Math with Yuzhao

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Probabilistic Aspects of Nonlinear Dispersive Equations
University of Birmingham

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Yuzhao Wang (Ph.D. in 2010 under B. Wang and C. Kenig)

- 2021 - , Associate professor at the University of Birmingham
- 2017 - 2021, Assistant professor at the University of Birmingham
- Jan. 2016 - Aug. 2017

  Postdoctoral researcher at the University of Edinburgh

  under my ERC Starting Grant "ProbDynDispEq" (2015 - 2020)

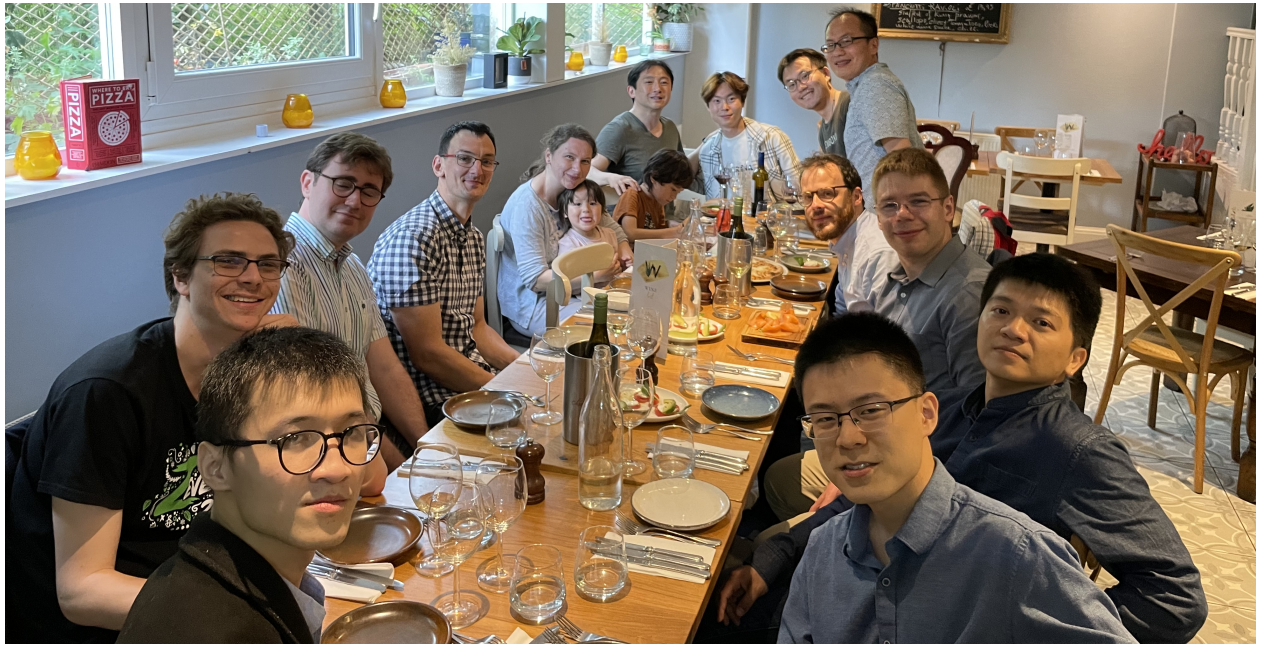
  ⋮
- 2013, **Our first collaboration** (together with Zihua Guo (IAS))
  when Yuzhao was in China and I was in Princeton

Over the last 9 years, we have written **15 joint papers** and more ongoing works

Since 2016, **Yuzhao contributed greatly in teaching** my Ph.D. students & postdocs

- **11 Ph.D. students**
- **4 postdocs:** Tristan Robert (Nancy, France), Guangqu Zheng (Liverpool from Sep. 2022), ...

Together with **Oana Pocovnicu** (Heriot-Watt), we formed an internationally visible group in the UK, working on **dispersive PDEs** with strong interest in **stochastics**


PIZZA

Our collaboration since 2013 -

Nonlinear dispersive PDEs on $\mathbb{R}^d$, $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, or M

- **Nonlinear Schrödinger equation (NLS):** $i\partial_t u = \Delta u \mp |u|^{p-1}u$
- **Nonlinear wave equation (NLW):** $\partial_t^2 u = \Delta u \mp u^k$
- **generalized KdV equation (gKdV):** $\partial_t u = \partial_x^3 u \mp \partial_x(u^k)$

Our main interests

Well-posedness (existence, uniqueness, and stability under perturbation of solutions)

- linear estimates (Strichartz estimates) '13
- nonlinear mechanism
  - high-to-low energy transfer: ill-posedness (norm inflation) '17, '22
  - short-time Fourier restriction norm method '17
  - normal form reduction '17, '18
- special structure: complete integrability '18, positivity / sign-definite structure '21
- stochastic perturbation: random data Cauchy theory, stochastic PDEs '17, ...
- math physics: Euclidean QFT, stochastic quantization, Liouville quantum gravity

Out of our 15 joint papers, I gave talks only on a few papers....

1. Strichartz estimates on irrational tori

Strichartz estimates on $\mathbb{R}^d$: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ with $2 \le q, r \le \infty$ and $(q, r, d) \neq (2, \infty, 2)$

(Str1) $$\|e^{-it\Delta} f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

- Strichartz '77 ($q = r$), Yajima '87, Ginibre-Velo '92, Keel-Tao '98 (endpoint)
- Non-endpoint case follows from a TT^* argument, the **dispersive estimate**:
$$\|e^{-it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)},$$
and the Hardy-Littlewood-Sobolev inequality (in time)

By interpolating (Str1) (with $q = r = \frac{2(d+2)}{d}$) and the trivial $L^\infty_{t,x}$-L^2_x bound by Bernstein's inequality, we obtain the following scaling-invariant inequality on $\mathbb{R}^d$:

(Str2) $$\|e^{-it\Delta} \mathbf{P}_{\le N} f\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|\mathbf{P}_{\le N} f\|_{L^2(\mathbb{R}^d)}$$

for $\frac{2(d+2)}{d} \le p \le \infty$

Q: Do (Str1) and (Str2) hold on the square torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$?

