Lecture 10 Tuesday March 29, 2022

Today, we fixed
- B Banach space
- K separable (real) Hilbert space
- $(\Omega, \mathcal{A}, \mathbb{P})$ prob. space

Ω sample space, $\mathcal{A}$ σ-algebra, $\mathbb{P}$ probability measure

Rem. Uncorrelation within a jointly Gaussian family $\Rightarrow$ independence

Example: $G \sim N(0,1)$, $\varepsilon \sim$ symmetric Rademacher (± 1 with prob. $\frac{1}{2}, \frac{1}{2}$) } indep

$\Rightarrow$ $\varepsilon G \sim N(0,1)$ is uncorrelated with G

But εG not indep of G.

Def. Given a (real) separable Hilbert space $\mathcal{H}$. We say

$W: \mathcal{H} \to L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ is a $\mathcal{H}$-isonormal process

if $\{W(h) : h \in \mathcal{H}\}$ is a centered (jointly) Gaussian

family indexed by $\mathcal{H}$, with $\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}$ $\forall h_1, h_2 \in \mathcal{H}$.

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Example {Wiener integral} see Lecture 1. let $(Y_t : t \geq 0)$ standard real Brownian motion

$\mathcal{H} = L^2(\mathbb{R}_+; \mathbb{R})$

$\left\{ W(h) = \int_0^\infty h(t)\, dY_t \,,\ h \in \mathcal{H} \right\}$ is $L^2(\mathbb{R}_+;\mathbb{R})$- isonormal.

Def. We say W is a <u>K-cylindrical Wiener process</u> if W is $L^2(\mathbb{R}_+; K)$-isonormal.

$\Rightarrow$ $\underset{h \in K}{W_t(h)} \equiv W(\mathbb{I}_{[0,t]} \otimes h)$ is a centered Gaussian with $\underset{2^{nd}\text{ moment}}{\text{variance}}$ $\|\mathbb{I}_{[0,t]} \otimes h\|^2_{L^2(\mathbb{R}_+;K)} = t\|h\|_K^2$.

AND $\mathbb{E}\left[W_t(h_1) W_s(h_2)\right] = \langle \mathbb{I}_{[0,t]} \otimes h_1, \mathbb{I}_{[0,s]} \otimes h_2 \rangle_{L^2(\mathbb{R}_+;K)} = (s \wedge t) \langle h_1, h_2 \rangle_K$ $\forall t,s \in \mathbb{R}_+$ $\forall h_1, h_2 \in K$.

That is, $\{W_t(h) : t \geq 0\}$ is a multiple of Brownian motion.

Book ref. ① Le Gall "Brownian motion, Martingales, and Stochastic Calculus" Eng version or French version

② Revuz and Yor

freely available on springerlink through university library.

Rem. if either $(s,t] \cap (s',t'] = \phi$ or $\langle h_1, h_2 \rangle_K = 0$,

we have $W(\mathbb{I}_{(s,t]} \otimes h_1)$ indep. of $W(\mathbb{I}_{(s',t']} \otimes h_2)$.

Def. ① $K \otimes B = \left\{ \underbrace{\sum_{j=1}^{N} h_j \otimes b_j}_{} : h_j \in K,\ b_j \in B,\ N \in \mathbb{N} \right\}$

finite-rank operator from K to B

$$(h \otimes b)(\varphi) = \langle h, \varphi \rangle_K \, b \in B$$

h: K, b: B, φ: K

② $\gamma(K,B)$ the space of γ-Radonifying operators from K to B is the completion of $K \otimes B$

under the norm

$$\left\| \sum_{n=1}^{N} h_n \otimes b_n \right\|_{\gamma(K,B)} := \left(\mathbb{E}\left[\left\| \sum_{n=1}^{N} G(h_n) b_n \right\|^2 \right] \right)^{\frac{1}{2}}$$ where G is K-isonormal.

or equivalently (by assuming $\{h_n\}_{n=1}^{N}$ is an orthonormal system in K)

$$\text{LHS} = \left(\mathbb{E}\left[\left\| \sum_{n=1}^{N} g_n b_n \right\|^2 \right] \right)^{\frac{1}{2}}$$ with $(g_n)_n$ iid real $N(0,1)$.

We say $T \in \mathcal{L}(K,B)$ is γ-Radonifying if $\|T\|_{\gamma(K,B)}$ is finite.

Rem. When B is Hilbert, $\gamma(K,B) = HS(K,B)$. the space of Hilbert-Schmidt operators from K to B.

Def. For $p \in (0,2]$. A Banach space B is of type p if

$\exists \mu \in (0,\infty)$ s.t.

$$\boxed{\mathbb{E}\left[\left\|\sum_{n=1}^{N} \varepsilon_n b_n\right\|_B^p\right] \le \mu \sum_{n=1}^{N} \|b_n\|_B^p} \quad \#1$$

for any finite seq. (b_n) in B.

