Lec 13 : 20/04/22 (Wed)

- So far, we studied SPDE with multiplicative noises in the Ito sense (i.e. where we interpreted the stoch conv in $L^2(\Omega)$), random field soln theory
- We now turn our attention to the pathwise well-posedness theory.

Issue: Consider $dY = Y\,dB$

First guess: $Y_1 = \int_0^t B\,dB$. ⇐ we can interpret this as an Ito-Wiener integral

BUT, $\nexists$ separable Banach space $X \subset C([0,1])$ s.t.

(i) $B \in X$, a.s.

(ii) $(f,g) \longmapsto \int_0^\cdot f(t)\,\partial_t g(t)\,dt$, defined for smooth functions, extends continuously to a map: $X \times X \to C([0,1])$

i.e. we can <u>not</u> construct $\int_0^t B\,dB$ pathwise.

<u>Main idea</u>: Augment the data B by $\mathbb{B}(s,t) = \int_s^t B(s,r)\,dB(r)$

LHS <u>defines</u> RHS.

$\Leftarrow$ rough path theory by Lyons '98.

(See my course notes from 2020,

- $(f, g) \longmapsto I(f,g)(t) = \int_0^t f(s)\,\partial_s g(s)\,ds$

  $C^{\alpha} \times C^{\beta} \longrightarrow C^{\gamma}$ $\qquad$ C^{α} = Hölder space.

- <u>Differential calculus pt of view</u>

  $I(f,g)$ is the unique soln to

  $$\partial_t I(f,g) = f \cdot \partial_t g \,, \qquad I(0) = 0$$

· Finite increment pt of view

little o.

$$(*)\quad \begin{cases} I(t) - I(s) = f(s)\,(g(t) - g(s)) + o(|t-s|), & 0 \le s \le t \le 1 \\ I(0) = 0 \end{cases}$$

unif in s.t.

- $(*)$ is clearly satisfied if $g \in C^1$, $f \in C$

$$I(t) - I(s) - f(s)\,(g(t) - g(s))$$

$$= \int_s^t \underbrace{(f(r) - f(s))}_{\to\, 0 \text{ unif}}\, \partial_r g(r)\, dr$$

$\to 0$ unif since f is unif conti on $[0,1]$.

- $(*)$ determines I.

Suppose J satisfies $(*)$. Set $D = I - J$

$$D(t) - D(s) = o(|t-s|) \Rightarrow D(t) \equiv D(0)$$

but $D(0) = 0$. $\Rightarrow$ $J \equiv I$.

$\Rightarrow$ I is the only function whose increment matches the "germ" $f(s)\,(g(t)-g(s))$ modulo a negligible error

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- For $n \geq 1$, $\quad C_n(V) = C(\Delta_n ; V)$ $\quad$ Think of $V = \mathbb{R}$

$$\Delta_n = \Delta_n(0,1)$$

$$\Delta_n(s,t) = \{(s_1, \cdots, s_n) : s \leq s_1 \leq \cdots \leq s_n \leq t\}$$

- n-cochain = element in $C_n(V)$
- coboundary operator $\delta : C_n(V) \to C_{n+1}(V)$

given by $$\delta f(s_1, \cdots, s_{n+1}) = \sum_{k=1}^{n+1} (-1)^{n-k} f(s_1, \cdots, \cancel{s_k}, \cdots, s_{n+1})$$

<u>ex</u>: $$\delta f(s,t) = f(t) - f(s)$$

$$\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u)$$

FACT: $\delta \circ \delta = 0$. $\left(\Rightarrow \operatorname{Im} \delta_{n-1} \subset \operatorname{Ker} \delta_n \right)$

cochain complex: $0 \to \mathbb{R} \to C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \xrightarrow{\delta}$

FACT: This complex is <u>exact</u>

i.e. $\operatorname{Im} \delta_{n-1} = \operatorname{Ker} \delta_n$

cohomology $H^n = \dfrac{\operatorname{Ker} \delta_n}{\operatorname{Im} \delta_{n-1}} = \{0\}$

If $\delta f = 0$, then $f = \delta g$ for some g.

