Recall from Lec 9 P11 (1D Cubic SNLS)

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u + u \Phi \xi \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}) \end{cases}$$

ξ: space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$

Φ: $\gamma(L^2_x ; L^\infty_x)$ see Lec 9, P12

Truncation: with Cut-off $\eta = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x \ge 2 \end{cases}$

smooth nonnegative on $\mathbb{R}_+$

$$\eta_R(u)(t) = \eta\left(\frac{\|u\|_{C([0,t]; L^2_x)} + \|u\|_{L^8([0,t]; L^4_x)}}{R} \right)$$

(8, 4)-Schrödinger admissible see Lec 3, P2

From Lec 9 P13–P21, Construction of the solution in $C_T L^2_x \cap L^8_T L^4_x$.

If T_{max} := maximal time of existence $< \infty$,
then $\lim_{t \uparrow T_{max}} \|u(t)\|_{L^2_x} = +\infty$

$\Rightarrow$ we have GWP
if $\sup_{0 \le t \le T_{max}} \|u(t)\|_{L^2_x} < \infty$

Main goal of this Lecture.

Ref: Le Gall's book or Revuz-Yor

Ito formula for d-dimensional Ito process $X_t = (X_t^{(1)}, \dots, X_t^{(d)}) \in \mathbb{R}^d$

with $X_t^{(j)} = X_0^{(j)} + \int_0^t \underbrace{Y_s^{(j)}}_{\text{adapted}} ds + \int_0^t \underbrace{Z_s^{(j)}}_{\text{adapted}} dB_s^{(j)}$

$(B_t^{(1)}, \dots, B_t^{(d)})_{t \ge 0}$ d-dimensional Brownian motion.

for $G \in C^2(\mathbb{R}^d)$, we have

(#a) $$G(X_t) - G(X_0) = \int_0^t \langle \nabla G(X_s), dX_s \rangle_{\mathbb{R}^d} + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{ij} G(X_s)\, d\langle X^{(i)}, X^{(j)} \rangle_s.$$

As $dX_s^{(i)} = Y_s^{(i)} ds + Z_s^{(i)} dB_s^{(i)}$ AND $d\langle X^{(i)}, X^{(j)}\rangle_s = \begin{cases} 0 & i \neq j \\ |Z_s^{(i)}|^2 ds & i = j \end{cases}$

then from #a,

$$G(X_t) - G(X_0) = \sum_{j=1}^{d} \int_0^t \partial_j G(X_s) \left[Y_s^{(j)} ds + Z_s^{(j)} dB_s^{(j)} \right] + \frac{1}{2} \sum_{j=1,\dots,d} \int_0^t \partial_{jj} G(X_s) |Z_s^{(j)}|^2 ds$$

$$= \underbrace{\int_0^t \left(\langle \nabla G(X_s), Y_s \rangle_{\mathbb{R}^d} + \frac{1}{2} \sum_{j=1}^{d} \partial_{jj} G(X_s) |Z_s^{(j)}|^2 \right) ds}_{\text{Lebesgue integral with random integrand}}$$

$$+ \underbrace{\sum_{j=1}^{d} \int_0^t \partial_j G(X_s) Z_s^{(j)} dB_s^{(j)}}_{\text{Ito integral}}$$

#b

★ Stratonovich integral (similar/related to Ito integral, defined as the limit of $\boxed{\text{Riemann sums}}$)

[P143, Revuz-Yor 3ed]

e.g. $\int_0^T Y_s \circ dB_s = \lim \sum_i \frac{Y_{t_i} + Y_{t_{i+1}}}{2} (B_{t_{i+1}} - B_{t_i})$

Stratonovich product

Brownian

when mesh size of the partitions of $[0,T]$ goes to 0.

While $\int_0^T Y_s \overset{\text{Ito}}{\cdot} dB_s = \lim \sum_i Y_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right).$

$\rightsquigarrow$ Chain rule $\quad g(B_t) - g(B_0) = \int_0^t g'(B_s)\circ dB_s$

AND #b can be rewritten as

$$G(X_t) - G(X_0) = \int_0^t \langle \nabla G(X_s), Y_s \rangle_{\mathbb{R}^d}\, ds + \sum_{j=1}^{d} \int_0^t \partial_j G(X_s) \circ dU_s^{(j)}$$

$\left(U_t^{(j)}\right)_{t\geq 0}$ is an Ito process given by $U_t^{(j)} = \int_0^t Z_s^{(j)} dB_s^{(j)}$

Back to <u>1D Cubic SNLS.</u>

stratonovich product

#C
$$\begin{cases} i\partial_t u = \Delta u + |u|^2 u + u \circ \underline{\Phi\xi} \\ u|_{t_0} = u_0 \end{cases}$$

$\rightarrow$ When the noise is real-valued, the mass $M(u(t)) = \|u(t)\|_{L_x^2}$ is formally conserved.

e.g. NLS models the wave propagation in fiber optics (medium for telecommunication and computer networking ......) ⑤
the deterministic equation preserves the mass.

