Lec 11 05/04/22 (Tue)

- $W =$ K-cylindrical wiener process $\rightsquigarrow$ Filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ $\rightsquigarrow$ "usual conditions" $\left(\begin{array}{l} \mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} \ \forall t \\ \mathcal{F}_0 \text{ contains all } \mathbb{P}\text{-null sets} \end{array} \right)$
- For $\phi : \mathbb{R}_+ \times \Omega \to K \otimes B$ $\mathbb{F}$-adapted and simple,

$$\phi = \text{lin. comb. of } \mathbb{I}_{(s,t] \times \underset{\mathcal{F}_s}{A}} \otimes (\underset{K}{h} \otimes \underset{B}{b}), \quad s<t \quad \Rightarrow \quad \boxed{\int_0^\infty \phi \, dW := \text{lin. comb. of } \mathbb{I}_A W(\mathbb{I}_{(s,t]} \otimes h) b}$$

Lemma 2 (pg. Lec 10): B has MT_2 (martingale type 2) & ϕ simple adapted

Then
$$\boxed{\mathbb{E}\left[\left\| \int_0^\infty \phi \, dW \right\|_B^2\right] \lesssim \mathbb{E} \int_0^\infty \|\phi(t)\|^2_{\gamma(K,B)} dt.} \quad \#3$$

Today: **Thm (Doob's maximal ineq in MT_2 setting)**

(i) For ϕ adapted, simple: $\mathbb{E} \sup_{t \geq 0} \left\| \int_0^t \phi \, dW \right\|_B^2 \lesssim$ RHS of #3

(ii) Extension for progressive (progressively measurable) process with $\mathbb{E}\int_0^\infty \|\phi(t)\|^2_{\gamma(K,B)}dt < \infty$ #4

$\Downarrow$

$$M_t := \int_0^\infty \mathbb{1}_{[0,t]}\phi \, dW \in L^2(\Omega; C(\mathbb{R}_+;B))$$

Def. We say $\{\phi(s,\omega)\}_{s\in\mathbb{R}_+,\omega\in\Omega}$ is progressively measurable if $\forall T\in(0,\infty)$,

$$(s,\omega)\in[0,T]\times\Omega \mapsto \phi(s,\omega)\in\gamma(K,B) \text{ is } \mathcal{B}([0,T])\otimes\mathcal{F}_T\text{-measurable.}$$

In particular, simple, adapted processes are progressively measurable.

simple, adapted processes $\to$ generate the so-called predictable σ-algebra that is equivalent to the one generated by the adapted and (left) continuous processes.

(A predictable process is actually $\{\mathcal{F}_{t-} : t\in\mathbb{R}_+\}$-adapted, $\mathcal{F}_{t-} = \bigvee_{s<t}\mathcal{F}_s$)

$\bigvee_{s<t}\mathcal{F}_s$: smallest σ-algebra that contains $\bigcup_{s<t}\mathcal{F}_s$

simple, adapted processes $\xrightarrow{\text{dense in}}$

$$L^2(\text{prog.}) = \left\{\begin{array}{l}\text{progressively measurable}\\ \text{processes with \#4}\end{array}\right\}$$

$\Downarrow$

(ii) follows by standard density argument. Ch3 Ondreját '04 thesis.

<u>proof of (i)</u> From <u>page 10, lec 10</u>, if ϕ is simple, adapted process,

then $\displaystyle\int_0^\infty \phi\, dw$ = finite-sum of martingale differences

also with respect to the filtration generated by themselves.

$$= \sum_{k=1}^{N} d_k$$

i.e. $\mathbb{E}\left[d_{k+1} \mid \sigma\{d_1,\cdots,d_k\}\right] = 0$.

Then, by definition of $\int_0^\infty \phi\, dw$ for simple, adapted process,

we can see that

$$\int_0^t \phi\, dW = \int_0^\infty \mathbb{1}_{[0,t]}(s)\,\phi(s)\, dW(s) = \sum_{k=1}^{N_t} d_k .$$

N_t: just some integer nondecreasing in t.

d_k: of the same form as in Lec 10, p10

Then the inequality in (i) reduces to

$$\mathbb{E} \sup_{t\geq 0} \Big\| \underbrace{\sum_{k=1}^{N_t} d_k}_{} \Big\|_B^2 \lesssim \mathbb{E}\int_0^\infty \|\phi(t')\|^2_{\gamma(K,B)}\, dt' \overset{\text{Lec 10 p10}}{=} \lim_{N\uparrow\infty} \underbrace{\sum_{k=1}^{N} \mathbb{E}\|d_k\|_B^2}_{\text{Constant after some finite } N}$$

discrete sum

important!

