Lec 1 : 25 / 01 / 22 (Tue)

- Stochastic PDEs with multiplicative noises.

dispersive

- Stochastic nonlinear Schrödinger eqn (SNLS)

$$i\partial_t u = \Delta u + \underbrace{N(u)}_{\text{nonlin}} + \underbrace{\sigma(u)\Phi\xi}_{\text{noise}}$$

- Stochastic nonlinear wave equation (SNLW)

$$\partial_t^2 u = \Delta u + N(u) + \sigma(u)\Phi\xi$$

parabolic / dissipative

- Stochastic nonlinear heat equation (SNLH)

$$\partial_t u = \Delta u + N(u) + \sigma(u)\Phi\xi$$

Noise: $\sigma(u)\,\Phi\,\xi$.

- ξ = space-time white noise

  $$\mathbb{E}\left[\xi(t,x)\overline{\xi(s,y)}\right] = \delta(t-s)\,\delta(x-y)$$

  Gaussian, mean 0.

- Φ = smoothing operator (in x)

  bdd operator on $L^2(M)$ $\qquad M = \mathbb{R}^d,\ \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

- $\sigma(u) \equiv 1 \Rightarrow$ additive noise.

  if $\sigma(u) \not\equiv 1$ (ex. $\sigma(u) = u$, or $\sigma(u) = u^k$)

  $\Rightarrow$ multiplicative noise.

SNLS : $\begin{cases} i\partial_t u = \Delta u + N(u) + \sigma(u)\Phi\xi. \\ u|_{t=0} = u_0 \end{cases}$ ③

$\partial_t u = -i\Delta u$
$+ N(u)$
$+$ —

- We say that u is a (mild) soln to SNLS if u satisfies the following mild formulation (= Duhamel formulation):

$$\boxed{u(t) = S(t)u_0 - i\int_0^t S(t-t')N(u)(t')dt' - i\int_0^t S(t-t')\sigma(u)(t')\Phi\, dW(t')}$$

nonlinear part

linear soln

effect of stoch. forcing

$S(t) = e^{-it\Delta}$ $\qquad \partial_t u = -i\Delta u$

$\widehat{S(t)f}(\xi) = e^{it|\xi|^2}\hat{f}(\xi)$ $\qquad \underset{\text{F.T.}}{\Longrightarrow}$ $\partial_t \hat{u}(t,\xi) = i|\xi|^2\hat{u}(t,\xi)$

for each fixed ξ.

$W(t,x)$ = L^2-cylindrical Wiener process

(= Brownian motion)

On $\mathbb{R}^d$, let $\{e_n\}_{n\in\mathbb{N}}$ be an O.N.B. of $L^2(\mathbb{R}^d)$

$$\boxed{W(t) = \sum_{n\in\mathbb{N}} \beta_n(t)\, e_n}$$

$\{\beta_n\}_{n\in\mathbb{N}}$ = indep Brownian motions

For NLS, we may choose β_n to be a $\mathbb{C}$-valued B.M. i.e.

$$\beta_n = \mathrm{Re}\,\beta_n + i\,\mathrm{Im}\,\beta_n$$

indep $\mathbb{R}$-valued B.M.

On $\mathbb{T}^d$. $\quad W(t) = \sum_{n\in\mathbb{Z}^d} \beta_n(t)\, e^{2\pi i n\cdot x}$

Stochastic convolution:

$$\Psi(t) = \int_0^t \underline{S(t-t')} \left(\sigma(u)(t') \underline{\underline{\Phi}} \, dW(t') \right)$$

$$= \sum_{n \in \mathbb{N}} \underline{\underline{\int_0^t}} S(t-t') \left(\sigma(u)(t') \Phi(e_n) \right) \underline{\underline{d\beta_n(t')}}$$

$$dW = \sum_{n \in \mathbb{N}} e_n \, \underline{\underline{d\beta_n}}$$

stochastic integral

· **Brownian motion:** $\{B(t)\}_{t\in\mathbb{R}_+}$ = stochastic process

$$B(t,\omega) \qquad (t,\omega)\in\mathbb{R}_+\times\Omega$$

satisfying

(i) $B(0)=0$, a.s.

(ii) $B(t)-B(s) \sim \underset{\text{normal distri.}}{N}(\underset{\text{mean}}{0},\ \underset{\text{var}}{t-s})$, $t>s$

(iii) $B(t_1)-B(s_1)$, $B(t_2)-B(s_2)$, $t_1\ge s_1\ge t_2\ge s_2$

are independent.

Properties: ① BM is a.s. conti.

② $\mathbb{E}\left[\,|B(t)-B(s)|^{2k}\,\right] = \underbrace{\dfrac{(2k)!}{2^k k!}}_{=(2k-1)!!}(t-s)^k$, $t>s$

In general, $\mathbb{E}\left[|B(t)-B(s)|^p\right] \sim_p |t-s|^{p/2}$, $p\ge 1$

· **Kolmogorov's continuity criterion:**

$\{X_t\}$ with values in a metric space

Suppose $\mathbb{E}[d(X_s, X_t)^p] \le c_0 |t-s|^{1+\alpha}$

for some $p>1$, $\alpha > 0$.

