Lec 14 26/04/22 (Tue) ①

- Rough path theory was originally introduced in terms of V^p = functions of bdd p-variations.

  FACT: • $C^\alpha \subset V^{1/\alpha}$, $0 < \alpha < 1$

  • If $f \in V^p_c$, then $\exists$ reparametrization τ s.t. $f \circ \tau \in C^{1/p}$.

- Rough diff eqn (RDE), RPDE

  Given an SDE (or SPDE) with a noise X which is rough in time (C^α, $\alpha \le 1/2$, V^p, $p \ge 2$), we lift the noise X to a rough path $(X, \mathbb{X})$ and study the original eqn, where an integral is interpreted as a rough integral

Digression: paracontrolled distributions

(Gubinelli – Imkeller – Perkowski '15)

Consider $dY = Y\,dX$ $\quad X \in C^\alpha, \ \alpha \le 1/2$

- Paracontrolled ansatz:

$$\boxed{Y = Y' \otimes\!\!\!\!< X + R}$$

$Y' \in C^\alpha$ "Gubinelli deriv"

$R \in C^{2\alpha}$.

paraproduct.

- Paraproduct decomposition: (Bony '81)

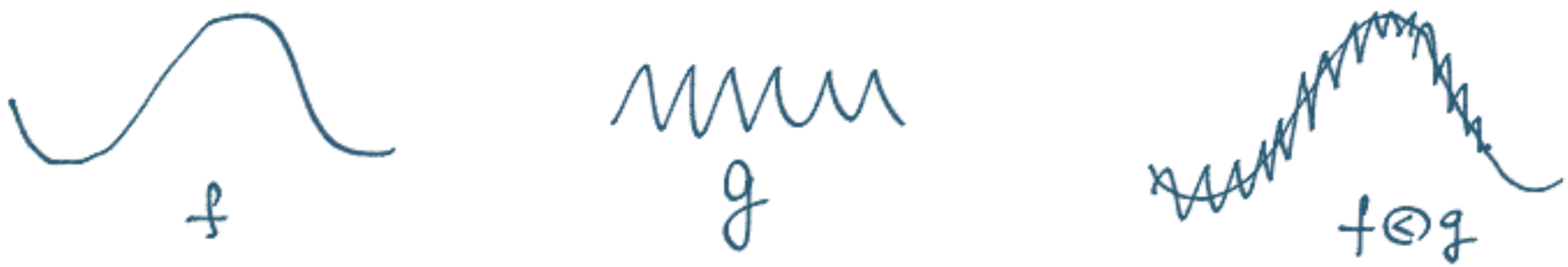
$$fg = \underline{f \,\circledless\, g} + f \circleddash g + f \circledgtr g$$

$$= \sum_{j<k-2} P_j(f)\, P_k(g) + \sum_{|j-k|\le 2} P_j(f)\, P_k(g) + \sum_{k<j-2} P_j(f)\, P_k(g)$$

- $f \olessthan g$ = paraproduct of g by f

  freq of $g \gg$ freq of f

locally looks like g.



Note: $\underset{\substack{\Uparrow \\ C^{\alpha_1}}}{f} \olessthan \underset{\substack{\Uparrow \\ C^{\alpha_2}}}{g}$ is always well defined.

$f \olessthan g \sim \min\{\alpha_2, \alpha_1 + \alpha_2\}$.

- $f \odot g$ = resonant product

  well defined if $\alpha_1 + \alpha_2 > 0$.

