MIGSAA advanced Ph.D. course

Singular stochastic dispersive PDEs

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Spring 2021

Contents:

LECTURE NOTES FOR SINGULAR SPDE

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1. Lecture 1

1.1. **Chapter 0. Introduction.** We begin by recalling a couple of examples of (deterministic) dispersive PDEs.

1. Non-linear Schrödinger Equation.
$$i\partial_t u - \Delta u \pm |u|^{k-1}u = 0 \tag{NLS}$$
where $k \in 2\mathbb{N}+1$ and u is $\mathbb{C}$-valued.
2. Non-linear Wave Equation.
$$\partial_t^2 u - \Delta u \pm u^k = 0 \tag{NLW}$$
where u is $\mathbb{R}$-valued.

Our goal is to understand how some given initial data is propogated under these non-linear dynamics. To this end we may ask the following questions:

1. Is the PDE well-posed?
By this we mean whether a solution to the PDE exists, is unique and is stable under perturbation. In the stochastic setting, one is mainly concerned with the first two requirements because stability under perturbation does not hold in the classical sense when the equation is driven by a rough noise.
   1.1. Is the PDE locally well-posed?
   By Local Well-Posedness (LWP), we mean that the PDE is well-posed for a short time, where the time may depend on the initial data.
   1.2. Is the PDE globally well-posed?
   By Global Well-Posedness (GWP), we mean that a unique solution to the PDE exists globally. Note that we aren't concerned with the stability criterion here.

By a process of randomisation we are able to get stochastic dispersive PDEs from deterministic ones. In the following list we collect some examples:

1. Stochastic Non-linear Schrödinger Equation.
$$i\partial_t u - \Delta u + |u|^{k-1}u = \phi\xi \tag{SNLS}$$
where ξ is a space-time white noise as in Definition 1.1 and ϕ is a bounded operator on L^2. In our case it will chosen so as to be a smoothing operator on the white noise).
2. Stochastic Non-linear Wave Equation.
$$\partial_t^2 u - \Delta u + u^k = \xi \tag{SNLW}$$
3. Stochastic damped Non-linear Wave Equation.
$$\partial_t^2 u + \partial_t u - \Delta u + u^k = \xi \tag{SdNLW}$$
In terms of LWP, (SNLW) and (SdNLW) are solved in the same way. Dampness helps when one is concerned with global in time behaviour.

Definition 1.1 (space-time White Noise)**.** *A space-time white noise is a Gaussian random distribution on $\mathbb{R} \times \mathbb{R}^d$ such that its covariance is given by the delta distribution.*

Rigorously, the white noise is a random distribution and should be thought of always with reference to the duality pairing $\langle \xi, \varphi \rangle$, where φ is a test function on $\mathbb{R} \times \mathbb{R}^d$. However we will formally deal with it as a random Gaussian function, $\xi(t, x)$, of the space-time coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and with the covariance structure:

$$\mathbb{E}\xi(t_1, x_1)\xi(t_2, x_2) = \delta(t_1 - t_2)\delta(x_1 - x_2)$$

This means that the (random) behaviour at different points is independent and as such the white noise is very rough. This is made definite in the notion of regularity. We postpone the rigorous definitions for now, but suffice it to say that regularity is a measure of the process' differentiability. It is known that for the space-time white noise, the regularity in x, is $-\frac{d}{2} - \varepsilon$ and in t it is given by $-\frac{1}{2} - \varepsilon$, for $\varepsilon > 0$. This roughness of the noise makes analysis of dispersive equations of this kind quite difficult. Nevertheless it is still a fruitful endeavour because of (at least) the following reasons:

1. They are analytically challenging. Given the difficulty in proving the local well-posedness of such equations with existing machinery, one can hope that the study of such equations can catalyse new advances in theories of Analysis, PDEs and Probability. Some notoriously difficult instances are:
   - SNLS 1-d cubic (i.e. $k = 3$), $\Phi = \mathrm{Id}$. The local well-posedness of this equation is open and understood to be critical.
   - SNLW 3-d cubic. The local well-posedness of this equation is again open but not considered critical. In fact there have been advances made in the past year.
2. SdNLW (SdNLW) formally preserves the Gibbs measure. It is important to understand the long term behaviour of the solutions of S(P)DEs and it is often possible to show that the dynamics converge to an invariant state. A related equation is Stochastic Non-linear Heat Equation (reaction-diffusion equation):

   $$\partial_t u - \Delta u + u^k = \xi \tag{SNLH}$$

   and it formally preserves the so-called Φ_d^{k+1}-measure (k refers to the degree of non-linearity and d is the underlying spatial dimension):

   $$d\rho = z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}dx} \underbrace{e^{-\frac{1}{2}\int |\nabla u|^2 dx} du}_{\text{Gaussian free field}}$$

   where du refers to the non-existent lebesgue measure. Construction of such measures was studied in the '70s and '80s. When $d \in \{1, 2\}$, measures for all $k \in 2\mathbb{N} + 1$ have been constructed. Note that in the $d = 2$ case, we require renormalisation. For $d = 3$, $k = 3$ is the only case that has been constructed: (Φ_3^4-measure).
   - Stochastic Quantisation. The idea behind this is to introduce a stochastic PDE which preserves the measure (on a function space) to investigate it. When the measure under consideration is Φ_d^{k+1} this is exactly the Stochastic Non-linear Heat Equation (SNLH) we have already seen. Well-posedness for (SNLH)

is easy for $d = 1$, but $d = 2$ requires renormalisation [1]. The concept of renormalisation will be a significant part of this course. The $d = 3$ ($k = 3$) case (with renormalisation) is given:

$$\partial_t u + (1 - \Delta)u + u^3 \underbrace{-\infty \cdot u}_{\text{renormalisation}} = \xi \tag{SQE}$$

Note that the introduction of 1 here is to preserve the invariant measure and it doesn't affect the LWP theory. Martin Hairer proved the local well-posedness of this dynamical Φ_3^4-model (parabolic Φ_3^4-model) in [2] using regularity structures which was followed by Gubinelli-Imkeller-Perkowski proving local well-posedness using paracontrolled calculus in [3], and also by Kupiainen in [4] using Renormalisation Groups. This equation is called the Stochastic Quantisation Equation and hence the label (SQE). The wave analogue is given by:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^3 - \infty \cdot u = \xi$$

Recall that the wave equation is a vectorial equation, so the solution is instead the pair $(u, \partial_t u)$. Then we have $(u, \partial_t u) \sim \underbrace{\Phi_3^4}_{u}$ $-$measure $\otimes \underbrace{\text{white noise}}_{\partial_t u}$. This is also referred to as the hyperbolic Φ_3^4 model.

Remark 1.2. *All the results mentioned until now have been on the d-dimensional torus, i.e.* $\mathbb{T}^d = (R\backslash\mathbb{Z})^d$. *In the sequel, the "$2\pi$" factor in the definition of the fourier transform will be taken for granted.*

Remark 1.3. *In the periodic setting spatial roughness is the only issue, but on* $\mathbb{R}^d$ *one also has to grapple with the fact that the noise doesn't decay as* $|x| \to \infty$*, and hence the solution is not integrable in the* $W^{s,p}$ *sense (definition 1.15). This however is not to say that there aren't useful results on the* $\mathbb{R}^d$*. Some will be explored in this course and the interested reader may alsp like to refer to* [5] *and* [6] *for the heat case and* [7] *for* $2-d$ *cubic SNLS.*

Let us now return to (SNLS) and discuss what we mean by a solution. We remark first that (SNLS) admits the following Itô formulation:

$$idu = (\Delta u - |u|^{k-1}u)dt + \phi dW \tag{1.1}$$

where $W(t, x)$ is the L^2-cylindrical Wiener Process, as in definition 1.5.