Then,
$$P\Bigg(\underbrace{\sup_{s \ne t} \frac{d(X_s, X_t)}{|s-t|^{\alpha/p - \varepsilon}}}_{\text{Hölder } C^{\frac{\alpha}{p}-\varepsilon}\text{-norm}} \ge \lambda \Bigg) \le \frac{c_1}{\lambda^p}$$

$\forall \; 0 < \varepsilon < \frac{\alpha}{p}$

$\forall \lambda > 0.$

i.e. X_t is a.s. $\left(\frac{\alpha}{p} - \varepsilon\right)$-Hölder continuous.

(in particular, continuous.)

ex: BM

$$\mathbb{E}[(B(t_2)-B(t_1))^2] = t_2 - t_1$$

$$\Rightarrow \mathbb{E}[|B(t_2)-B(t_1)|^p] \sim |t_2-t_1|^{p/2} \quad = 1+\alpha$$

$$\frac{\alpha}{p}- = \frac{\frac{p}{2}-1}{p}- = \frac{1}{2}-\frac{1}{p}- \to \frac{1}{2}- \quad \text{by taking } p\to\infty$$

i.e. BM is a.s. $(\frac{1}{2}-\varepsilon)$- Hölder conti.

$\Rightarrow$ white noise $dB \sim -\frac{1}{2}-$

Aside: $\xi \longrightarrow W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n$

(s.t. $\partial_t W$ is a space-time white noise

$$\beta_n(t) = \langle \mathbb{1}_{[0,t]} e_n, \xi \rangle_{L^2_{t,x}}$$

- <u>Wiener integral</u>:

$$I(f) = \int_a^b f(t)\, dB(t)$$

$\uparrow$ deterministic

① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}[(I(f))^2] = \|f\|^2_{L^2((a,b))} = \int_a^b |f(t)|^2\, dt$.

i.e. $I : L^2((a,b)) \to L^2(\Omega)$ is an isometry.

- <u>Step 1:</u> step func $f(t) = \sum_{j=1}^{n} a_{j-1} \mathbb{1}_{[t_{j-1}, t_j)}(t)$

$\Rightarrow$ Define $I(f) = \sum_{j=1}^{n} a_{j-1} \left(B(t_j) - B(t_{j-1}) \right)$

left endpt Riemann sum.

$\Leftarrow$ easy to check ① & ②

② : $\mathbb{E}[(I(f))^2] = \cdots = \sum_j a_{j-1}^2 \underbrace{\mathbb{E}[(B(t_j) - B(t_{j-1}))^2]}_{t_j - t_{j-1}} = \int_a^b |f|^2\, dt$

<u>Step 2</u>: Given $f \in L^2(a,b)$

approximate f by step functions. f_n

and define $I(f) = \lim_{n\to\infty} I(f_n)$ in $L^2(\Omega)$

<u>Rmk</u>: ① If $f \in C^1$, we can define $I(f)$ as

a Paley-Wiener-Zygmund integral

$$I(f) = \int_a^b f\, dB = -\int_a^b f'(t)\, dB(t) + f(b)B(b) - f(a)B(a)$$

② If $f \in C^{\frac{1}{2}+}$, then

we can define $I(f) = \int_a^b f\, dB$ as a <u>Young integral</u>

(a generalization of the Riemann-Stieltjes integral.)

③ If B is $\mathbb{C}$-valued,

$$\mathbb{E}\left[|I(f)|^2\right] = 2\|f\|^2_{L^2(a,b)}$$

· Ito integral:

σ-field

filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ s.t. $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subset \mathcal{F}$
$t_1 \le t_2$

$\sigma(B_\tau : \tau \le t)$

· We say $X(t)$ is adapted (non-anticipating) if $X(t)$ is $\mathcal{F}_t$ - measurable, $\forall\, t \ge 0$

· progressively meas :

$$[0,T] \times \Omega$$
$$\Psi$$
$$(t,\omega) \longmapsto X(t,\omega) \text{ is } \mathcal{B}_{[0,T]} \otimes \mathcal{F}_t \text{-meas}$$

· p.m. $\Rightarrow$ adapted

· adapted & left (or right) conti $\Rightarrow$ p.m.

ex : adapted & càdlàg $\Rightarrow$ p.m.

continue à droite
limite à gauche

"Assume" (i) $B(t)$ is $\mathcal{F}_t$ - meas

(ii) $B(t) - B(s)$ is indep of $\{\mathcal{F}_s\}_{s<t}$.

Set $L^2_{ad}([a\, b] \times \Omega) = \{ f(t, \omega) :$ · f is adapted to $\{\mathcal{F}_t\}$

· $\int_a^b \mathbb{E}[f^2(t)]\, dt < \infty \}$

define the Ito integral here.

can define it for a more general class:

ex $\mathcal{L}_{ad}(\Omega ; L^2([a, b]))$

(1) f_t adapted

(2) $\int_a^b |f(t,\omega)|^2\, dt < \infty$, a.s.

$$I(f) = \int_a^b f(t)\, dB(t)$$

① $\mathbb{E}[I(f)] = 0$

② $\mathbb{E}[(I(f))^2] = \int_a^b \mathbb{E}[f^2(t)]\, dt$ (Ito isometry)