- Let $A(s,t) = f(s)\, \delta g(s,t) = f(s)\left(g(t) - g(s)\right)$

$(*) \Longleftrightarrow \quad A = \delta I + R \qquad R = o(|t-s|)$

$\Rightarrow \quad \delta A = \delta R$

<u>Sewing map</u> allows us to recover R from $\delta A \in C_3$

- Topology on $\mathcal{C}_n$: We say $f \in \mathcal{C}_n^\alpha$ if

$$\|f\|_{\mathcal{C}_n^\alpha([s,t])} = \sup_{\Delta_n([s,t])} \frac{|f(s_1, \cdots, s_n)|}{|s_n - s_1|^\alpha} < \infty$$

See Gubinelli - Tindel AOP '10 for a variant norm

Set $\mathcal{C}_n^{\alpha+} = \bigcup_{\beta > \alpha} \mathcal{C}_n^\beta$

- Rmk : $\delta \mathcal{C}_1 \cap \mathcal{C}_2^{1+} = \{0\}$ $\quad (\Leftarrow \mathcal{C}_{\text{Hölder}}^{1+} = \{\text{const}\})$

Thm (Sewing Lemma) $\exists! \Lambda : \mathcal{C}_3^{1+} \cap \delta \mathcal{C}_2 \to \mathcal{C}_2^{1+}$

s.t. $\quad \delta \Lambda = \mathrm{Id}_{\mathcal{C}_3 \cap \delta \mathcal{C}_2}$

and $\quad \|\Lambda h\|_{\mathcal{C}_2^\alpha(I)} \leq c_\alpha \|h\|_{\mathcal{C}_3^\alpha(I)}$, $\quad h \in \mathcal{C}_3^\alpha \cap \delta \mathcal{C}_2$

$\forall$ closed interval $I \subset \mathbb{R}_+$

$\forall \alpha > 1.$

Pf: See my course notes from 2020.

or Gubinelli - Tindel '10.

Gubinelli JFA '04
Feyel - de la Pradelle
EJP '06

We have

$$A = \delta I + R$$

$$\Rightarrow \delta A = \delta R$$

$$\boxed{\text{Assume } \delta A \in C_3^{1+} \cap \delta C_2}$$

$$\Rightarrow \underbrace{R}_{o(|t-s|)} = \underbrace{\Lambda \delta A}_{1+} + \delta f \longrightarrow \delta f \equiv 0. \text{ b/c } \delta f = o(|t-s|)$$

$$\left(\Leftarrow \delta(R - \Lambda\delta A) = \delta R - \underbrace{(\delta\Lambda)}_{= Id}\delta A = \delta A - \delta A = 0 \right.$$

$$\Rightarrow R = \Lambda \delta A.$$

$$\Rightarrow \delta I = (Id - \Lambda\delta) A$$

$\boxed{\text{· Young integral:}}$ $f \in C^{\alpha}$, $g \in C^{\beta}$. $\boxed{\alpha + \beta > 1}$

$$\delta I(f,g)(s,t) = \int_s^t f(u)\, dg(u)$$

$$\circledast \Rightarrow \quad = \underbrace{\underbrace{\int_s^t f(s)\, dg(u)}_{= f(s)\,\delta g(s,t)}}_{= A} + \underbrace{o(|t-s|)}_{= -R}$$

$$\cdot \quad \delta A(s,u,t) = -\underbrace{\delta f(s,u)}_{O(|u-s|^{\alpha})}\, \underbrace{\delta g(u,t)}_{O(|t-u|^{\beta})} \in C_3^{\overbrace{\alpha+\beta}^{>1}}$$