$$\overset{\text{Monotone cvg.}}{=} \lim_{N\uparrow\infty} \mathbb{E} \sup_{n\leq N} \Big\| \sum_{k=1}^{n} d_k \Big\|_B^2$$

It remains to show for each $N \geq 1$,

$$\mathbb{E} \sup_{n \leq N} \Big\| \sum_{k=1}^{n} d_k \Big\|_B^2 \underset{\text{indep. of } N}{\lesssim} \sum_{k=1}^{N} \mathbb{E}\left[\|d_k\|_B^2\right].$$

In fact, $\left\{ X_n = \Big\| \sum_{k=1}^{n} d_k \Big\|_B : n \geq 1 \right\}$ is a real valued sub-martingale

$$\mathbb{E}\left[X_{n+1} \,\middle|\, d_1, \dots, d_n \right] = \mathbb{E}\left[\Big\| \sum_{k=1}^{n+1} d_k \Big\|_B \,\middle|\, d_1, \dots, d_n \right]$$

$$\underset{\text{Jensen}}{\geq} \left\| \mathbb{E}\left[\sum_{k=1}^{n+1} d_k \,\middle|\, d_1, \dots, d_n \right] \right\|_B = X_n .$$

P8, pisier '16 book

Jacod protter "probability Essentials" Thm 26.3

Then by Doob's L^2-inequality

$$\mathbb{E}\left(\max_{n \leq N} X_n^2 \right) \leq \left(\frac{2}{2-1} \right)^2 \mathbb{E}\left[X_N^2 \right]$$

$$\Rightarrow \mathbb{E} \sup_{n \leq N} \Big\| \sum_{k=1}^{n} d_k \Big\|_B^2 \leq 4\, \mathbb{E}\left[\Big\| \sum_{k=1}^{N} d_k \Big\|_B^2 \right] \underset{MT_2}{\lesssim} \sum_{k=1}^{N} \mathbb{E}\left[\|d_k\|_B^2\right]$$

$\blacksquare$

"Localization": For ϕ progressively measurable with $\int_0^\infty \|\phi(t)\|^2_{\gamma(K,B)} dt < \infty$ almost surely.

(1) Define $\tau_n = \inf\left\{ t \geq 0 : \int_0^t \left(1 + \|\phi(s)\|^2_{\gamma(K,B)}\right) ds \geq n \right\}$ $[\tau_n \uparrow +\infty]$

which is a stopping time. $\Rightarrow$ $\mathbb{I}_{[0,\tau_n]}$ is progressively-measurable.

$\{\tau_n$

(2) Define $\int_0^T \phi \, dW := \int_0^\infty \underbrace{\mathbb{I}_{[0,\tau_n]}\phi}_{\text{progressively measurable}} \, dW$ on $\underbrace{\{T \leq \tau_n\}}_{\text{an event with probability } \uparrow 1 \text{ as } n \to +\infty}$

stochastic integral well defined.

so $\int_0^T \phi \, dW$ is an almost surely well-defined object

(3) $M_t = \int_0^t \phi\, dW \Rightarrow M_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} \phi\, dW = \int_0^\infty \mathbb{I}_{[0,\, t\wedge\tau_n]} \phi\, dW \in L^2(\Omega; B)$

Moreover, $(M_{t\wedge\tau_n} : t \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_{t\wedge\tau_n} : t\geq 0)$, for each n.

- τ_n stopping time $\Rightarrow$ so is $\tau_n \wedge t$
- For a $\mathbb{F}$-stopping time τ, $\mathcal{F}_\tau = \{A : A\cap\{\tau\leq t\}\in\mathcal{F}_t, \forall t\geq 0\}$ is a σ-algebra = all information up to the random time τ.

<u>lemma 3.6 in Ondreját '04 thesis</u>

let τ be a $\mathbb{F}$-stopping time, ϕ progressively-mes in $\gamma(K,B)$

s.t. $\mathbb{E}\int_0^\infty \|\phi(s)\|^2_{\gamma(K,B)}\, ds < \infty$.

Then $M_t = \int_0^t \phi(s)\, dW$, $t\geq 0$ defines a martingale and $M_{t\wedge\tau} = \int_0^t \mathbb{I}_{[0,\tau]}\phi\, dW$ almost surely.