The above SNLS $\approx$ NLS over random medium

stochastic forcing $u \circ \Phi \xi$

Also preserves the mass if the noise is real.

Explanation: formally for smooth u :

complex conjugate

$$\partial_t M(u)(t) = \partial_t \int_{\mathbb{R}} u(t)\,\bar{u}(t)\,dx$$

$$= \int_{\mathbb{R}} \partial_t [u(t)\,\overline{u(t)}]\,dx = 2\,\mathrm{Re} \int_{\mathbb{R}} \bar{u}(t)\,\partial_t u(t)\,dx$$

$$\overset{\text{equation}}{=} 2\,\mathrm{Re} \int_{\mathbb{R}} \bar{u}\,[\underbrace{-i\Delta u}_{\text{integration by part} \to i\|\partial_x u\|^2_{L^2(\mathbb{R})}} - i|u|^2 u - i u \underbrace{\Phi \xi}_{\text{real}}]\,dx = 2\,\mathrm{Re}(\text{pure imaginary number})$$

$$= 0.$$

The Ito formulation of #c $\Rightarrow$

$$\begin{cases} i\partial_t u = \Delta u + |u|^2 u + \underbrace{u \cdot \Phi\xi}_{\text{Ito product}} - \frac{i}{2} u F_{\Phi} \\ u|_{t=0} = u_0 \end{cases}$$

Stratonovich SNLS #d

De Bouard and Debussche '03 (real-valued noise ; Later we'll consider more general case)

$\Phi\xi$ = distributional derivative $\partial_t \Phi W$

$$\Phi W(t,x) = \sum_{k=0}^{\infty} \underbrace{\beta_k(t)}_{\text{iid real Brownian motions}} (\Phi e_k)(x), \quad (e_k)_{k\geq 0} = \text{orthonormal basis of } L^2(\mathbb{R};\mathbb{R})$$

$$\Phi \in \underbrace{\gamma}_{\gamma\text{-Radonifying}}(L^2(\mathbb{R};\mathbb{R}) ; L^\infty(\mathbb{R};\mathbb{R}))$$

$$\underbrace{F_\Phi(x) := \sum_{k=0}^{\infty} (\Phi e_k)^2(x)}_{\text{does not depend on the particular choice of orthonormal basis.}}$$

$\Bigg(\{h_k\}_{k\geq 0}$ another O.B. $\Rightarrow \displaystyle\sum_{k=0}^{\infty} (\Phi h_k)^2 = \sum_{k=0}^{\infty} \Big(\Phi \sum_{j=0}^{\infty} \langle h_k, e_j\rangle e_j\Big)^2$

$$= \sum_{k=0}^{\infty} \sum_{j,l=0}^{\infty} \langle h_k, e_j\rangle\langle h_k, e_l\rangle (\Phi e_j)\Phi e_l = \sum_{j=0}^{\infty} (\Phi e_j)^2. \Bigg)$$

Fact: ① Given Θ Fourier multiplier with symbol $\theta : \mathbb{R} \to [0,1]$ being an even function,

$$\widehat{\Theta v}(\xi) = \theta(\xi)\hat{v}(\xi).$$

Then, Θ is a bounded linear operator from $L^2(\mathbb{R};\mathbb{R})$ to $L^2(\mathbb{R};\mathbb{R})$.

Indeed,

$$\|\Theta v\|_{L^2(\mathbb{R})} \underset{\text{Plancherel}}{\sim} \|\theta(\xi)\hat{v}(\xi)\|_{L^2_\xi} \le \|\hat{v}(\xi)\|_{L^2_\xi} \sim \|v\|_{L^2_x}.$$

AND for $v \in L^2(\mathbb{R};\underline{\underline{\mathbb{R}}})$,

$$(\Theta v)(x) \sim \int_{\mathbb{R}} \theta(\xi)\hat{v}(\xi)e^{ix\cdot\xi}\,d\xi \in \mathbb{R}$$

$\hat{v}(-\xi) = \overline{v(\xi)}$, $\theta(-\xi) = \theta(\xi)$

② Θ as in ① AND assume $\Phi \in \gamma(L^2(\mathbb{R};\mathbb{R}); L^\infty(\mathbb{R};\mathbb{R})) \cap \underline{\gamma(L^2;L^2)}$ appear in P12 Lec9

$$\Rightarrow \quad F_{\Phi\Theta}(x) := \sum_{k=0}^{\infty} (\Phi\Theta e_k)^2(x) \in L^\infty(\mathbb{R}).$$