On an irrational torus $\mathbb{T}^d_{\boldsymbol{\alpha}} = \prod_{j=1}^d \mathbb{R}/(\alpha_j \mathbb{Z})$? $\alpha_j > 0$, rationally independent

- must be local in time due to (quasi-/almost) periodicity of a linear solution

(Str2) $$\|e^{-it\Delta}\mathbf{P}_{\le N}f\|_{L^p_{t,x}([0,1]\times M)} \overset{?}{\lesssim} N^{\frac{d}{2}-\frac{d+2}{p}}\|\mathbf{P}_{\le N}f\|_{L^2(M)}$$

Q: Do (Str1) and (Str2) hold on the square torus $M=\mathbb{T}^d$?

On an irrational torus $M=\mathbb{T}^d_{\boldsymbol{\alpha}}=\prod_{j=1}^d \mathbb{R}/(\alpha_j\mathbb{Z})$? $\alpha_j>0$, rationally independent

- Very challenging due to the **lack of the dispersive estimate**

  $\Longrightarrow$ No hope to prove (Str1) with general admissible (q,r) on $\mathbb{T}^d$ in general
- Zygmund '74: $d=1$ and $q=r=4$
- Kenig-Ponce-Vega '91: local smoothing estimate on $\mathbb{T}$ for a very short time $\sim N^{-1}$

Bourgain '93 made the first substantial progress on the square torus $\mathbb{T}^d$

- proved (Str2) for $d=1,2$ except for the endpoint $p=\frac{2(d+2)}{d}$
- **disproved** (Str2) at the endpoint (2-d case: Takaoka-Tzvetkov '01)

  $\Longleftarrow$ ε-loss is necessary at the endpoint
- partial range for $d\ge 3$: (i) $p>4$ when $d=3$ and (ii) $p>\frac{2(d+4)}{d}$ when $d\ge 4$
- yielded the first low regularity well-posedness of NLS on $\mathbb{T}^d$

$\Longleftarrow$ remained open until Bourgain-Demeter '15 (also ε-removal by Killip-Vişan '16)

Bourgain '93: Strichartz estimates on the square torus $\mathbb{T}^d$

- **number-theoretic tools:** divisor counting estimate, Hardy-Littlewood circle method
- non-trivial for PDE people, especially the HL circle method part
- $d = 1$ and $p = 6$: The counter example to the L^6-Strichartz estimate on $\mathbb{T}$:

$$\Big\| \sum_{|n|\leq N} e^{i(nx+n^2t)} \Big\|_{L^6([0,1]\times\mathbb{T})} \gtrsim (\log N)^{\frac{1}{6}} \underbrace{N^{\frac{1}{2}}}_{\sim \|\mathbf{1}_{|n|\leq N}\|_{\ell^2_n}}$$

  also follows from the HL circle method (Diophantine approximation, Weyl sum, Gauss sum)
  $\Longleftarrow$ Bourgain just refer to the book by Vinogradov '54...
- In 2011, E. Stein and L. Pierce gave a two-week summer course on the basic HL circle method, which allowed me to at least understand Bourgain's counter example (see a note on my website) but not the entire paper...
- Burq-Gérard-Tzvetkov '02-: many papers on Strichartz estimates on compact manifolds

Bourgain '07 studied Strichartz estimates on an irrational torus ($d = 3$)

$$\mathbb{T}^d_{\boldsymbol{\alpha}} = \prod_{j=1}^d \mathbb{R}/(\alpha_j \mathbb{Z}), \quad \alpha_j > 0, \text{ rationally independent}$$

- proved (Str2) for $(d, p) = (3, 4)$ with $\frac{1}{12}$-derivative loss
  - based on Hausdorff-Young's inequality and the lattice counting estimate by Jarník '26
  - "unquestionably deserve to be studied more"
- Catoire-W.-M. Wang '10: proved (Str2) for $(d, p) = (2, 4)$ with $\frac{1}{6}$-derivative loss
- Bourgain '13: proved (Str2) with ε-loss for $p \leq \frac{2(d+1)}{d}$ $(< \frac{2(d+2)}{d})$
  - based on the induction on scales (Bourgain-Guth '11) and the multilinear restriction theorem (Bennett-Carbery-Tao '06)

Theorem 1: Guo-Oh-Wang, Proc. Lond. Math. Soc. '14

The scaling invariant Strichartz estimate (Str2) hold on $\mathbb{T}^d_{\boldsymbol{\alpha}}$ for a wider range of p (and also new Strichartz estimates with ε-loss)

- In 2012, Yuzhao wrote a note for the higher-dimensional case
- Thanks to this work, we finally(!) worked out all the details of the Bourgain '93 paper
- Theorem 1 yielded new local well-posedness results for NLS on $\mathbb{T}^d_{\boldsymbol{\alpha}}$ in (sub-)critical spaces (including the energy-critical NLS on a partially irrational torus $\mathbb{T}^3_{\boldsymbol{\alpha}}$)

Ideas of the proof:

- higher-dimensional case: Hausdorff-Young's inequality (as in Bourgain '07) and the Hardy-Littlewood circle method: for $r = \frac{dp}{4} > 4$

$$\int_I \Big| \sum_{0 \le n \le N} e^{2\pi i n^2 t} \Big|^r dt \sim N^{r-2}$$

- When $r = \frac{dp}{4} = 4$, the above bound comes with $(\log N)^2$ (Hua '38) $\Longrightarrow$ ε-loss
- $d = 2$: repeat the argument in Bourgain '93 & bound the following term

$$\mathbf{K}(x,t) = \chi(t) \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} \sigma_{n_1} \cdot \mathbf{1}_{|n_2|\le N} \cdot e^{i(n\cdot x + (n_1^2 + \theta n_2^2)t)}$$

  - in n_1: by a Weyl sum type argument (Bourgain '93)
  - in n_2: bound by N (very crude!!)