Thm. (BDG in MT_2 setting) '04 Ondreját thesis

$0<p<\infty$, B martingale type 2 separable Banach space, $\exists\, C_{p,B} \in (0,\infty)$ such that

for any $\mathbb{F}$-adapted stopping time τ, F progressively measurable process in $\gamma(K,B)$.

we have :

#5 $$\mathbb{E} \sup_{0\le t\le \tau} \left\| \int_0^t F(t')\,dW(t') \right\|_B^p \le C_{p,B}\, \mathbb{E}\left[\left(\int_0^\tau \|F(t')\|_{\gamma(K,B)}^2 dt' \right)^{\frac{p}{2}} \right].$$

W : K-cylindrical Wiener process

Proof : Define $M(r) = \left\| \int_0^r F(t')\,dW(t') \right\|_B$ $\quad M_0 = 0$

running maximum $\rightarrow M^*(r) = \sup_{s\le r} M(s)$, $\quad N(r) = \left[\int_0^r \|F(t')\|_{\gamma(K,B)}^2 dt' \right]^{\frac{1}{2}}$

choose (later) $\beta>1$, $\delta>0$, $\lambda>0$, $t\geq 0$ (Good λ inequality, due to Burkholder '73)

define $\tau_1 = \inf\{r \geq 0 : M(r) \geq \beta\lambda\}$
$\tau_2 = \inf\{r \geq 0 : M(r) \geq \lambda\}$ $\Rightarrow$ $\tau_2 < \tau_1$

$\sigma = \inf\{r \geq 0 : \underline{N(r) \geq \delta\lambda}\}$

$\rho_n = \inf\{r \geq 0 : M(r) \geq n\}$ $\Rightarrow$ $\boxed{M^*(t \wedge \rho_n) \leq n}$

Define $\underline{A_1 = \{ M^*(t \wedge \rho_n) \geq \beta\lambda \text{ and } N(t) < \delta\lambda \}}$

$$A_2 = \left\{ \left\| \int_0^{t \wedge \tau_1 \wedge \sigma \wedge \rho_n} F(t')\, dW(t') - \int_0^{t \wedge \tau_2 \wedge \sigma \wedge \rho_n} F\, dW \right\|_B \geq \lambda(\beta - 1) \right\}.$$

On A_1: $\tau_2 \leq \tau_1 \leq t \wedge \rho_n \leq t \leq \sigma$.
$\left\{ M(\tau_1) = \beta\lambda,\ M(\tau_2) = \lambda \right.$ $\parallel$

9

$$\Rightarrow \begin{cases} t\wedge\tau_1\wedge\sigma\wedge\rho_n = \tau_1 \\ t\wedge\tau_2\wedge\sigma\wedge\rho_n = \tau_2 \end{cases} \Rightarrow \left\| \int_0^{t\wedge\tau_1\wedge\sigma\wedge\rho_n} F\,dw - \int_0^{t\wedge\tau_2\wedge\sigma\wedge\rho_n} F\,dw \right\|_B = \left\| \int_0^{\tau_1} F\,dw - \int_0^{\tau_2} F\,dw \right\|_B$$

$$\geq M(\tau_1) - M(\tau_2) = (\beta - 1)\lambda.$$

That is, we have $\underline{A_1 \subseteq A_2}$.

Next $(\tau_2 < \tau_1)$

$$\mathbb{E}\left[\left\| \int_0^{t\wedge\underline{\underline{\tau_1}}\wedge\sigma\wedge\rho_n} F\,dw - \int_0^{t\wedge\underline{\underline{\tau_2}}\wedge\sigma\wedge\rho_n} F\,dw \right\|_B^2 \right] = \mathbb{E}\left[\left\| \int_0^t \mathbb{1}_{(\tau_2\wedge\sigma\wedge\rho_n,\, \tau_1\wedge\sigma\wedge\rho_n]}(s)\, F(s)\,dW \right\|_B^2 \right]$$

$$\underset{\text{BDG}(p=2)}{\lesssim} \mathbb{E}\left[\int_0^t \underbrace{\mathbb{1}_{(\tau_2\wedge\sigma\wedge\rho_n,\, \tau_1\wedge\sigma\wedge\rho_n]}(s)}_{\text{nonzero only when}} \|F(s)\|_{\gamma(K,B)}^2\, ds \;\; \mathbb{1}_{M^*(t\wedge\rho_n)\geq\lambda} \right]$$