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By definition,

$$F_{\Phi\Theta}(x) = \sum_{k=0}^{\infty} \left(\Phi\Theta e_k\right)^2(x) = \mathbb{E}\left| \sum_{k=0}^{\infty} \underbrace{g_k}_{\text{iid real } N(0,1)} \Phi\Theta e_k(x) \right|^2$$

$$\Rightarrow \|F_{\Phi\Theta}\|_{L^q(\mathbb{R})} \underset{\substack{\text{Minkowski} \\ 2\le q<\infty}}{\le} \mathbb{E}\left[\left\| \sum_{k=0}^{\infty} g_k \Phi\Theta e_k \right\|^2_{L^{2q}(\mathbb{R})} \right]$$

$$\underset{\text{Hölder}}{\le} \|\Phi\Theta\|^{\frac{2}{q}}_{\gamma(L^2;L^2)} \|\Phi\Theta\|^{\frac{2(q-1)}{q}}_{\gamma(L^2;L^\infty)}$$

Letting $q \uparrow +\infty$, we get

$$\underline{\|F_{\Phi\Theta}\|_{L^\infty(\mathbb{R})} \le \|\Phi\Theta\|_{\gamma(L^2;L^\infty)}} \lesssim \|\Phi\|_{\gamma(L^2;L^\infty)}$$

$\Theta: L^2 \circlearrowleft$ bdd lin. ← Fact ①

use <u>ideal property</u>, <u>Lec 8, P7</u>

Now we look at the Ito- 1D Cubic SNLS as in previous Lectures

#e
$$\begin{cases} i\partial_t u = \Delta u + |u|^2 u + u \cdot \Phi \xi \\ u\big|_{t=0} = u_0 \in L^2_x \end{cases}$$

Ito product

Compare with the Stratonovich SNLS #d p6

Lec 7, p1. $u = u_R$ for $t \in [0, t_R]$, $t_* = \lim_{R \uparrow +\infty} t_R$

stopping time

solution to the truncated version of #e:

$$u(t) = S(t) - i\int_0^t S(t-t')\eta_R(u)(t')(|u|^2 u)(t')dt' - i\int_0^t S(t-t')\eta_R(u)(t')\, u(t')\Phi dW(t')$$

Ito integration

see also Lec 6, p6

Next: Further regularization in order to apply Ito formula.

Recall the cut-off function η from P1

define the following Fourier multipliers

$$\widehat{\Theta_k v}(\xi) = \eta\left(\frac{|\xi|}{k}\right)\hat{v}(\xi) \qquad k \in \mathbb{N} = \{1, 2, \dots\}$$

$$\widehat{S_k(t)v}(\xi) = \widehat{\Theta_k S(t) v}(\xi) = \eta\left(\frac{|\xi|}{k}\right) e^{it|\xi|^2}\hat{v}(\xi)$$

Consider $u = u_{R,m,N}$ with $m = (m_1, m_2) \in \mathbb{N} \times \mathbb{N}$

solution to the following regularized equation (in Duhamel formulation)

#Reg

$$u(t) = S_{m_1}(t)u_0 - i\int_0^t S_{m_1}(t-t')\,\eta_R(u)(t')\,\Theta_{m_2}\left[|\Theta_{m_2}u|^2\,\Theta_{m_2}u\right](t')\,dt'$$

$$- i\int_0^t S_{m_1}(t-t')\,\eta_R(u)(t')\,u(t')\,\Phi\,\Theta_{m_2}\,d\widetilde{W}_N(t')$$

noise with finite many components $\{\beta_0, \dots, \beta_N\}$

where the $\underbrace{\text{stochastic integral}}$ is understood as

$$\underbrace{-i\sum_{k=0}^{N}\int_0^t S_{m_1}(t-t')\,\eta_R(u)(t')\,u(t')\,(\Phi\,\Theta_{m_2}e_k)(x)\,d\beta_k(t')}_{\text{Ito integral}}.$$

$$\int_{\mathbb{R}}dx\int_0^t\left|S_{m_1}(t-t')\,\eta_R(u)(t')\,u(t')\,\underbrace{(\Phi\,\Theta_{m_2}e_k)(x)}_{L^\infty_x}\right|^2 dt' \lesssim 1 \quad \Uparrow$$