$(\varepsilon_n)_n$ iid. sym. Radmacher random variables

± 1 w.p. $\frac{1}{2}, \frac{1}{2}$.

Cotype p for $p \in [2,\infty]$ if "$\le \mu$" is replaced by "$\ge \tilde{\mu}$".

Book ref. G. Pisier 2016 "Martingales in Banach spaces"

Remark. (i) $p>2$? Take $b_1 = \cdots = b_N \neq 0$ in #1

$$\underbrace{\left(\mathbb{E}\left[\left|\sum_{n=1}^{N} \varepsilon_n\right|^2\right]\right)^{p/2}}_{N^{p/2}} \underset{\text{Jensen}}{\le} \mathbb{E}\left[\left|\sum_{n=1}^{N} \varepsilon_n\right|^p\right] \le \mu N$$

fails for $N \gg 1$.

(ii) Every Banach space is of type 1.

(iii) Type p implies Type q for $q \le p$.

Indeed,

$$\left(\mathbb{E}\left\|\sum_{n=1}^{N} \varepsilon_n b_n\right\|_B^q\right)^{\frac{1}{q}} \underset{\text{Jensen}}{\le} \left(\mathbb{E}\| \quad \|_B^p\right)^{\frac{1}{p}} \underset{\text{Type } p}{\le} \mu^{\frac{1}{p}} \left(\sum_{n=1}^{N} \|b_n\|_B^p\right)^{\frac{1}{p}}$$

$$\le \mu^{\frac{1}{p}} \left(\sum_{n=1}^{N} \|b_n\|_B^q\right)^{\frac{1}{q}}$$

Since $\|(a_n)\|_{\ell_n^p} \le \|(a_n)\|_{\ell_n^q}$ for $p \ge q$

① Banach space of type p / cotype q } measures "how far" it is from being Hilbert.

② Von Neumann: parallelogram law only holds on Hilbert spaces.

$$\|x+y\|_B^2 + \|x-y\|_B^2 = 2\|x\|_B^2 + 2\|y\|_B^2 \quad \forall x,y \in B$$

$\Leftrightarrow$ B is a Hilbert space.

③ '72 Kwapień proved: B cotype 2 & type 2 $\Leftrightarrow$ B is isometrically a Hilbert space

④ L. Schwartz '69: Radonifying map $K \xrightarrow{\gamma} B \rightsquigarrow$ true Radon Prob on B.

L. Gross '67: Abstract Wiener space (Lec 4)

Now we fix W to be a K-cylindrical Wiener process.

Def. We say $\phi : \mathbb{R}_+ \to K \otimes B$ is (deterministic) simple if

$$\phi = \text{lin. Comb. of } \mathbb{I}_{(s,t]} \otimes (h \otimes b)$$

ref. Kuo 1975 book. Bogachev 1998 book.

Define $$\int_0^\infty \phi \, dW := \text{lin. Comb. of } \underbrace{W(\mathbb{I}_{(s,t]} \otimes h)}_{\text{real Gaussian random variable}} b$$ (this is a Banach space B-valued Gaussian random variable)

Lemma 1. B type 2 & ϕ (deterministic) simple.

THEN,

$$\boxed{\mathbb{E}\left[\left\| \int_0^\infty \phi \, dW \right\|_B^2 \right] \lesssim \int_0^\infty \|\phi(t)\|^2_{\gamma(K,B)} \, dt} \quad \#2$$

Rem: After obtaining #2, one can extend $\int_0^\infty \phi \, dW$ for deterministic $\phi \in L^2(\mathbb{R}_+ ; \gamma(K,B))$ by a standard density argument. AND #2 is still valid for such general integrand.

proof WOLOG, Consider

$$\phi = \sum_{n=1}^{N} \mathbb{1}_{(t_{n-1}, t_n]} \otimes \sum_{j=1}^{k} h_j \otimes b_{jn} \quad \begin{cases} (h_j) \text{ orthonormal in } K \\ b_{jn} \in B \\ 0 = t_0 < t_1 < \cdots < t_N < \infty \end{cases}$$