Subsequent developments:

- Demeter '14: unpublished note (incidence theory and multilinear restriction)
- Bourgain-Demeter '15: resolution of the ℓ^2-decoupling conjecture
  $\Longrightarrow$ **full range of the Strichartz estimates** on $\mathbb{T}^d_{\boldsymbol{\alpha}}$ (with ε-loss): $p \ge \frac{2(d+2)}{d}$
- Killip-Vişan'16: ε-removal in the non-endpoint case $p > \frac{2(d+2)}{d}$ (**need number theory!!**)

Q: Periodic Strichartz estimates for KdV on $\mathbb{T}$? Bourgain '93: $q = r = 6$ with ε-loss

- $q = r = 6$ *without* ε-loss?
- $q = r = 8$ with ε-loss? $q = r = 10$ with ε-loss: Hughes-Wooley '21

2. Ill-posedness: norm inflation

- **Well-posedness:** existence, uniqueness, and stability under perturbation
- **Ill-posedness:** one of the criteria above fails
- **Scaling heuristics:** We "expect" ill-posedness for $s < s_{\text{scaling}}$ (sometimes even above s_{scaling})

Norm inflation (Christ-Colliander-Tao '03**):** given $\varepsilon > 0$, there exist a (smooth) solution u_ε and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u_\varepsilon(0)\|_{H^s} < \varepsilon \qquad \text{but} \qquad \|u_\varepsilon(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}$$

- implies discontinuity of the solution map: $H^s \to C([0,T]; H^s)$ at $u_0 = 0$

Kappeler '15 asked me if I could prove norm inflation for the 1-d cubic NLS on $\mathbb{T}$

- $\boxed{s_{\text{scaling}} = -\frac{1}{2}}$
- Christ-Colliander-Tao '03: norm inflation on $\mathbb{R}$ for $s \lneq -\frac{1}{2}$
- Carles-Kappeler '17: norm inflation on $\mathbb{T}$ for $s < -\frac{2}{3}$ (also in the Fourier-Lebesgue spaces)
- I asked Kishimoto and he gave a proof overnight (Fourier analytic approach)

Theorem 2: Oh-Wang, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. '18

Let $s \leq s_{\text{critical}} = -\frac{1}{2}$. Then, norm inflation holds in $H^s(\mathbb{T})$ for the cubic NLS on $\mathbb{T}$

- also for the fractional NLS (Choffrut-Pocovnicu '18: even above s_{scaling})
- $s < 0$: need to exhibit **high-to-low energy transfer**
- Christ-Colliander-Tao '03: **ODE approach**
  - use an explicit solution of the dispersionless problem on $\mathbb{R}$ (which exhibits high-to-low energy transfer) and a **scaling argument**, but this argument is *not* so robust, leaving a gap (i) $s = -\frac{1}{2}$ when $d = 1$ and (ii) $-\frac{d}{2} < s < 0$ when $d \geq 2$
  - A similar argument (with *low-to-high energy transfer*) works for $s > 0$: CCT '03, Burq-Gérard-Tzvetkov '05
- Iwabuchi-Ogawa '15, Kishimoto '19: Fourier analytic approach, much more robust
  - Oh '17: norm inflation at generic initial data
- In Theorem 2, we adapted the ODE approach
  - subcritical case ($s < -\frac{1}{2}$):
    scaling argument $\implies$ work on a dilated torus $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$ (with $L \to \infty$)
  - critical case ($s = -\frac{1}{2}$):
    directly establish robust high-to-low energy transfer for the dispersionless model

Cubic nonlinear heat equation (NLH) on $M = \mathbb{R}^d$ or $\mathbb{T}^d$:

$$\partial_t u - \Delta u \pm u^3$$

- canonical equation in both deterministic and stochastic analysis
- Local well-posedness in $\mathcal{C}^s(M) = B^s_{\infty,\infty}(M)$ for $s > -\frac{2}{3}$

Theorem 3: Chevyrev-Oh-Wang '22

Let $s \leq -\frac{2}{3}$ and $\sigma \in \mathbb{R}$. Then, norm inflation with **infinite loss of regularity** holds for the cubic NLH in $\mathcal{C}^s$. Namely, given $\varepsilon > 0$, there exist a solution u_ε and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u_\varepsilon(0)\|_{\mathcal{C}^s} < \varepsilon \qquad \text{but} \qquad \|u_\varepsilon(t_\varepsilon)\|_{\mathcal{C}^\sigma} > \varepsilon^{-1}$$

- $s_{\text{scaling}} = -1 \implies$ norm inflation *above* the scaling critical regularity
- sharp in the regularity $s \in \mathbb{R}$ since NLS is well-posed for $s > -\frac{2}{3}$
- Fourier analytic approach as in Oh '17 with initial data supported on a single mode
- $s = -\frac{2}{3}$: Combine norm inflation mechanisms at lacunary scales ($\sim 2^{2^k} N$)
- norm inflation also holds in $B^{-\frac{2}{3}}_{\infty,q}$ for $3 < q \leq \infty \impliedby$ sharp in terms of q