Sewing lemma

$$\Longrightarrow \quad \delta I(f,g) = (\mathrm{Id} - \Lambda\delta)(f\,\delta g)$$

$$\Rightarrow \underline{I(f,g)(t)} = I(f,g)(t) - I(f,g)(0)$$

$$= \sum_{\substack{P \\ \uparrow \text{ partition of } [0,t]}} \Big(f(t_i)(g(t_{i+1}) - g(t_i)) + \underbrace{\Lambda\delta(f\delta g)(t_i, t_{i+1})}_{|t_{i+1}-t_i|^{1+\varepsilon}} \Big)$$

$$= \underline{\lim_{|P|\downarrow 0} \sum_P f(t_i)(g(t_{i+1}) - g(t_i))}$$

ex: fBM, $H > 1/2$

- Now, consider $dY = Y\,dX$ $\qquad X \in C^\alpha,\ \alpha > 1/2$

$$\Rightarrow Y_t = Y_r + \int_r^t Y_u\,dX_u$$

Assume $Y \in C^\alpha$

$$= Y_r + \int_r^t Y_r\,dX_u + \boxed{\int_r^t \delta Y_{ur}\,dX_u}$$

$A_{tr} = A(r,t)$

$$\int_r^t Y_u \, dX_u = \underbrace{\int_r^t Y_r \, dX_u}_{Y_r \delta X_{tr}} + \square$$

Take δ

$$\Longrightarrow \quad \delta \square_{t_1 t_2 t_3} = -\delta(Y\delta X)_{t_1 t_2 t_3} = \delta X_{t_1 t_2} \delta Y_{t_2 t_3}$$

$$= \mathcal{O}(|t_1 - t_3|^{2\alpha})$$

$2\alpha > 1$

Sewing lemma

$$\Longrightarrow \quad \square = -\Lambda \delta (Y \delta X)$$

$$\Rightarrow \int_r^t Y_u \, dX_u = (Id - \Lambda\delta)(Y\delta X)_{tr}$$

$\Rightarrow$ We arrive at the following fixed pt problem:

$$Y_t - Y_r = (Id - \Lambda\delta)(Y\delta X)_{tr} =: (\delta \Gamma Y)_{tr}$$

$$\frac{|Y_t - Y_r|}{|t-r|^\alpha} \le \|Y\|_{L^\infty} \|X\|_{C^\alpha} + \underbrace{\|\Lambda(\delta Y \delta X)\|_{C_2^\alpha}}$$

$0 \le r \le t \le T \le 1$

$$\lesssim T^\alpha \|\Lambda(\delta Y \delta X)\|_{C_2^{2\alpha}}$$

$$\lesssim T^\alpha \|\delta Y \delta X\|_{C_3^{2\alpha}}$$

$$\lesssim T^\alpha \|Y\|_{C^\alpha} \|X\|_{C^\alpha}$$

Q: What about BM? $\quad X = \text{BM} \in C^\alpha \setminus C^{1/2}$, a.s. $\alpha < 1/2$.

$X \in C^\alpha(\mathbb{R}_+, V)$, $\alpha < 1/2$

$\mathbb{X} \in C_2^{2\alpha}(\mathbb{R}_+, V \otimes V)$, satisfying Chen's relation.

$$\mathbb{X}_{t_1 t_3} - \mathbb{X}_{t_1 t_2} - \mathbb{X}_{t_2 t_3} = \delta X_{t_1 t_2} \otimes \delta X_{t_2 t_3}$$

$(X, \mathbb{X})$, rough path.

· Controlled rough path : Gubinelli '04

Impose a structure on $Y \in C^\alpha$:

Gubinelli derivative

$$(\ast\ast) \qquad \delta Y_{tr} = Y'_r X_{tr} + R^Y_{tr} \qquad \boxed{Y' \in C^\alpha} \quad R^Y \in C_2^{2\alpha}.$$

" Locally, Y behaves like X".

