \in \mathcal{B}_{\mathbb{R}}$, $A \in \mathcal{G}$.

$\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$

(iv) If Y is $\mathcal{G}$-meas, $\mathbb{E}[XY] < \infty$, then $\mathbb{E}[XY \mid \mathcal{G}] = Y\,\mathbb{E}[X \mid \mathcal{G}]$.

· Pf of ① & ② (on page ①)

① $$\mathbb{E}[\underbrace{a_{j-1}\,(B(t_j)-B(t_{j-1}))}]$$

$$\overset{(i)}{=} \mathbb{E}[\,\mathbb{E}[\,\cdots \mid \mathcal{F}_{t_{j-1}}]\,]$$

$$\overset{(iv)}{=} \mathbb{E}[\,a_{j-1}\,\underbrace{\mathbb{E}[B(t_j)-B(t_{j-1}) \mid \mathcal{F}_{t_{j-1}}]}\,] = 0$$

$$\overset{(iii)}{=} \mathbb{E}[\,B(t_j)-B(t_{j-1})] = 0$$

② $i < j$

$$\mathbb{E}[\,a_{i-1}\,a_{j-1}\,(B(t_i)-B(t_{i-1}))(B(t_j)-B(t_{j-1}))]$$

$$\overset{(i),(iv)}{=} \mathbb{E}[\,a_{i-1}\,a_{j-1}\,(B(t_i)-B(t_{i-1}))\,\underbrace{\mathbb{E}[B(t_j)-B(t_{j-1}) \mid \mathcal{F}_{t_{j-1}}]}_{=0}\,]$$

$$= 0$$

- $\underline{i=j}$ $\quad \mathbb{E}\left[\, a_{j-1}^2 \,(B(t_j) - B(t_{j-1}))^2 \,\right]$

(i),(iv),(iii)

$$= \mathbb{E}[\, a_{j-1}^2 \,]\,(t_j - t_{j-1})$$

$\Rightarrow$ sum over j.

$\boxtimes$

<u>Step 2:</u> FACT: Given $f \in L^2_{ad}([a,b]\times\Omega)$,

$\exists \{f_n\}$ of step stoch processes

converging to f in $L^2([a,b]\times\Omega)$.

$\Rightarrow$ Define the Ito integral:

$$I(f) = \int_a^b f(t)\, dB$$

$$:= \lim_{n\to\infty} I(f_n).$$

Properties: 0 I, linear

1 $\mathbb{E}[I(f)] = 0$

2 $\mathbb{E}[(I(f))^2] = \int_a^b \mathbb{E}[(f(t))^2]\, dt$ (Ito isometry)

3 $\mathbb{E}\left[\int_a^b f(t)\, dB \int_a^b g(t)\, dB\right]$

$$= \int_a^b \mathbb{E}[f(t)\, g(t)]\, dt$$

$I : L^2_{ad}([a,b] \times \Omega) \to L^2(\Omega)$ is an isometry.

· Ito's lemma: $dX = f\, dt + g\, dB$.

consider $F(X)$

"Taylor expansion of order 2"

Then, $\boxed{dF = \partial_x F\, dX + \frac{1}{2} \partial_x^2 F\, (dx)^2 = \partial_x F(f\, dt + g\, dB) + \frac{1}{2} \partial_x^2 F\; g^2\, dt}$

$(dt)^2 = 0$

$(dt)(dB) = 0$

$(dB)^2 = dt$

ex: $F(x) = \frac{x^2}{2}$. $\quad X = B$

$$\Rightarrow \quad \frac{B^2}{2}(t) = \int_0^t B\,dB + \frac{1}{2}t.$$

$$\Leftrightarrow \quad \int_0^t B\,dB = \frac{B^2}{2}(t) \underbrace{- \frac{1}{2}t}_{\text{Ito correction.}}$$

· Stratonovich integral: defined by the midpt sum.

$\Rightarrow$ NO Ito correction

ex: $\displaystyle\int_0^t B \circ dB = \frac{B^2(t)}{2}.$

Function spaces:

- Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

$$\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} = \text{Japanese bracket.}$$

- $W^{s,p}(\mathbb{R}^d)$ Sobolev space
  (= Bessel potential space L^p_s)

$$\|f\|_{W^{s,p}} = \| \langle \nabla \rangle^s f \|_{L^p}$$

$\underline{p=2}$

$H^s = W^{s,2}$

$$\widehat{\langle \nabla \rangle^s f}(\xi) = \langle \xi \rangle^s \hat{f}(\xi)$$

$\uparrow$ Bessel potential of order $-s$.

Note: standard in the study of dispersive PDEs but slightly different from $W^{s,p}$ obtained by real interpolation. $\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left(\iint \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}} \right)^{1/p}$, $0<s<1$

- Homogeneous Sobolev space

- $\dot{H}^s(\mathbb{R}^d)$: with $|\xi|^{2s}$ instead of $\langle\xi\rangle^{2s}$

- $\dot{W}^{s,p}(\mathbb{R}^d)$: $\|f\|_{\dot{W}^{s,p}} = \| \, |\nabla|^s f \, \|_{L^p}$

$$\underbrace{\widehat{|\nabla|^s f}}_{\text{Riesz potential of order } -s}(\xi) = |\xi|^s \hat{f}(\xi).$$

($\dot{W}^{s,p} = \dot{L}^p_s$ = Riesz potential space

Note: $\dot{H}^s$ is NOT a norm but is a semi-norm.

If $\hat{f}$ is a distribution supported at $\xi = 0$,

then $\|f\|_{\dot{H}^s} = 0$. $\quad s > 0$

Dirac δ and its derivatives

polynomials $\xleftarrow{\mathcal{F}^{-1}}$

$\Leftarrow$ Need to quotient out by polynomials.