3. Normal form approach

Poincaré-Dulac Theorem: Consider a differential equation

$$\partial_t x = Ax + F(x) = Ax + \sum_{j=a}^{\infty} f_j(x), \quad x = (x_1, x_2, \ldots, x_N),$$

where $f_j(x)$ denotes nonlinear terms of degree j in x

- Under some assumption, we can introduce a sequence of changes of variables:

$$\begin{aligned} z_1 &= x + y_1, \\ z_2 &= z_1 + y_2 = x + y_1 + y_2, \\ &\vdots \\ z &= z_\infty = x + \sum_{j=1}^{\infty} y_j, \end{aligned}$$

  to reduce the system to the canonical form:

$$\partial_t z = Az + G(z) = Az + \sum_{j=a}^{\infty} g_j(z)$$

- Canonical form:

$$\partial_t z = Az + G(z) = Az + \sum_{j=a}^{\infty} g_j(z)$$

where $g_j(z) =$ *resonant* monomials of degree j in z

- After the Jth step,

$$\partial_t z_J = Az_J + G_J(z_J),$$

where monomials of degree up to $J(a-1)+a-2$ in $G_J(z_J)$ are all resonant

- Interaction representation: $\widetilde{x}(t) = e^{-tA}x(t)$, etc.

$$\partial_t x = Ax + F(x) \qquad \Longrightarrow \qquad \partial_t \widetilde{x} = e^{-tA}F(e^{tA}\widetilde{x})$$

Also, the resulting canonical equations become

$$\begin{cases} \partial_t \widetilde{z}_J = e^{-tA}G_J(e^{tA}\widetilde{z}_J), & \text{after the } J\text{th step} \\ \partial_t \widetilde{z} = e^{-tA}G(e^{tA}\widetilde{z}), & J = \infty \end{cases}$$

- Canonical form:

$$\partial_t z = Az + G(z) = Az + \sum_{j=a}^{\infty} g_j(z)$$

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- After integrating in time, we obtain

$$\begin{cases} \widetilde{z}_J(t) = \widetilde{z}_J(0) + \int_0^t e^{-t'A} G_J(e^{t'A}\widetilde{z}_J(t'))dt', & \text{after the } J\text{th step} \\ \widetilde{z}(t) = \widetilde{z}(0) + \int_0^t e^{-t'A} G(e^{t'A}\widetilde{z}(t'))dt', & J = \infty \end{cases}$$

The main goal point of the classical Poincaré-Dulac normal form reductions is to renormalize the flow so that it is expressed in terms of resonant terms. We, however, introduce the following change of viewpoint to study dispersive PDEs

Generalized Duhamel formulation:

- After the Jth step:

$$\widetilde{x}(t) = \widetilde{x}(0) - \sum_{j=1}^{J} \left[\widetilde{y}_j(t) - \widetilde{y}_j(0)\right] + \int_0^t e^{-t'A} G_J(e^{t'A}\widetilde{z}_J(t'))dt'$$

- With $J = \infty$:

$$\widetilde{x}(t) = \widetilde{x}(0) - \sum_{j=1}^{\infty} \left[\widetilde{y}_j(t) - \widetilde{y}_j(0)\right] + \int_0^t e^{-t'A} G(e^{t'A}\widetilde{z}(t'))dt'$$

Original Duhamel formulation: $\widetilde{x}(t) = \widetilde{x}(0) + \int_0^t e^{-t'A} F(e^{t'A}\widetilde{x}(t'))dt'$

This change of viewpoint turned out to be useful in various settings:

- **Unconditional uniqueness** for dispersive PDEs in low regularities
  - uniqueness in the entire $C([0,T];H^s)$
  - construction of solutions *without* any auxiliary functions spaces such as Strichartz spaces or the $X^{s,b}$-spaces
  - precursor (two iterations & no mention of NF): Babin-Ilyin-Titi '11, Kwon-Oh '12
  - infinite iterations: Guo-Kwon-Oh '13, Oh-Wang '21, Kishimoto '21, etc.
- **Nonlinear smoothing** $\Longrightarrow$ upper bound on the growth of high Sobolev norms
  - two iterations: Erdogăn-Tzirakis '13
- **Improved energy estimates** for proving uniqueness, quasi-invariance, etc.
  - NF reductions applied to the evolution equation for an energy
  - uniqueness: Oh-Wang '18
  - quasi-invariance: Oh-Tzvetkov '17, O.-Sosoe-Tz. '18, Oh-Seong '21
  - adding correction terms to the I-method: CKSTT '02 (NF view point: Guo-Oh '18)
- **Reducibility** to the linear equation
  - weak form of *integrability*
  - Chung-Guo-Kwon-Oh '17: quadratic derivative NLS on $\mathbb{T}$

Renormalized cubic NLS on $\mathbb{T}$: $i\partial_t u = \partial_x^2 u + \big(|u|^2 - 2\fint |u|^2\big)u$

- Fourier-Lebesgue space $\mathcal{F}L^p(\mathbb{T}) = \{f \text{ on } \mathbb{T} : \widehat{f}(n) \in \ell^p_n\}$
- sensible weak solution: $u_{0,n} \to u_0$ implies (i) $u_n \to u$ and (ii) $\mathcal{N}(u_n)$ converges to some limit in $\mathcal{D}'_{t,x}$ (independent of the approximating sequence $\{u_{0,n}\}$)