If $X \in C^\alpha([0,T]; V)$ and $Y \in C^\alpha([0,T]; W)$
then $Y' \in C^\alpha([0,T]; \underbrace{L(V;W)}_{\text{lin bdd operator}})$

Consider $I_{tr} = \int_r^t Y_u \, dX_u$

$\Rightarrow$ By $(\ast\ast)$, we formally have

$$I_{tr} = \underbrace{Y_r \int_r^t dX_u}_{= \delta X_{tr} \cdot Y_r} + \underbrace{Y'_r \overbrace{\int_r^t X_{ur} \, dX_u}^{\text{undefined}}}_{= Y'_r \mathbb{X}_{tr}} + \boxed{\int_r^t R^Y_{ur} \, dX_u}$$

NOT welldefined

So, we $\boxed{\text{define}}$ I by $I(0) = 0$ and

$$(+) \qquad I_{tr} = \delta X_{tr} Y_r + \mathbb{X}_{tr} Y'_r + o(|t-r|)$$

As in ⊛ on page 3, ⊕ characterizes I_{tr}.

$\boxed{\text{Also, when } Y \text{ and } X \text{ are smooth, } \int_r^t Y_u\, dX_u \text{ satisfies } \oplus \text{ where } \mathbb{X}_{tr} = \int_r^t X_{ur}\, dX_u}$

$$\Rightarrow I_{tr} = \int_r^t Y_u\, dX_u$$

$$= \delta X_{tr} Y_r + \mathbb{X}_{tr} Y'_r + \boxed{}$$

$$\Rightarrow \delta \boxed{}_{t_1t_2t_3} = -\delta \left(\delta X_{tr} Y_r + \mathbb{X}_{tr} Y'_r \right)_{t_1t_2t_3}$$

$$= \delta X_{t_1t_2}\, \delta Y_{t_2t_3} - \delta(\mathbb{X}_{tr} Y'_r)_{t_1t_2t_3} =: \mathrm{I} - \mathrm{II}$$

- $\mathrm{I} \overset{(**)}{=} \cancel{\delta X_{t_1t_2}\, \delta X_{t_2t_3} Y'_{t_3}} + \underline{\delta X_{t_1t_2} R^Y_{t_2t_3}} \in C_3^{3\alpha}$
- $\mathrm{II} = \mathbb{X}_{t_1t_3} Y'_{t_3} - \mathbb{X}_{t_1t_2} Y'_{t_2} - \mathbb{X}_{t_2t_3} Y'_{t_3}$

$$= \underbrace{(\mathbb{X}_{t_1t_3} - \mathbb{X}_{t_2t_3}) Y'_{t_3}}_{\overset{\text{Chen}}{=} \mathbb{X}_{t_1t_2} Y'_{t_3} + \delta X_{t_1t_2} \delta X_{t_2t_3} Y'_{t_3}} - \mathbb{X}_{t_1t_2} Y'_{t_2} = \underbrace{-\mathbb{X}_{t_1t_2}\, \delta Y'_{t_2t_3}}_{\in C_3^{3\alpha}} + \cancel{\delta X_{t_1t_2}\, \delta X_{t_2t_3} Y'_{t_3}}$$

$$\star \Rightarrow \delta \,\square = \delta X_{t_1t_2} R^Y_{t_2t_3} - \mathbb{X}_{t_1t_2}\, \delta Y'_{t_2t_3} \in C_3^{3\alpha} \cap \delta C_2$$

Hence, for $\alpha > 1/3$, we can apply the sewing lemma

$$\Rightarrow \square = -\Lambda\delta\,(\delta X_{t\cdot r} Y_r + \mathbb{X}_{tr} Y'_r)$$

$$\Rightarrow \boxed{I_{tr} = \int_r^t Y_u\, dX_u = \left[(\mathrm{Id} - \Lambda\delta)(\delta X_{t_1t_2} Y_{t_2} + \mathbb{X}_{t_1t_2} Y'_{t_2})\right]_{tr}}$$

- Now, consider the following RDE (rough differential equ):

$$\begin{cases} dY = Y\,dX \\ Y|_{t=0} = Y_0 \end{cases}, \quad X \in C^\alpha, \quad \frac{1}{3} < \alpha \le 1/2$$

$$\Rightarrow Y_t - Y_r = \int_r^t Y_u\, dX_u \qquad \left(\Longrightarrow \delta Y_{tr} = Y_r \delta X_{tr} + \text{"error"} \Rightarrow Y' = Y \right.$$

$$(\dagger\dagger) \qquad = \left[(\mathrm{Id} - \Lambda\delta)(\delta X \cdot Y + \mathbb{X} \cdot Y')\right]_{tr}$$

- Study (++) for a controlled path $(Y, Y') \in \mathcal{D}_X^{2\alpha}$ satisfying (**) on page 12.