  If well defined, then $f \odot g \sim \alpha_1 + \alpha_2$

Back to the paracontrolled ansatz:

$$(*) \qquad Y = Y' \prec X + R \qquad\qquad X \in C^{\alpha}, \quad \alpha < 1/2.$$

$$=: Z + R$$

$$\underbrace{\partial_t Y = Y \,\partial_t X}_{(1/2-)+(-1/2-)\,<\,0} \quad = \underbrace{Y \prec \partial_t X}_{-1/2-} + \underbrace{Y \odot \partial_t X}_{\text{NOT well defined}} + \overbrace{Y \succ \partial_t X}^{0-}$$

$$\Longrightarrow \quad \partial_t Z = (Z + R) \prec \partial_t X \qquad \Longrightarrow \quad Z \sim 1/2-$$

$$\partial_t R = \underbrace{(Z + R) \succ \partial_t X}_{0-} + \overset{-1/2-}{Z \odot \partial_t X} + \underline{R \odot \partial_t X}.$$

$$\left(\text{i.e.} \quad \partial_t Y = \square \prec \partial_t X + \partial_t R \right)$$

By ignoring the resonant products, we expect $R \sim 1-$

$$\Longrightarrow \quad \underbrace{R \odot \partial_t X}_{(1-)+(-\frac{1}{2}-)\,>\,0} \quad \text{well defined.}$$

- <u>Issue:</u> $Z \circleddash \partial_t X$

$$(\tfrac12 -) + (-\tfrac12 -) < 0.$$

- Use the structure of Z.

$$Z(t) = \underset{\substack{\uparrow \\ \text{const}}}{Z_0} + \underbrace{\int_0^t (Z+R) \circledless \partial_t X(t')\, dt'}$$

const $\Rightarrow$ $Z_0 \circleddash \partial_t X$ is well defined.

- $$\left(\int_0^t (Z+R) \circledless \partial_t X(t')\, dt'\right) \circleddash \partial_t X$$

$$= \left[(Z+R) \circledless \delta X_{t0}\right] \circleddash \partial_t X + \underbrace{\overset{\frac12 + 2\varepsilon}{\mathrm{com}_1} \circleddash \overset{-\frac12 - \varepsilon}{\partial_t X}}_{\text{good}}$$

$$= (Z+R) \circledless \big(\underbrace{\delta X_{t0} \circleddash \partial_t X}\big) + \mathrm{com}_1 \circleddash \partial_t X + \mathrm{com}_2$$

$$\mathrm{com}_2 = [\circledless, \circleddash](Z+R, \delta X_{t0}, \partial_t X)$$

View this as a pad of given data.

$$\mathbb{X}_{tr} = \int_r^t \delta X_{ur}\, dX_u$$

Now, all the terms make sense.

$$\partial_t Z = (Z+R) \circledcirc\!\!\!<\, \partial_t X$$

$$\partial_t R = (Z+R) \circledcirc\!\!\!> \partial_t X + R \circleddash \partial_t X + Z_0 \circleddash \partial_t X$$
$$+ (Z+R) \circledcirc\!\!\!< (\delta X_{t_0} \circleddash \partial_t X) + \mathrm{com}_1 \circleddash \partial_t X + \mathrm{com}_2$$

- Solve the system for Z and R by a standard contraction argument.

  (strictly speaking, we need to multiply by a (smooth) cutoff in time to prove LWP.

<u>Summary:</u>

paracontrolled: $X \overset{①}{\longmapsto} (\partial_t X, \delta X_{t_0} \circleddash \partial_t X) \overset{②}{\longmapsto} (Z, R)$

$\longmapsto Y = Z + R$

controlled path: $X \overset{①}{\longmapsto} (X, \mathbb{X}) \overset{②}{\longmapsto} (Y', R) \;(\text{or } (Y, Y'))$

$\longmapsto Y$

① : stochastic analysis to lift X to an enhanced data set / rough path

② : deterministic analysis (<u><u>NO</u></u> probability)

- Pathwise local well-posedness of SNLS with a multip. noise on $\mathbb{T}^d$.

$$\begin{cases} i\partial_t u = \Delta u + \mathcal{N}(u) + \underline{u}\,\Phi\,\xi & \text{on } \mathbb{T}^d \\ u|_{t=0} = u_0 \end{cases}$$

linear in u. (but nonlinear in noise)

- Interaction representation : $v(t) = S(-t)\,u(t) = e^{it\Delta}u(t)$
- For simplicity, set $\mathcal{N}(u) = 0$.