Theorem 4: Oh-Wang, J. Anal. Math. '21

- Let $1 \le p < \infty$. Then, the renormalized cubic NLS (rNLS) on $\mathbb{T}$ is globally well-posed in $\mathcal{F}L^p(\mathbb{T})$ in the sense of sensible weak solutions
- When $1 \le p \le \frac{3}{2}$, unconditional uniqueness holds (even without the renormalization)

- The condition $p \le \frac{3}{2}$ is necessary in making sense of $|u|^2 u$, i.e. $u \in \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}) \subset L^3(\mathbb{T})$
- construction by an infinite iteration of NF reductions
  - Guo-Kwon-Oh '13: construction in $L^2(\mathbb{T})$ and UU in $H^{\frac{1}{6}}(\mathbb{T})$
- $p = \infty$ is scaling-critical $\implies$ Theorem 4 is almost optimal

  Note: $\mathcal{F}L^\infty(\mathbb{T})$ does not admit approximation by smooth functions and thus is not suitable for studying well-posedness

General setup:

1. Separate the nonlinear part into *nearly resonant* and *non-resonant* parts
2. "Eliminate" the non-resonant part $\Longrightarrow$ introduces *higher* order terms
3. Repeat (or terminate the process at some finite step)

"Eliminate": by *integration by parts*

Cubic NLS on $\mathbb{T}$: Interaction representation $v(t) = S(-t)u(t)$:

$$\partial_t \widehat{v}_n = -i \sum_{n=n_1-n_2+n_3} e^{-i\phi(\bar{n})t} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} =: -i\mathcal{N}^{(1)}(v)_n,$$

where $\quad \phi(\bar{n}) := n^2 - n_1^2 + n_2^2 - n_3^2 = 2(n_2 - n_1)(n_2 - n_3)$

- Given a parameter $K = K(\|u(0)\|_{L^2}) > 0$, write

  $$\mathcal{N}^{(1)}(v) = \underbrace{\mathcal{N}^{(1)}_{\text{res}}(v)}_{\text{nearly resonant}} + \underbrace{\mathcal{N}^{(1)}_{\text{nr}}(v)}_{\text{non-resonant}}, \text{ depending on } |\phi(\bar{n})| \leq K \text{ or } > K$$

- Nearly resonant part $\mathcal{N}^{(1)}_{\text{res}}(v)$ satisfies a good $\mathcal{F}L^p$-estimate
- **No** estimate is available for the (highly) non-resonant part $\mathcal{N}^{(1)}_{\text{nr}}(v)$

  $\Longrightarrow$ Apply a NF reduction to $\mathcal{N}^{(1)}_{\text{nr}}(v)$

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  $\Longrightarrow$ Apply a NF reduction to $\mathcal{N}^{(1)}_{\text{nr}}(v)$

- **1st step of NF reductions** (= differentiation by parts)

$$\begin{aligned}
\mathcal{N}^{(1)}_{\text{nr}}(v)_n &= \sum_{\substack{n=n_1-n_2+n_3 \\ |\phi(\bar{n})|>K}} e^{i\phi(\bar{n})t}\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3} \\
&= \partial_t\Big[\sum_{*}\frac{e^{i\phi(\bar{n})t}}{\phi(\bar{n})}\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3}\Big] \\
&\quad - \sum_{*}\frac{e^{i\phi(\bar{n})t}}{\phi(\bar{n})}\partial_t\big(\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3}\big) \\
&=: \partial_t \underbrace{\mathcal{N}^{(2)}_{\text{bd}}(v)_n}_{\text{easy}} + \underbrace{\mathcal{N}^{(2)}(v)_n}_{\text{quintic}}
\end{aligned}$$

- Divide the quintic term $\mathcal{N}^{(2)}(v)$ into

  (i) nearly resonant part $\mathcal{N}^{(2)}_{\text{res}}(v)$: bounded in $\mathcal{F}L^p$

  $\Longleftarrow$ modulation restriction + divisor counting argument

  (ii) non-resonant part $\mathcal{N}^{(2)}_{\text{nr}}(v)$: no estimate available

  $\Longrightarrow$ 2nd step of NF reductions

- Repeat the process indefinitely

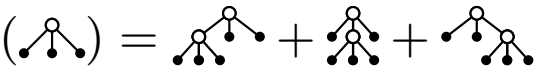
Difficulty:

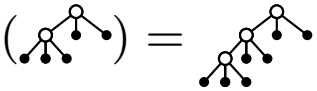
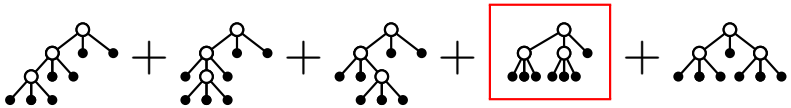
- When we apply differentiation by parts, the time derivative may fall on any of the factors $\widehat{v}_{n_j}$. In general, the structure of such terms can be very complicated, depending on where the time derivative falls

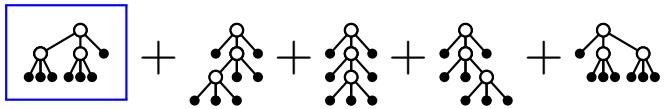
We use *ordered trees for indexing* such terms arising in the general steps of the NF reductions

- ordered trees = (ternary) trees "with memory"

  $\Longleftarrow$ The order in which time derivative fall matters!!

  Example: $\partial_t($  $)=$ $\Longrightarrow$

  $\partial_t($  $)=$ 

  $\partial_t($  $)=$ 

  $\partial_t($  $)=\cdots$

  As ordered trees,  and  are different!!