OR $\underbrace{du(t)}_{\text{Ito differential}} = -i\Theta_{m_1}\Delta u - i\eta_R(u)\,\Theta_{m_2}\left[|\Theta_{m_2}u|^2\Theta_{m_2}u\right] - i\,\eta_R(u)\,u\,\Phi\,\Theta_{m_2}\,dW_N(t)$

Ito differential for $u = u_{R,m,N}$, $u|_{t=0} = u_0$

$\underline{M(u)(t) = \int_{\mathbb{R}} u(t)\,\bar{u}(t)\,dx}$, and we apply Ito formula for $|u(t,x)|^2$

that is, $u(t,x)\,\bar{u}(t,x) = |u_0(x)|^2 + 2\,\mathrm{Re}\int_0^t u(t',x)\, d^{\text{Ito}} u(t',x)$ $\quad$ (12)

$+ \underbrace{\langle u(x), \bar{u}(x)\rangle_t}_{\text{co-variation process.}}$

$\displaystyle = \sum_{k=0}^{N}\int_0^t |u(t')|^2 \left(\overline{\Phi}\Theta_{m_2}e_k\right)^2(x)\,dt'$

$\displaystyle = \sum_{k=0}^{N}\left(\overline{\Phi}\Theta_{m_2}e_k\right)^2(x)\int_0^t |u(t',x)|^2\,dt'$

$\displaystyle \mathrm{Re}\int_0^t u(t',x)\, d^{\text{Ito}} u(t',x)$

$\displaystyle = \mathrm{Re}\int_0^t u(t',x)\Big[\, i\Theta_{m_1}\Delta u(t',x) + i\eta_R(u)(t')\,\Theta_{m_2}\big[|\Theta_{m_2}u|^2\Theta_{m_2}u\big](t')\Big]dt'$

$\displaystyle + \underbrace{\mathrm{Re}\sum_{k=0}^{N} i\int_0^t |u(t',x)|^2\,\eta_R(u)(t')\,\overline{\left(\Phi\Theta_{m_2}e_k\right)(x)}\, d\beta_k(t')}$ $\quad$ #f

→ Zero when the noise is real. ($\overline{\Phi}\Theta_{m_2}e_k$ real, see Fact ①, P7)

→ later, we consider non-real setting.

Therefore, (real noise setting), we have

#9
$$M(u)(t) = M(u_0) + 2\,\mathrm{Re}\int_0^t\int_{\mathbb{R}} u(t',x)\Big[\, i\,\overline{\Theta_{m_1}\Delta u(t',x)} + i\,\eta_R(u)(t')\,\overline{\Theta_{m_2}\left[|\Theta_{m_2}u|^2\Theta_{m_2}u\right](t')}\,\Big]\,dx\,dt'$$

Complex conjugate

by (i) & (ii)

purely imaginary

$$+\int_0^t\left(\int_{\mathbb{R}} \boxed{\sum_{k=0}^{N}\left(\Phi\Theta_{m_2}e_k\right)^2(x)}\;|u(t',x)|^2\,dx\right)dt'$$

L^∞_x-norm $\lesssim 1$, see P7,8

this term would be Cancelled in the Stratonovich-SMS see De Bouard et Debussche '03

(i) $\displaystyle -\int_{\mathbb{R}} \overline{u(t',x)}\,\Theta_{m_1}\Delta u(t',x)\,dx = \int_{\mathbb{R}} \Theta_{m_1}(\xi)\,|\xi|^2\,|\hat{u}(t',\xi)|^2\,d\xi \in \mathbb{R}$

(ii) $\displaystyle \underbrace{\eta_R(u)(t')}_{\text{real}}\int_{\mathbb{R}} \overline{u(t',x)}\,\Theta_{m_2}\left[|\Theta_{m_2}u|^2\Theta_{m_2}u\right](t')\,dx = \eta_R(u)(t')\int_{\mathbb{R}} |\Theta_{m_2}u|^4\,dx \in \mathbb{R}$

Thus #9 reduces to

$$M(u)(t) = M(u_0) + \ldots$$

$$\leq M(u_0) + C\int_0^t M(u)(t')\,dt'$$

→ the constant C does not depend on R, m_1, m_2, or N.

Grönwall $\Longrightarrow$ $\boxed{M(u)(t) \leq M(u_0)\, e^{Ct}}$

Because of cut-off $\eta_R(u) = \eta\left(\frac{\|u\|_{C([0,t];L^2_x)} + \|u\|_{L^8([0,t];L^4_x)}}{R}\right)$

with $\eta(x) = 0$ for $x \geq 2$

Finally send $m_1 \to +\infty$, then $m_2 \to +\infty$, $N \uparrow +\infty$ [i.e. remove regularizations]

$S_{m_1}(t)v \xrightarrow{L^2_x} S(t)v$

$\forall v \in L^2(\mathbb{R})$

send $R \uparrow +\infty$ ⟶ Goal on P2.