$$v(t) = u_0 - i\int_0^t S(-t')\left(S(t')v(t')\underbrace{\Phi\, d\xi(t')}_{=dX_{t'}}\right)$$

Set $Y_t = v(t)$ and write

$$Y_t = Y_r - i\int_r^t \underline{S_{-t'}}\, dX_{t'}\, \underline{S_{t'}}\,\underline{\underline{Y_{t'}}}$$

$$D_t^\alpha e^{it|n|^2} \sim \underline{\underline{|n|^{2\alpha}}}\, e^{it|n|^2}$$

make things harder : need to lose spatial reg. to compute C_t^α-norms.

Young case: $X \in C^\alpha_t H^s$, $\alpha > 1/2$ $\qquad \Phi \in HS(L^2; H^s)$

$$\int_r^t S_{-t'}\, dX_{t'}\, S_{t'}\, Y_{t'} = \int_r^t S_{-t'}\, dX_{t'}\, S_{t'}\, Y_r$$

$$+ \boxed{-i \int_r^t S_{-t'}\, dX_{t'}\, S_{t'} \int_r^{t'} S_{-t''}\, dX_{t''}\, S_{t''}\, Y_{t''}}$$

$$\Rightarrow \quad \delta \boxed{} = -\delta \Big(\underbrace{\boxed{\int_r^t S_{-t'}\, dX_{t'}\, S_{t'}}}_{=: X^1_{tr} \text{ operator}} Y_r \Big)$$

$$= X^1_{t_1 t_2}\, \delta Y_{t_2 t_3} = X^1 \delta Y$$

(XX) • If X^1 maps H^s to H^s with $\boxed{\| X^1_{tr} \|_{\mathcal{L}(H^s; H^s)} \lesssim |t-r|^\alpha}$,

then

$$\delta \boxed{} = X^1 \delta Y \in C_3^{\boxed{2\alpha} > 1}$$

$$|t_1 - t_2|^\alpha \qquad |t_2 - t_3|^\alpha$$

denote the class by $C^\alpha_2 \mathcal{L}(H^s; H^s)$

By the sewing lemma, we obtain

$$\int_r^t S_{-t'}\, dX_{t'}\, S_{-t'} Y_{t'} = (Id - \Lambda\delta)\, \underset{=}{X'} Y$$

operator

- $\underline{Q}$: How to verify (**) ?

$$\widehat{X'_{tr} f}(n) = \sum_{n = n_1 + n_2} \int_r^t e^{-it'(|n|^2 - |n_2|^2)} \underbrace{\phi(n_1)\, d\beta^H_{n_1}(t')}_{\Leftarrow\, dX} \hat{f}(n_2)$$

- when X is a fBM in time with a Hurst parameter $H > 1/2$

(slightly smoother in time than a BM

<u>Assume:</u>

- $\widehat{\Phi f}(n) = \phi(n) \hat{f}(n)$

and $\Phi \in HS(L^2; H^s)$, $\underline{s > d/2}$

algebra

Main tool: random matrix / tensor estimate

from Deng–Nahmod–Yue . Invent. Math. '21
Oh – Wang – Zine SPDE '22.

$\Rightarrow \quad X' \in C_2^{\alpha} \mathcal{L}(H^s; H^s), \quad \alpha > 1/2 \qquad (\alpha = H-)$

$\rightarrow$ perform a contraction argument as in Lec 13.

· Rough case : $X = \Phi W, \quad \Phi \in HS(L^2; H^s), \quad s > d/2.$

L^2- cylindrical Wiener process

$$Y_t = Y_r - i \underbrace{\int_r^t S_{-t'}\, dX_{t'}\, S_{t'}}_{X'_{tr}} (D^{-\varepsilon} Y_{t'})$$

Controlled path : $$\boxed{\begin{aligned} \delta Y_{tr} &= X'_{tr} Y'_r + R^Y_{tr} \\ &= -i\, X'_{tr} D^{-\varepsilon} Y_r + R^Y_{tr} \end{aligned}}$$