- Indexing via ordered trees allows us to handle combinatorial complexity

After the Jth step:

$$\partial_t v(t) = \partial_t \left(\sum_{j=2}^{J+1} \mathcal{N}_{\text{bd}}^{(j)}(v) \right) + \sum_{j=1}^{J+1} \mathcal{N}_{\text{res}}^{(j)}(v) + \underbrace{\mathcal{N}_{\text{nr}}^{(J+1)}(v)}_{\text{bad}}$$

- In order to justify the formal computations, we consider frequency truncated initial data $\mathbf{P}_{\leq N} u(0)$ and the associated *smooth* solutions
- In general, we only have

$$\left|\mathcal{N}_{\text{nr}}^{(J+1)}\right| \leq F(N, J) \quad \text{with} \quad \lim_{N \to \infty} F(N, J) = \infty \text{ for each fixed } J \in \mathbb{N}$$

  This, however, does not cause an issue since we also show

$$\lim_{J \to \infty} F(N, J) = 0$$

  for each *fixed* $N \in \mathbb{N}$. Therefore, by **first taking the limit $J \to \infty$ and then $N \to \infty$**, we conclude that the error term $\mathcal{N}_{\text{nr}}^{(J+1)}$ vanishes in the limit

Putting all together, we obtain the **normal form equation:**

$$\boxed{v(t) = v(0) + \sum_{j=2}^{\infty} \mathcal{N}_{\text{bd}}^{(j)}(v) \Big|_0^t + \int_0^t \sum_{j=1}^{\infty} \mathcal{N}_{\text{res}}^{(j)}(v)(t')dt'}$$

$\Longleftarrow$ $\mathcal{N}_{\text{bd}}^{(j)}(v)$ of deg $2j-1$, $\mathcal{N}_{\text{res}}^{(j)}(v)$ of deg $2j+1$: both bounded in $\mathcal{F}L^p$

Application on energy estimates

Integration by parts is often useful in establishing a good energy estimate.

$\Longleftrightarrow$ NF reduction on the evolution equation $\partial_t E(v) = \cdots$ satisfied by the (non-conserved) energy functional $E(v)$

Renormalized 4NLS on $\mathbb{T}$: $i\partial_t u = \partial_x^4 u + \big(|u|^2 - 2 \fint |u|^2\big)u$

Theorem 5: Oh-Wang, Forum Math. Sigma '18

Let $s > -\frac{1}{3}$. Then, the renormalized 4NLS is globally well-posed in $H^s(\mathbb{T})$

- **Existence part:** short-time Fourier restriction norm method
- **Uniqueness part:** infinite iteration of NF reductions on the H^s-energy functionals (of the difference of solutions with the same initial condition) and obtained an **modified energy of an infinite order**
- For bookkeeping, we use "ordered bi-trees" that grow in two directions

After an infinite iteration of NF reductions:

$$\|u(t)\|_{H^s}^2 - \|u(0)\|_{H^s}^2 = \sum_{j=2}^{\infty} \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \mathcal{N}_{\text{bd}}^{(j)}(u)(n,t')\Big|_0^t$$
$$+ \int_0^t \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \mathcal{N}_{\text{res}}^{(j)}(u)(n,t')dt'$$

Then, by defining a **modified energy** $E_\infty(u)$ **of an infinite order** by

$$\boxed{E_\infty(u) = \|u\|_{H^s}^2 - \sum_{j=2}^{\infty} \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \mathcal{N}_{\text{bd}}^{(j)}(u)(n),}$$

we obtain

$$E_\infty(u)(t) - E_\infty(u)(0) = \int_0^t \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \mathcal{N}_{\text{res}}^{(j)}(u)(n,t')dt',$$

where RHS satisfies good estimates

Moral: This infinite iteration of NF reductions allows us to **exchange analytical difficulty with algebraic/combinatorial difficulty**

- relevant analysis involves simple Cauchy-Schwarz inequality

4. Complete integrability

Q: *What is integrability?*

- Solvability via *integration* by quadratures (i.e. in an explicit manner)

Finite dimensional Hamiltonian dynamics on $\mathbb{R}^{2N}$:

$$\frac{dp}{dt} = \frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = -\frac{\partial H}{\partial p}$$

- There exist $H_1 (= H)$, $H_2, \ldots, H_N$ all in involution: $\{H_j, H_k\} = 0$
  $\Longrightarrow$ (Liouville) The system is *integrable*
- Action-angle variables (Liouville-Arnold):

$$(p, q) \overset{\text{symplect.}}{\mapsto} (I, \varphi) \text{ such that } \frac{dI}{dt} = 0, \ \frac{d\varphi}{dt} = c(I)$$

Infinite dimensional case (= PDEs): various notions of integrability

- infinitely many conservation laws $\quad (\Longleftarrow$ bi-Hamiltonian structure)
- **Lax pair formulation**
- Action-angle coordinates

$$\vdots$$

- Reducibility (to the linear equation)

  Note: *No* Hamiltonian structure required

1-d cubic NLS on $M = \mathbb{R}$ or $\mathbb{T}$: $i\partial_t u = \partial_x^2 u \mp 2|u|^2 u$

- completely integrable: **Lax pair formulation**

$$\frac{d}{dt} L(t;\kappa) = \big[P(t,\kappa), L(t;\kappa)\big], \quad \text{where } L(t;\kappa) = \begin{pmatrix} -\partial_x + \kappa & iu \\ \mp i\overline{u} & -\partial_x - \kappa \end{pmatrix}$$