NOW: Non-Conservative Noise (i.e. the noise is not real-valued)

From the discussion on P12, we have ($u = u_{R,m,N}$)

$$M(u)(t) = M(u_0) + \mathrm{Re}\, i \sum_{k=0}^{N} \int_{\mathbb{R}} dx \int_0^t |u(t',x)|^2 \, \eta_R(u)(t') \, \overline{(\Phi \Theta_{m_2} e_k)(x)} \, d\beta_k(t') \quad \#f'$$

$$+ \int_0^t \left(\int_{\mathbb{R}} \sum_{k=0}^{N} (\Phi \Theta_{m_2} e_k)^2(x) \, |u(t',x)|^2 dx \right) dt' .$$

L^∞_x-norm $\lesssim 1$, see P7, 8

Fix any $q \in [2, \infty)$, $[a+b+c]^q \lesssim a^q + b^q + c^q \quad \forall a,b,c \in \mathbb{R}_+$

$$\Rightarrow M(u)(t)^q \lesssim M(u_0)^q + \left(t \int_0^t \frac{1}{t} \, \eta_R(u)(t') \, M(u)(t') \, dt' \right)^q + |\#f'|^q$$

$$\underset{\text{Jensen}}{\leq} M(u_0)^q + t^{q-1} \int_0^t \eta_R(u)(t') \, M(u)(t')^q \, dt' + |\#f'|^q . \quad \Bigg\} \#h$$

For any finite time $T>0$:

$$\mathbb{E}\Big[\sup_{0\le t\le T} \underbrace{\big|\#f'(t)\big|^{q}}_{\text{we ignore } \mathrm{Re}\, i}\Big] \le \mathbb{E}\left\{\left|\sup_{t\le T}\sum_{k=1}^{N}\int_0^t \eta_R(u)(t')\left[\int_{\mathbb{R}} \underbrace{\overline{(\Phi\Theta_{m_2}e_k)(x)}}_{\lesssim 1}\, |u(t',x)|^2\, dx\right] d\beta_k(t')\right|^{q}\right\}$$

$$\overset{\text{BDG}}{\lesssim_q} \mathbb{E}\left\{\left|\int_0^T \eta_R(u)^2(t')\underbrace{\sum_{k=1}^{N}\left(\int_{\mathbb{R}} \overline{\Phi\Theta_{m_2}e_k(x)}\,|u(t',x)|^2\, dx\right)^2}_{\left(\left\| \overline{\Phi\Theta_{m_2}e_k(x)}\, |u(t',x)|^2 \right\|_{\ell^2_{k\le N} L^1_x}\right)^2} dt'\right|^{\frac{q}{2}}\right\}$$

$$\underset{\text{Minkowski}}{\le} \Big\| \underbrace{\big\|\Phi\Theta_{m_2}e_k(x)\big\|_{\ell^2_k}}_{\text{in } L^\infty_x} |u(t',x)|^2 \Big\|^2_{L^1_x} \lesssim M(u)(t')^2$$

clean-up ....

$$\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le T} |\#f'(t)|^{q}\Big] \lesssim \mathbb{E}\left[\left|\int_0^T \eta_R(u)^2(t')\, M(u)^2(t')\,dt'\right|^{\frac{q}{2}}\right]$$

$$\overset{\text{Jensen}}{\le} T^{\frac{q}{2}-1}\,\mathbb{E}\int_0^T \underbrace{\eta_R(u)^{q}(t')}_{\le 1}\, M(u)^{q}(t')\,dt'$$

With #h in mind and $M^*_t = \sup_{r\le t} M(u)(r)$,

We deduce that

$$\mathbb{E}\,|M^*_T|^{q} \lesssim |M(u_0)|^{q} + T^{q-1}\int_0^T \mathbb{E}\big(|M^*_t|^{q}\big)\,\eta_R(u)(t)\,dt$$

$$+\; T^{\frac{q}{2}-1}\int_0^T \mathbb{E}\big(|M^*_t|^{q}\big)\,\eta_R(u)(t)\,dt$$

ensures finiteness of integrals

Grönwall

$$\rightsquigarrow \mathbb{E}\left[|M_T^*|^q\right] \leq \underline{C_1}\, M(\mu_0)^q \exp\left[\left(T^{q-1}+T^{\frac{q}{2}-1}\right)\underline{C_2}\right]$$

Constants do not depend on R, m_1, m_2, or N.

"passing these parameters to ∞" $\rightsquigarrow$ Goal on P2