(***)

$$\int_r^t S_{-t'}\, dX_{t'}\, S_{-t'}\, D^{-\varepsilon} Y_{t'}$$

$$\overset{(***)}{=} \underbrace{\int_r^t S_{-t'}\, dX_{t'}\, S_{-t'}}_{X^1_{tr}} D^{-\varepsilon} Y_r$$

$$- i \underbrace{\int_r^t S_{-t_1}\, dX_{t_1}\, S_{t_1}\, X^1_{t' r}}_{= \mathbb{X}^2_{tr}} D^{-2\varepsilon} Y_r$$

$$+ \boxed{X^1_{tr} R_{\cdot r}}$$

Remark: $(X^1, \mathbb{X}^2)$ is an operator-valued rough path adapted to the Schrödinger flow, satisfying

$$\mathbb{X}^2_{t_1 t_3} - \mathbb{X}^2_{t_1 t_2} - \mathbb{X}^2_{t_2 t_3} = X_{t_1 t_2} \circ X_{t_2 t_3}$$

composition of operators.

$$\delta\,\square = -\delta(X^1 D^{-\varepsilon} Y) + i\,\delta(\mathbb{X}^2 D^{-2\varepsilon} Y)$$

- $$-\delta(X^1 D^{-\varepsilon} Y)_{t_1 t_2 t_3} = X^1_{t_1 t_2}\,\delta D^{-\varepsilon} Y_{t_2 t_3}$$

$\alpha + 2\alpha = 3\alpha > 1$

$$\overset{(***)}{=} \cancel{-i\, X^1_{t_1 t_2} \circ X^1_{t_2 t_3} D^{-2\varepsilon} Y_{t_3}} + \underline{\underline{X^1_{t_1 t_2} D^{-\varepsilon} R_{t_2 t_3}}}$$

- $$i\,\delta(\mathbb{X}^2 D^{-2\varepsilon} Y) = i\,\mathbb{X}^2_{t_1 t_3} D^{-2\varepsilon} Y_{t_3} - i\,\mathbb{X}^2_{t_1 t_2} D^{-2\varepsilon} Y_{t_2} - i\,\mathbb{X}^2_{t_2 t_3} D^{-2\varepsilon} Y_{t_3}$$

$$\overset{\text{Chen}}{=} \underline{\underline{-i\,\mathbb{X}^2_{t_1 t_2}\,\delta D^{-2\varepsilon} Y_{t_2 t_3}}} + \cancel{i\, X_{t_1 t_2} \circ X_{t_2 t_3} D^{-2\varepsilon} Y_{t_3}}$$

$2\alpha + \alpha = 3\alpha > 1$ if $\alpha > 1/3$ (see next page)

$\Rightarrow$ Apply the sewing lemma:

$$\int_r^t S_{-t'}\, dX_{t'}\, S_{t'}\, Y_{t'} = (\mathrm{Id} - \Lambda\delta)(X^1 Y - i\,\mathbb{X}^2 Y)$$

- For this computation, we need

$$\mathbb{X}^1 \in C_2^\alpha \mathcal{L}(H^s; H^{s-\varepsilon})$$
$$\mathbb{X}^2 \in C_2^\alpha \mathcal{L}(H^s; H^{s-2\varepsilon})$$

$\Leftarrow$ fails when $\varepsilon = 0$

Thm: (Oh-Zheng '22). Let $\varepsilon \geq 0$, $\Phi \in HS(L^2; H^s)$, $s > d/2$.

Then, SNLS:

$$\begin{cases} i\partial_t u = \Delta u + \mathcal{N}(u) + u\Phi\xi \\ u|_{t=0} = u_0 \end{cases}$$

$\mathcal{N}(u)$: $|u|^{p-1}u$, $p \in 2\mathbb{N}+1$.

ξ: space-time white noise

- When $\varepsilon = 0$, this argument fails and we need another idea.
- heat case: Gubinelli-Tindel AOP '10.

Also, see Hairer-Pardoux JMSJ '15. for the regularity structure approach.