$\mathcal{D}_X^{2\alpha}$ is a Banach space under $(Y,Y') \longmapsto |Y_0| + |Y_0'| + \|(Y,Y')\|_{X,2\alpha}$

- Endow $\mathcal{D}_X^{2\alpha}$ with the seminorm

$$\|(Y,Y')\|_{X,2\alpha} = \|Y'\|_{C^\alpha} + \|R^Y\|_{C_2^{2\alpha}}$$

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const in time (smooth in time)

Set
$$\Gamma(Y,Y')(t) = \Big(\underbrace{\overset{=: Z_t}{\int_0^t Y_u \, dX_u}}_{\text{interpreted as in (++)}}, \underline{Y_t} \Big) + (Y_0, 0)$$

Z'_t, see next page.

In order to estimate the seminorm $\|\cdot\|_{X,2\alpha}$ of $\Gamma(Y,Y') = (\Gamma_1, \Gamma_2)$ we need to write Z_t (and hence Γ_1) as a controlled path

$$\delta Z_{tr} = \underline{Z'_r \, \delta X_{tr}} + R^Z_{tr}$$

Since $Z_t = \int_r^t Y_u \, dX_u$, we have (from $\oplus$ on page 12)

$$\underline{Z' = Y.}$$

- What is R^Z?

$$\boxed{R^Z_{tr}} = \delta Z_{tr} - Z'_r \, \delta X_{tr}$$

$$= \int_r^t Y_u \, dX_u - Y_r \, \delta X_{tr}$$

$$\overset{(++)}{=} \boxed{\mathbb{X}_{tr} Y_r - \left[\Lambda \delta \left(\delta X \cdot Y + \mathbb{X} Y' \right)\right]_{tr}}$$

- By writing $\Gamma(Y, Y') = (\Gamma_1(Y, Y'), \Gamma_2(Y, Y'))$, we have

$$\|\Gamma_1(Y, Y')\|_{C^\alpha} = \|Z'\|_{C^\alpha} = \|Y\|_{C^\alpha}$$

(A) $$\underset{(**) \text{ on p.12}}{\le} \underbrace{\|Y'\|_{L^\infty_T} \|X\|_{C^\alpha}}_{\le \left(|Y'_0| + T^\alpha \|Y'\|_{C^\alpha}\right) \|X\|_{C^\alpha}} + T^\alpha \|R^Y\|_{C_2^{2\alpha}}$$

$$\|\Gamma_2(Y,Y')\|_{C_2^{2\alpha}} = \|R^Z\|_{C_2^{2\alpha}}$$

B
$$\le \|\mathbb{X}\|_{C_2^{2\alpha}} \underbrace{\|Y\|_{L_T^\infty}} + \underbrace{\|\delta X_{t_1t_2} R^Y_{t_1t_2} - \mathbb{X}_{t_1t_2}\delta Y'_{t_2t_3}\|_{C^{2\alpha}}}_{T^\alpha\left(\|X\|_{C^\alpha}\|R^Y\|_{C_2^{2\alpha}} + \|\mathbb{X}\|_{C_2^{2\alpha}}\|Y'\|_{C^\alpha}\right)}$$

Note that

C
$$\|Y\|_{L_T^\infty} \le |Y_0| + T^\alpha \boxed{\|Y\|_{C^\alpha}} \leftarrow \text{use A to estimate}$$