- Killip-Vişan-Zhang '18 studied the following perturbation determinant $\alpha(\kappa; u)$:

$$\alpha(\kappa; u) = \mathrm{Re} \sum_{j=1}^{\infty} \frac{(\mp 1)^{j-1}}{j} \operatorname{tr} \Big\{ \big[(\kappa - \partial_x)^{-\frac{1}{2}} u (\kappa + \partial_x)^{-1} \overline{u} (\kappa - \partial_x)^{-\frac{1}{2}} \big]^j \Big\}$$

  - $\alpha(\kappa; u)$ is conserved under NLS
  - leading order term $= \int_{\mathbb{R}} \frac{2\kappa |\widehat{u}(\xi)|^2}{4\kappa^2 + \xi^2} d\xi$ on $\mathbb{R}$
  - sum of $\alpha(\kappa; u)$ over $\underbrace{\text{dyadic } \kappa \geq \kappa_0}_{\sim \text{ scaling symmetry of NLS}}$ $\sim H^s$-norm for $s > s_{\text{scaling}} = -\frac{1}{2}$

$\Longrightarrow$ a priori global-in-time H^s-norm bound for $s > s_{\text{scaling}} = -\frac{1}{2}$

- Modulation space $M^{2,p}(\mathbb{R}) = \big\{ f \text{ on } \mathbb{R} : \mathcal{F}_x^{-1}\big(\mathbf{1}_{(-\frac{1}{2},\frac{1}{2}]} \cdot \widehat{f}(\cdot - n)\big) \in \ell^p_n L^2_x \big\}$
- On $\mathbb{T}$, $M^{2,p}(\mathbb{T}) = \mathcal{F}L^p(\mathbb{T})$
- compatible with the Galilean symmetry of NLS (i.e. translations on the freq. space)

Theorem 6: Oh-Wang, J. Differential Equations '20

Let $2 \leq p < \infty$. Then, the (renormalized) cubic NLS on $M = \mathbb{R}$ or $\mathbb{T}$ is globally well-posed in $M^{2,p}(M)$

- almost critical spaces (critical $p = \infty$)
- local well-posedness: S. Guo '17 on $\mathbb{R}$, Grünrock-Herr '08 on $\mathbb{T}$
- We used **both scaling and Galilean symmetries** of NLS to sum up $\alpha(\kappa; u)$

Q: Can we establish a priori global-in-time bound on the $\mathcal{F}L^p(\mathbb{R})$-norm, $p < \infty$?

Complex-valued modified KdV (mKdV): $\partial_t u + \partial_x^3 u \pm 6|u|^2 \partial_x u = 0$

- The same a priori global-in-time bound on the $M^{2,p}$-norm holds for mKdV on $M = \mathbb{R}$ and $\mathbb{T}$ (Oh-Wang '20)

Theorem 7: Oh-Wang, Discrete Contin. Dyn. Syst. A '21

Let $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. Then, mKdV is globally well-posed in $M_s^{2,p}(\mathbb{R})$

- $s < \frac{1}{4}$: solution map is not locally uniformly continuous (in the focusing case)
- On $\mathbb{T}$?: Kappeler-Molnar '17 proved local well-posedness of the *real-valued* defocusing mKdV in $\mathcal{F}L^p(\mathbb{T})$, $p < \infty$, in the sense of sensible weak solutions. However, the local existence time is *not* characterized by the size of initial data...

Subsequent development:

- Harrop-Griffiths, Killip, and Vişan '20 proved global well-posedness (in the sense of sensible weak solutions) of NLS and mKdV in $H^s(\mathbb{R})$, $s > -\frac{1}{2}$

Q: On $\mathbb{T}$?

- Non-existence for $s < 0$ *without* renormalization: Guo-Oh '18

5. Random data Cauchy theory & stochastic PDEs

Nonlinear wave equations (NLW):

$$\partial_t^2 u = \Delta u - u^{k-1}$$

Nonlinear Schrödinger equations (NLS):

$$i\partial_t u = \Delta u - |u|^{k-2} u$$

Main interest: Study these equations with rough ***stochastic forcing*** and/or ***random initial data***

Well-posedness: existence, uniqueness, and *stability under perturbation*

Q1: How to make sense of the nonlinearity u^{k-1} as a distribution?

- *Impose a structure* on the unknown function u

Q2: For rough initial data / noise, the map: *data* $\longmapsto u$ is ill defined

- *Decompose* the classically ill-posed solution map into two steps:
  1. Construction of an *enhanced data set* (as in rough path theory)
  2. Deterministic continuous map from enhanced data set to u

Stochastic quantization equation

Study an SPDE which preserves a target measure (Parisi-Wu '81, Ryang-Saito-Shigemoto '85)

- **Parabolic Φ^k_d-model:** $\partial_t u = \Delta u - u^{k-1} + \sqrt{2}\xi$

- **Hyperbolic Φ^k_d-model:** stochastic damped NLW on $\mathbb{T}^d$

$$\partial_t^2 u + \partial_t u = \Delta u - u^{k-1} + \sqrt{2}\xi$$

  - **canonical stochastic quantization equation** for the Φ^k_d-measure (= Hamiltonian SQE, given as a hypoelliptic Langevin equation)

- **Dispersive Φ^k_d-model:** *deterministic* NLS on $\mathbb{T}^d$ with Gibbsian initial data

(NLS + dissipation + ξ $\Longrightarrow$ SCGL (parabolic techniques apply))

Goal: Construct global-in-time dynamics and prove invariance of the Φ^k_d-measure

$$\boxed{\text{parabolic } \Phi^k_d\text{-model} \ll \text{hyperbolic } \Phi^k_d\text{-model} \ll \text{dispersive } \Phi^k_d\text{-model}}$$

- **Main difficulty: local well-posedness**

$\boldsymbol{d = 2}$**:**

- parabolic Φ^k_2-model: Da Prato-Debussche '03
- hyperbolic Φ^k_2-model: Gubinelli-Koch-Oh '18, Tolomeo '21, GKOT '21
- dispersive Φ^k_2-model: Bourgain '96, Deng-Nahmod-Yue '19

$\boldsymbol{d = 3}$**:** only $k = 4$ is possible in the defocusing case

- parabolic Φ^4_3-model: Hairer '14, etc.
- hyperbolic Φ^4_3-model: Bringmann-DNY '22 (Oh-Wang-Zine '22 with $\langle\nabla\rangle^{-\varepsilon}\xi$)
- dispersive Φ^4_3-model: **open** (critical!!)