- Sobolev inequality: $\boxed{\frac{s}{d} = \frac{1}{p} - \frac{1}{q}}$ $\quad 1 < p < q < \infty$

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)}$$

$$\text{On } \mathbb{T}^d, \quad \left\| f - \fint f\,dx \right\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{\dot{W}^{s,p}(\mathbb{T}^d)}$$

- Sobolev embedding: $\boxed{sp > d}$

$$\|f\|_{L^\infty} \lesssim \|f\|_{W^{s,p}}.$$

- Algebra property : $\underline{sp > d}$

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}} \|g\|_{W^{s,p}} \qquad 1 < p < \infty$$

($\underline{p=2}$: follows from Young's ineq, triangle ineq, and C-S.

$\underline{p \neq 2}$: Littlewood-Paley theory, paraproducts, vector-valued inequality

$$\left\| \left(\sum |P_k f_k|^2\right)^{1/2} \right\|_{L^p} \lesssim \left\| \left(\sum |f_k|^2\right)^{1/2} \right\|_{L^p}$$

$1 < p < \infty$.)

Now, back to the stochastic convolution

$$\Psi(t) = \int_0^t S(t-t')\big(\sigma(u)(t')\,\Phi\, dW(t')\big)$$

$$= \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\big(\sigma(u)(t')\,\Phi(e_n)\big)\, d\beta_n(t')$$

Φ smoothing operator

($\Leftarrow$ soln to linear stoch. Schrödinger eqn:

$$i\partial_t u - \Delta u = \sigma(u)\,\Phi\,\xi.$$

- Given an operator $T: X \to Y$, X, Y, Hilbert sp
we say T is Hilbert-Schmidt if

$$\|T\|_{HS(X;Y)} = \Big(\sum_n \|Te_n\|_Y^2\Big)^{1/2} < \infty$$

$\mathcal{L}^2(X,Y)$ $\qquad \{e_n\} = $ O.N.B. of X.

· **Additive case:** $\sigma(u) \equiv 1$

$$\Psi(t) = \int_0^t S(t-t')\,\Phi\, dW(t')$$

Prop: On $\mathbb{R}^d$. $\Phi \in HS(L^2; H^s)$, $s \in \mathbb{R}$.

Then, (i) $\Psi \in C_t H^s_x$, a.s

$H^1 \subset L^r$

(ii) Given any $1 \le q < \infty$ and

finite $r \ge 2$ s.t. $r \le \dfrac{2d}{d-2}$ when $d \ge 3$,

we have

$$\Psi \in L^q_T W^{s,r}_x = L^q([0,T]; W^{s,r}(\mathbb{R}^d))$$

a.s. for any $T > 0$

Rmk: On $\mathbb{T}^d$,

(i) holds true.

and we also have $\Psi \in C_t W_x^{s-\varepsilon,\infty}(\mathbb{T}^d)$, $\forall \varepsilon > 0$
a.s.

Pf: (i) $\mathbb{E}\left[\|\Psi(t)\|_{H^s}^2\right] = \mathbb{E}\left[\|\langle\nabla\rangle^s \Psi(t)\|_{L^2}^2\right] = \iint \cdots \, dx \, dP$

$$\langle\nabla\rangle^s \Psi(t) = \sum_{n\in\mathbb{N}} \int_0^t S(t-t') \langle\nabla\rangle^s \Phi(e_n) \, d\beta_n(t')$$

$$= \int \sum_n \sum_m \mathbb{E}\left[\int_0^t \cdots d\beta_n(t_1) \, \overline{\int_0^t \cdots d\beta_m(t_2)} \right] dx$$

$$= 2 \sum_n \int_0^t \| \underbrace{S(\cancel{t-t'})}_{\text{multiplication by } e^{-i(t-t')|\xi|^2} \text{ on the Fourier side}} \Phi(e_n) \|_{H^s}^2 \, dt'.$$

$$= 2t \|\Phi\|_{HS(L^2; H^s)}^2.$$

To show $\Psi \in C_t H^s_x$ a.s, we use Kolmogorov's continuity criterion.

$$\mathbb{E}\left[\| \Psi(t+h) - \Psi(t) \|^2_{H^s}\right]$$

$$\begin{cases} \Psi(t+h)\overline{\Psi(t+h)} \to 2(\cancel{t}+h)\|\Phi\|^2_{HS} \\ -\Psi(t+h)\overline{\Psi(t)} \to \cancel{2t\|\Phi\|^2_{HS}} \\ -\Psi(t)\overline{\Psi(t+h)} \to \cancel{2t\|\Phi\|^2_{HS}} \\ +\Psi(t)\overline{\Psi(t)} \to \cancel{2t\|\Phi\|^2_{HS}} \\ \underline{h>0} \quad \mathbb{E}\left[\int_0^{t+h} \cdots d\beta_n(t') \overline{\int_0^t \cdots d\beta_m(t')}\right] \\ \to 2t \end{cases} \quad \underline{h>0}$$

$$= 2h\|\Phi\|^2_{HS}$$

Note that $\Psi(t)$ is Gaussian.

$$\mathbb{E}\left[\| \Psi(t+h) - \Psi(t) \|_{H^s}^p \right]$$

$$\leq C_p \left(\mathbb{E}\left[\| \Psi(t+h) - \Psi(t) \|_{H^s}^2 \right] \right)^{p/2}$$

$$\lesssim |h|^{p/2} \, \|\Phi\|_{HS(L^2; H^s)}^p$$

Take $p \gg 1$

$$\overset{\text{Kolmogrov}}{\Longrightarrow} \quad \Psi \in C_t H_x^s, \quad \text{a.s.}$$