- Hence, with $|||(Y,Y')||| := |Y_0| + |Y_0'| + \|(Y,Y')\|_{X,2\alpha}$, we have

$$|||\Gamma(Y,Y')||| \overset{A,B,C}{\le} \boxed{C_0(1+\|\mathbb{X}\|_{C_2^{2\alpha}})|Y_0| + \|X\|_{C^\alpha}\|\mathbb{X}\|_{C_2^{2\alpha}}|Y_0'|} =: \tfrac{1}{2}R$$

$\Gamma|_{t=0} = (Y_0, Y_0)$

$$+ \; c(\|X\|_{C^\alpha}, \|\mathbb{X}\|_{C_2^{2\alpha}})\, T^\alpha \big(\underbrace{\|R^Y\|_{C_2^{2\alpha}} + \|Y'\|_{C^\alpha}}_{= \|(Y,Y')\|_{X,2\alpha}}\big)$$

for any $0 \le T \le 1$.

Set $R := 2\left(c(1+\|\mathbb{X}\|_{C_2^{2\alpha}})|Y_0| + \|X\|_{C^\alpha}\|\mathbb{X}|_{C_2^{2\alpha}}|Y_0'|\right).$ 18

Then, by choosing $T = T(R, \|X\|_{C^\alpha}, \|\mathbb{X}\|_{C_2^{2\alpha}}) > 0$ suff. small,

we have

(D)
$$|||\Gamma(Y,Y')||| \leq \tfrac{1}{2}R + C(\|X\|_{C^\alpha}, \|\mathbb{X}\|_{C_2^{2\alpha}})T^\alpha \underbrace{\|(Y,Y')\|_{X,2\alpha}}_{\leq |||(Y,Y')|||}$$
$$\leq R$$

for any $(Y,Y') \in B_R \subset \underbrace{\mathcal{D}_X^{2\alpha} \cap \{(Y,Y')|_{t=0} = (Y_0, Y_0')\}}_{\text{closed subset in } \mathcal{D}_X^{2\alpha}.}$

initial data $Y|_{t=0}$ given in the problem.

- Difference estimate:

(A′)
$$\|\Gamma_1(Y,Y') - \Gamma_1(\tilde{Y},\tilde{Y}')\|_{C^\alpha} = \|Y - \tilde{Y}\|_{C^\alpha}$$
$$\leq T^\alpha \|Y' - \tilde{Y}'\|_{C^\alpha}\|X\|_{C^\alpha} + T^\alpha \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\alpha}}$$

for any $(Y,Y'), (\tilde{Y},\tilde{Y}') \in B_R$ ← same initial dat

B′ $$\|\Gamma_2(Y,Y') - \Gamma_2(\tilde{Y},\tilde{Y}')\|_{C_2^{2\alpha}}$$
$$\leq \|\mathbb{X}\|_{C_2^{2\alpha}} \underline{\|Y-\tilde{Y}\|_{L_T^\infty}}$$
$$+ T^\alpha \left(\|X\|_{C^\alpha} \|R^Y - R^{\tilde{Y}}\|_{C_2^{2\alpha}} + \|\mathbb{X}\|_{C_2^{2\alpha}} \|Y' - \tilde{Y}'\|_{C^\alpha} \right)$$

C′ $$\|Y-\tilde{Y}\|_{L_T^\infty} \leq T^\alpha \underbrace{\|Y-\tilde{Y}\|_{C^\alpha}}_{\leftarrow \text{ use } A'}$$

$\Rightarrow$ By A′, B′, and C′, we have

D′ $$|||\Gamma(Y,Y') - \Gamma(\tilde{Y},\tilde{Y}')||| \leq \underbrace{C(\|X\|_{C^\alpha}, \|\mathbb{X}\|_{C_2^{2\alpha}})\, T^\alpha}_{\leq 1/2 \text{ by choosing } T \ll 1.} |||(Y,Y') - (\tilde{Y},\tilde{Y}')|||.$$

- From D & D′, we conclude that Γ is a contraction on B_R.