By definition,

$$\int_0^\infty \phi \, dW = \sum_{n=1}^{N} \sum_{j=1}^{k} \underbrace{W\left(\mathbb{1}_{(t_{n-1}, t_n]} \otimes h_j\right)}_{=: \sqrt{t_n - t_{n-1}}\, g_{jn}} b_{jn}$$

variance $(t_n - t_{n-1}) \|h_j\|_K^2 = t_n - t_{n-1}$

By Rem. on Page 2 & _____, $(g_{jn})_{j,n}$ iid $N(0,1)$

⚠ g_{jn} indep of $g_{j'n'}$ for $(j,n) \neq (j',n')$

$$= \sum_{n=1}^{N} \sum_{j=1}^{k} \sqrt{t_n - t_{n-1}}\, g_{jn} b_{jn}$$

Therefore, LHS of #2

$$= \mathbb{E}\left[\left\| \sum_{n=1}^{N} \sum_{j=1}^{k} \sqrt{t_n - t_{n-1}}\, g_{jn} \varepsilon_n b_{jn} \right\|_B^2 \right]$$

law unchanged

(ε_n) iid sym. Rad. indep of $(g_{jn})_{jn}$

adding (ε_n) does not change the law of random object inside $\|\cdot\|_B^2$

First integrate out the randomness of (ε_n) & use type 2 def.

$$\lesssim \sum_{n=1}^{N} \mathbb{E}\left[\left\| \sum_{j=1}^{k} (t_n - t_{n-1})^{\frac{1}{2}} g_{jn} b_{jn} \right\|_B^2 \right]$$

$$= \sum_{n=1}^{N} (t_n - t_{n-1})\, \mathbb{E}\left[\left\| \sum_{j=1}^{k} g_{j,n} b_{j,n} \right\|_B^2 \right] = \sum_{n=1}^{N} (t_n - t_{n-1}) \left\| \sum_{j=1}^{k} h_j \otimes b_{jn} \right\|_{\gamma(K,B)}^2 = \int_0^\infty \|\phi(t)\|_{\gamma(K,B)}^2 \, dt$$

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Def: $p \in [1,2]$ we say B has martingale type p [MT_p for short] if $\exists\, \mu_p \in (0,\infty)$ s.t.

$$\mathbb{E}\left[\left\|\sum_{n=1}^{N} d_n\right\|_B^2\right] \le \mu_p \sum_{n=1}^{N} \mathbb{E}\left[\|d_n\|_B^p\right]$$

for any finite sq. $\{d_n\}_{n=1}^{N}$ of martingale difference in $L^p(\Omega, \mathcal{A}, \mathbb{P}; B)$

"$\mathbb{E}[d_n \mid d_1, \dots, d_{n-1}] = 0$" $\forall n \ge 1$

Digression

ⓐ $Fin(B) = \{$random variables on Ω that take only finitely many values$\}$

ⓑ We say $f : \Omega \to B$ is (Bochner) measurable if $\exists\, f_n \in Fin(B)$ s.t. $f_n(\omega) \xrightarrow{n \uparrow \infty} f(\omega)$ $\forall \omega \in \Omega$.

for $f \in Fin(B)$, $\|f\|_{L^p(\Omega;B)} = \left(\int_\Omega \|f(\omega)\|_B^p \, \mathbb{P}(d\omega)\right)^{\frac{1}{p}}$ is well-defined. $p = \infty \to$ Ess-sup norm

$L^p(\Omega;B) =$ Completion of $Fin(B)$ under $L^p(\Omega;B)$-norm.

Book ref: D. Williams probability with martingales

ⓒ $X \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ $\mathcal{G} \subseteq \mathcal{A}$ σ-algebra

$\mathbb{E}^{\mathcal{G}}[X]$ Conditional expectation of X given $\mathcal{G}$ defines a <u>norm-1 operator</u> on $L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$

AND positive $\mathbb{E}^{\mathcal{G}}[X] \ge 0$ for $X \ge 0$ a.s.

THEN. [see section 1.2 in pisier doob book]

$$\left(\mathbb{E}^{\mathcal{G}}\otimes \mathbb{I}_B\right)(\underbrace{X}_{L^p(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})}\otimes \underbrace{b}_{B}) = \mathbb{E}^{\mathcal{G}}[X]\, b$$ extends to a bounded linear operator on $L^p(\Omega,\mathcal{A},\mathbb{P};B)$

$p\in[1,\infty]$

Useful property: $\mathbb{E}^{\mathcal{G}}[XY] = Y\,\mathbb{E}^{\mathcal{G}}[X]$ for any $X\in L^p(\Omega,\mathcal{A},\mathbb{P};B)$, $Y\in L^\infty(\Omega,\mathcal{G},\mathbb{P};\mathbb{R})$, $\mathcal{G}\subseteq\mathcal{A}$ ☆

Filtration $\mathbb{F}$ generated by the K-cylindrical Wiener process W.

$\mathbb{F} = \{\mathcal{F}_t : t\geq 0\}$ where $\mathcal{F}_t = \sigma$-algebra generated by $\{W(\mathbb{1}_{[0,s]}\otimes h) : s\leq t, h\in K\}$.

Clearly $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$ $\forall t\leq t'$.