Oh-Okamoto-Tolomeo '21: $k = 3$ (non-defocusing) for the parabolic and hyperbolic Φ^3_3-model

- Main difficulty when $k = 3$: measure (non)-construction
- local well-posedness: paracontrolled approach (Gubinelli-Koch-Oh '18)

Q: What about non-polynomial nonlinearities?

Q: What about non-polynomial nonlinearities?

- sine-Gordon, $\sin(\beta u)$:
  - $d = 1$, wave: McKean '81
  - $d = 2$, parabolic, local in time: Hairer-Shen '16, Chandra-Hairer-Shen '18
- exponential nonlinearity (Liouville model), $e^{\beta u}$:
  - $d = 2$: Garban '20
- regularities of the stochastic terms depend sensitively on $\beta^2 > 0$.
- stochastic terms are "of infinite degree"

Theorem 8: Oh-Robert-Sosoe-Wang, '21, '21

- Hyperbolic sine-Gordon model on $\mathbb{T}^2$, $0 < \beta^2 < 2\pi$
- Parabolic sine-Gordon model on $\mathbb{T}^2$, $0 < \beta^2 < 4\pi$

Theorem 9: Oh-Robert-Wang, Comm. Math. Phys. '21

- Hyperbolic Liouville model on $\mathbb{T}^2$, $0 < \beta^2 < 0.86\pi$
- Parabolic Liouville model on $\mathbb{T}^2$, $0 < \beta^2 < 4\pi$

- Hoshino-Kawbi-Kusuoka '21, '22: parabolic Liouville model for $0 < \beta^2 < 8\pi$
- Oh-Robert-Tzvetkov-Wang, '20: Liouville quantum gravity model

Cubic SNLW with almost space-time white noise on $\mathbb{T}^3$:

$$(\mathrm{SNLW}_\varepsilon) \qquad \partial_t^2 u - \Delta u + u^3 = \langle \nabla \rangle^{-\varepsilon} \xi, \qquad \varepsilon > 0$$

Theorem 10: Oh-Wang-Zine, Stoch. Partial Differ. Equ. Anal. Comput. '22

Let $\varepsilon > 0$. Then, $\mathrm{SNLW}_\varepsilon$ is locally well-posed.

Second order expansion: $\boxed{u = \mathbf{1} - \Psi + v}$ ***without*** paracontrolled ansatz

- $\mathbf{1} = \mathcal{I}(\langle \nabla \rangle^{-\varepsilon} \xi) \sim \varepsilon - \frac{1}{2}- > -\frac{1}{2} \qquad \Longrightarrow \qquad v \sim \frac{1}{2}+$
- $\Psi = \mathcal{I}(\Psi) \sim \varepsilon- \quad \Longleftarrow \quad$ gain from $\frac{1}{2}$-regularity by multilinear smoothing

Fixed point problem for v:

$$v = S(t)(u_0, u_1) + \mathcal{I}\big(- v^3 + 3(\Psi - \mathbf{1})v^2 - 3\Psi^2 v\big)$$
$$+ 6\mathcal{I}\big((\Psi \mathbf{1})v\big) - 3\mathfrak{I}^{\vee}(v) + \mathcal{I}\big(\Psi^3\big) + 3\,\Psi - 3\,\Psi$$

Fixed point problem for v:

$$v = S(t)(u_0, u_1) + \mathcal{I}\big(- v^3 + 3(\Psi - \uparrow)v^2 - 3\Psi^2 v\big)$$
$$+ 6\mathcal{I}\big((\Psi\uparrow)v\big) - 3\mathfrak{I}^{\vee}(v) + \mathcal{I}\big(\Psi^3\big) + 3\,\Psi - 3\,\Psi$$

Multilinear smoothing of $(\frac{1}{2}+)$-regularity:

- $\Psi = \mathcal{I}(\Psi\vee)$ and $\Psi = \mathcal{I}(\Psi^2\uparrow)$
- Random operator: $\mathfrak{I}^{\vee}(v) = \mathcal{I}(\vee v)$ "$\vee v \sim -1-$" but $\mathfrak{I}^{\vee}(v) \sim \frac{1}{2}+$
  random matrix estimates: Bourgain '96, '97, Richards '16, Deng-Nahmod-Yue '20
- reduced analysis to that in Bringmann '20 on the Hartree cubic nonlinearity

Factorization of the ill-posed solution map:

$$(u_0, u_1, \langle\nabla\rangle^{-\varepsilon}\xi)$$
$$\xmapsto{①} \big(u_0, u_1, \uparrow, \Psi, \Psi\uparrow, \Psi, \Psi, \mathfrak{I}^{\vee}\big)$$
$$\xmapsto{②} v$$
$$\longmapsto u = \uparrow - \Psi + v$$

Thank you for all the help you provided to the group members, and I hope that we can have fun talking about math for many more years!!

(Unfortunately, Justin Forlano, Andreia Chapouto, Guangqu Zheng, and Oana Pocovnicu could not join this photo)