nonzero only when

$\tau_2 \leq \sigma\wedge\rho_n\wedge\tau_1\wedge t \Rightarrow \tau_2 \leq t\wedge\rho_n$

ok to add the above indicator

$$\leq \mathbb{E}\left[\underbrace{\int_0^{\underline{\underline{t\wedge\sigma}}} \|F(s)\|^2_{\gamma(K,B)}\,ds}_{\text{bounded by } \delta^2\lambda^2 \text{ by def of } \sigma} \quad \underbrace{\mathbb{1}_{M^*(t\wedge\rho_n)\geq\lambda}}\right]$$

$$\leq C_1\delta^2\lambda^2\,\mathbb{P}\left(M^*(t\wedge\rho_n)\geq\lambda\right).$$

Therefore $\mathbb{P}(A_1)\leq\mathbb{P}(A_2)\underset{\substack{\uparrow\\ \text{Chebyshev's ineq.}}}{\leq}\dfrac{C_2\delta^2\cancel{\lambda^2}\,\mathbb{P}\left(M^*(t\wedge\rho_n)\geq\lambda\right)}{\cancel{\lambda^2}(\beta-1)^2}$

$$\Rightarrow \mathbb{P}\left(M^*(t\wedge\rho_n)\geq\beta\lambda\right)\leq\mathbb{P}\left(M^*(t\wedge\rho_n)\geq\beta\lambda,\ N(t)<\delta\lambda\right)+\mathbb{P}\left(N(t)\geq\delta\lambda\right)$$

$$\leq\mathbb{P}\left(N(t)\delta^{-1}\geq\lambda\right)+\mathbb{P}(A_1)$$

$$\leq\mathbb{P}\left(N(t)\delta^{-1}\geq\lambda\right)+\frac{C_2\delta^2}{(\beta-1)^2}\mathbb{P}\left(M^*(t\wedge\rho_n)\geq\lambda\right)$$

Fact: (Layer-Cake theorem)

For $Z \geq 0$ in $L^p(\Omega)$, $p>0$: $\mathbb{E}[Z^p] = \mathbb{E}\int_0^Z p\,\lambda^{p-1}\,d\lambda = \mathbb{E}\int_0^\infty \mathbb{1}_{[0,Z]}(\lambda)\, p\lambda^{p-1}\,d\lambda$

$$= \int_0^\infty \mathbb{P}(Z \geq \lambda)\, p\,\lambda^{p-1}\,d\lambda$$

$\Downarrow$

$$\mathbb{E}\left[\left(M^*(t\wedge\rho_n)\beta^{-1}\right)^p\right] \leq \mathbb{E}\left[\left(N(t)\delta^{-1}\right)^p\right] + \frac{C_1\delta^2}{(\beta-1)^2}\,\mathbb{E}\left[M^*(t\wedge\rho_n)^p\right]$$

$$\Leftrightarrow \mathbb{E}\left[M^*(t\wedge\rho_n)^p\right] \leq \beta^p\delta^{-p}\,\mathbb{E}\left[N(t)^p\right] + \underbrace{\boxed{\frac{C_1\beta^p\delta^2}{(\beta-1)^2}}}_{=\frac{1}{2}\text{ for some } \delta\ll 1}\underbrace{\mathbb{E}\left[M^*(t\wedge\rho_n)^p\right]}_{\text{bounded by } n^p<\infty \text{ so can be moved to LHS.}}$$

$$\Rightarrow \quad \mathbb{E}\left[M^*(t\wedge\rho_n)^p\right] \underset{\substack{\uparrow \\ \text{indep } n}}{\lesssim} \mathbb{E}\left[N(t)^p\right].$$

Recall $\rho_n \uparrow +\infty$ as $n \uparrow +\infty$

the running maximum $M^*(t\wedge\rho_n) \le M^*(t)$

AND $M^*(t\wedge\rho_n) \uparrow M^*(t)$.

By monotone convergence theorem

$$\mathbb{E}\left[M^*(t)^p\right] \lesssim \mathbb{E}\left[N(t)^p\right]$$

we can set $t=\infty$

Also we can replace F by $F\mathbb{1}_{[0,\tau]}$ $\rightarrow$ #5 P7.

That is, $$\mathbb{E}\sup_{r\le t}\left\|\int_0^r F(t')\,dW(t')\right\|_B^p \le C_{p,B}\,\mathbb{E}\left|\int_0^t \|F(t')\|^2_{\gamma(K,B)}\,dt'\right|^{p/2}$$

$\blacksquare$