Definition 1.4 (Brownian Motion)**.** *A* ($\mathbb{R}$*-valued*) *standard Brownian Motion is a stochastic process* $B = (B_t)_{t\geq 0}$ *such that the following holds:*

- $W_0 = 0$ *a.s..*
- $W_t - W_s \sim N(0, t - s)$ *for all* $0 \leq s < t$.
- *For any* $0 \leq t_0 < t_1 < \cdots < t_n$*, the random variables* $W_{t_1} - W_{t_0}, \cdots, W_{t_n} - W_{t_{n-1}}$ *are independent.*
- W *is almost surely continuous.*

Definition 1.5 (L^2-cylindrical Wiener Process)**.** *$W(t,x)$ is a L^2-cylindrical Wiener Process, if it admits the following decomposition:*

$$W(t,x) = \sum_{n\in\mathbb{Z}^d} \beta_n(t)e^{in\cdot x} = \sum_{n\in\mathbb{Z}^d} \beta_n(t)e_n(x) \tag{1.2}$$

where $\beta_n(t)_{n\in\mathbb{Z}^d}$ is a collection of independent, $\mathbb{C}$-valued standard Brownian Motions (i.e. $\beta_n = \mathrm{Re}\beta_n + \mathrm{Im}\beta_n$, with both $\mathrm{Re}\beta_n$ and $\mathrm{Im}\beta_n$ being $\mathbb{R}$-valued standard Brownian Motions) and $e_n(x) \overset{\text{def}}{=} e^{in\cdot x}$.

Remark 1.6. *While reading (1.2) one should keep in mind our convention of dropping all expressions involving 2π when using Fourier methods. By definition, we need a complete orthonormal system and hence the summation here would need to be normalised. Further, in the interest of notational ease, we will drop the 2 and just assume that $\mathrm{Var}\beta_n = t$.*

Remark 1.7. *For (SNLW) and (SNLH), we need a real valued noise and hence will further require that $\beta_{-n} = \overline{\beta}_n$. This condition makes sense because for a real valued function, the fourier coefficient at frequency $-n$, is equal to the conjugate of the fourier coefficient at frequency n.*

As is often the case in the theory of differential equations, one doesn't try to solve (SNLS) (or (1.1) for that matter) but instead an analogue of the integral formulation called the mild formulation (or Duhamel's formulation):

$$u(t) = S(t)u_0 - \int_0^t S(t-t')|u|^{k-1}u(t')dt' + \underbrace{\int_0^t S(t-t')\phi \, dW(t')}_{\text{Stochastic Convolution } (\Psi_{\text{Sch}})} \tag{1.3}$$

Here $S(t) = e^{-it\Delta}$, is the so-called linear Schrödinger propagator defined as below. It is a fourier multiplier with the following effect at frequency n: $\widehat{S(t)f}(n) = e^{-it|n|^2}\widehat{f(n)}$.

Those who are familiar with the well-posedness theory for the (NLS) should recognise the above formulation, except for the extra term we call **Stochastic Convolution** (Ψ_{Sch}) (the subscript is meant to emphasise the link to (SNLS). They should also not be surprised that we use similar kind of Banach Fixed Point arguments for (1.3). Before we may get to the alluded to proof, let us study the stochastic convolution more closely. Note first that if ϕ is a diagonal operator in the sense that $\widehat{\phi(e_n)} = \hat{\phi}_n e_n$, with $\hat{\phi}_n$ a constant, then:

$$\begin{aligned}\Psi_{\text{Sch}}(t) &= \int_0^t S(t-t')\phi dW(t') \\ &\overset{(1.2)}{=} \sum_{n\in\mathbb{Z}^d} e_n \int_0^t e^{i(t-t')|n|^2}\hat{\phi}_n d\beta_n(t')\end{aligned} \tag{1.4}$$

This means that each summand is a Wiener Integral. We refrain from going into the construction of the Wiener Integral, but the following theorem showcases the property of the course we will be needing in the course.