Def. We say (a random time) $\tau:\Omega\to[0,\infty]$ is a $\mathbb{F}$-stopping time if $\{\tau\leq t\} = \{\omega\in\Omega : \tau(\omega)\leq t\}\in\mathcal{F}_t$, $\forall t\geq 0$.

We call τ predictable if $\exists\, \tau_n$ $\mathbb{F}$-stopping times s.t. $\tau_n < \tau_{n+1} < \tau$ $\forall n$ and $\tau_n \uparrow \tau$ as $n\uparrow\infty$.

Def: we say $\phi : \mathbb{R}_+ \times \Omega \to K \otimes B$ is $\mathbb{F}$-adapted and simple if $\phi(s)$ is $\mathcal{F}_s$-measurable $\forall s \in \mathbb{R}_+$ and $\phi(w)$ is a simple function.

$$\Leftrightarrow \quad \phi = \text{lin. Comb. of } \mathbb{1}_{(s,t]\times A} \otimes (\underset{K}{h} \otimes \underset{B}{b}), \quad s<t$$

the event $A \in \mathcal{F}_s$

Define $\displaystyle\int_0^\infty \phi \, dW = \text{lin. Comb. of } \mathbb{1}_A \, W(\mathbb{1}_{(s,t]} \otimes h) \, b$

indep.

Lemma 2. B has MT_2 & ϕ simple $\mathbb{F}$-adapted. THEN,

$$\mathbb{E}\left[\left\|\int_0^\infty \phi \, dW\right\|_B^2\right] \lesssim \mathbb{E}\int_0^\infty \|\phi(t)\|^2_{\gamma(K,B)} \, dt \qquad \#3$$

proof: Consider $\phi = \sum_{n=1}^{N} \mathbb{1}_{(t_{n-1},t_n]} \sum_{m=1}^{M} \mathbb{1}_{F_{mn}} \otimes \sum_{j=1}^{k} h_j \otimes b_{jmn}$

$0 = t_0 < \cdots < t_N < \infty$

For each $n \in \{1,\cdots,N\}$ the events $\{F_{mn}\}_{m=1}^{M}$ are mutually disjoint and in $\mathcal{F}_{t_{n-1}}$ ☆1

h_j: orthonormal in K

b_{jmn}: B

$$\int_0^\infty \phi \, dW = \sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{j=1}^{k} \mathbb{1}_{F_{mn}} W\left(\mathbb{1}_{(t_{n-1},t_n]} \otimes h_j\right) b_{jmn}$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{j=1}^{k} \mathbb{1}_{F_{mn}} (t_n - t_{n-1})^{\frac{1}{2}} g_{jn} \, b_{jmn} =: \sum_{n=1}^{N} d_n$$

with $d_n = (t_n - t_{n-1})^{\frac{1}{2}} \sum_{m=1}^{M}\sum_{j=1}^{k} \mathbb{1}_{F_{mn}} g_{jn} b_{jmn}$ is $\mathcal{F}_{t_n}$-measurable

$\mathbb{1}_{F_{mn}}$: $\mathcal{F}_{t_{n-1}}$-measurable

g_{jn}: indep of $\mathcal{F}_{t_{n-1}}$

$\Rightarrow d_1, \cdots, d_n$ adapted to $\mathcal{F}_{t_1}, \cdots, \mathcal{F}_{t_n}$ is a martingale difference.

Using ☆ on P8, one has

$\mathbb{E}[d_n \mid \mathcal{F}_{t_{n-1}}]$

$= \cdots \underline{\mathbb{E}[g_{jn} \mid \mathcal{F}_{t_{n-1}}]} \cdots$

$= 0$

therefore

$$\text{LHS of } (\#3) = \mathbb{E}\left[\left\|\sum_{n=1}^{N} d_n\right\|_B^2\right] \overset{MT_2}{\lesssim} \sum_{n=1}^{N} \mathbb{E}\left[\|d_n\|_B^2\right] = \sum_{n=1}^{N} (t_n - t_{n-1}) \mathbb{E}\left[\left\|\sum_{m=1}^{M} \mathbb{1}_{F_{mn}} \sum_{j=1}^{k} g_{jn} b_{jmn}\right\|_B^2\right]$$

$$\overset{☆1}{=} \sum_{n=1}^{N}\sum_{m=1}^{M} \left[\mathbb{E}\, \mathbb{1}_{F_{mn}}\right] \mathbb{E}\left[\left\|\sum_{j=1}^{k} g_{jn} b_{jmn}\right\|_B^2\right] = \sum_{n=1}^{N}\sum_{m=1}^{M} \left(\mathbb{E}\,\mathbb{1}_{F_{mn}}\right) \left\|\sum_{j=1}^{k} h_j \otimes b_{jmn}\right\|_{\gamma(K,B)}^2 = \text{RHS of } (\#3) \quad \blacksquare$$