Theorem 1.8 ($\mathbb{R}$-valued Wiener Integral)**.** *Let $f \in L^2([a,b])$, be a deterministic function. Then for a standard Brownian Motion, B, and the function f, the Wiener Integral - $I(f) = \int_a^b f(t)dB(t)$ - is a centred Gaussian random variable with the variance:*

$$\underbrace{\mathrm{Var}(I(f))}_{\mathbb{E}|I(f)|^2} = \|f\|^2_{L^2([a,b])}$$

For a complete construction the interested reader may refer to the classic [9] for an exhaustive overview or the more recent [8] for a very readable introduction.

Remark 1.9. *To deal with $\mathbb{C}$-valued functions we simply decompose it into its real and imaginary parts and use Theorem 1.8 on either of the components. Due to our assumption on the normalisation of the variance of the $\mathbb{C}$-valued Brownian motion, it too has the same variance as in the real case (otherwise there would be a factor of 2).*

Remark 1.10. *With reference to the variance of the Wiener integral, we can conclude that I is an isometry from $L^2([a,b])$ onto its image in $L^2(\Omega)$.*

A less general formulation in the case that f is a bit more regular is given by the Paley-Wiener-Zygmund Integral:

Definition 1.11 (Paley-Wiener-Zygmund Integral)**.** *If $f \in C^1([a,b])$, such that $f(a) = f(b) = 0$ then the Paley-Wiener-Zygmund Integral is defined by:*

$$I(f) = \int_a^b f dB = -\int_a^b f'(t)B(t)dt \text{ pathwise} \tag{1.5}$$

While the Paley-Wiener-Zygmund integral is less general than the Wiener Integral, it gives us pathwise integration. Indeed the integrand on the RHS of (1.5) is continuous (a.s.) and there is no difficulty in understanding the integral in the Riemann sense. Of course where they both exist, the integrals must coincide. Having defined (the integral that defines) Stochastic Convolution, we now collect some definitions and results we will need to formulate our result on its regularity:

Theorem 1.12 (Kolmogorov Continuity Criterion)**.** *For a stochastic process $X = \{X_t\}_{t\geq 0}$ taking values in a complete metric space, suppose that the following holds:*

$$\mathbb{E}[d(X_s, X_t)^p] \leq C_0|s-t|^{1+\alpha}$$

for some $p, \alpha > 0$. Then:

$$\mathbb{P}\left(\sup_{s\neq t} \frac{d(X_s, X_t)}{|s-t|^{\alpha/p-\varepsilon}} \geq \lambda\right) \leq \frac{C_1}{\lambda^p} \qquad \forall\ 0 < \varepsilon < \frac{\alpha}{p}$$

That is to say that X is $(\frac{\alpha}{p} - \varepsilon)$-Hölder continuous.

Example 1 (Brownian Motion)**.**

$$\mathbb{E}|B(t_2) - B(t_1)|^2 = t_2 - t_1 \Rightarrow \mathbb{E}|B(t_2) - B(t_1)|^p \sim |t_2 - t_1|^{p/2}$$

By setting $p/2 = 1 + \alpha$, we get:

$$\frac{\alpha}{p} - = \frac{p/2 - 1}{p} - = \frac{1}{2} - \frac{1}{p} -$$

By the expression $\beta-$ we mean all $\beta - \varepsilon$, for $\varepsilon > 0$. As the above is true for all p, we can conclude that the Brownian Motion is a.s. $(\frac{1}{2} - \varepsilon)$-Hölder continuous.

We will also need the knowledge of the following operators and spaces from Functional Analysis:

Definition 1.13 (Hilbert-Schmidt Operators)**.** *Let X and Y be two Hilbert Spaces and T be a bounded linear operator from X to Y. We say that T is a Hilbert-Schmidt Operator if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of U.*

One can check that the above sum is actually independent of the choice of orthonormal basis. It is also easy to see that space of such operators are linear. In fact more is provable:

Proposition 1.14. *The space of Hilbert-Schmidt operators from X to Y (which are both as before), $\mathrm{HS}(X;Y)$, is a Hilbert space with scalar product and norm defined by:*

$$\langle T, S\rangle_{\mathrm{HS}(X;Y)} = \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_Y \qquad \|T\|_{\mathrm{HS}(X;Y)} = \sum_{k=1}^{\infty} (\|Te_k\|_Y^2)^{\frac{1}{2}}$$

Definition 1.15 (Sobolev spaces (Bessel Potential Space); $W^{s,p}$)**.** *We denoted by $W^{s,p}$ the space of those functions, for which the following norm is finite:*

$$\begin{aligned} \|f\|_{W^{s,p}} &= \|\langle\nabla\rangle^s f\|_{L^p} & \langle\nabla\rangle &= \sqrt{1-\Delta} \\ &= \|\mathcal{F}^{-1}(\langle n\rangle^s \hat{f}(n))\|_{L^p} & \langle\cdot\rangle &= \sqrt{1+|\cdot|^2} \end{aligned}$$

with $s \in \mathbb{R}$ and $1 \le p \le \infty$. When $p = 2$, by Plancherel's identity we have $W^{s,2} = H^s$, with:

$$\|f\|_{H^s} = \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2n} |\hat{f}(n)|^2\right)^{\frac{1}{2}}$$

Theorem 1.16 (Sobolev Embedding Theorems)**.** *Let $1 < p < q < \infty$ be such that $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$. Then one has:*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^d)}$$

If the function is further assumed to be mean zero, we get the same result on $\mathbb{T}^d$.
For $sr > d$, the following inequality holds on both $\mathbb{R}^d$ and $\mathbb{T}^d$:

$$\|f\|_{L^\infty} \lesssim \|f\|_{\dot{W}^{s,r}}$$

where by $\dot{W}^{s,r}$ we mean the Sobolev space, wherein $\langle\cdot\rangle$ has been replace by $|\cdot|$

Proof. Refer to [13] □

We are now able to state the first result of this course:

Proposition 1.17. *If $\phi \in HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$ for $s \in \mathbb{R}$ then $\Psi_{\text{Sch}} \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha,r}(\mathbb{T}^d)$ for $r \leq \infty$ and $\alpha > 0$ a.s.. For $r = 2$ in particular $\Psi_{\text{Sch}} \in C_t H_x^s$, a.s.*

By $C_T H^s$ one means the space $C([0,T]; H^s)$ and the norm on the space is the natural mixed norm. The proof is postponed till the next lecture.

Remark 1.18. *In the Banach setting we have the same result as proposition 1.17 but with the hypothesis of ϕ being a Hilbert-Schmidt operator being replaced with γ-radonifying operator. The reader may refer to appendix of* [10] *for further details.*

Remark 1.19. *On $\mathbb{R}^d$, we have the result $\Psi \in L^q W_x^{s,r}$, for $q < \infty$, $r \leq \frac{2d}{d-2}$. We don't prove this but the interested reader may want to refer to* [11] *or* [12].

2. Lecture 2

We remark first that assumption of diagonality on ϕ implies:

$$\begin{aligned} \|\phi\|_{HS(L^2;H^s)} &= \Big(\sum_{n\in\mathbb{Z}^d} \|\phi(e_n)\|_{H^s}^2\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{\phi}|^2\Big)^{\frac{1}{2}} \end{aligned}$$

before starting the proof of proposition 1.17.

Proof. (of Proposition 1.17) Assume $t \leq \tau$.

$$\begin{aligned} \mathbb{E}[\Psi_{\text{Sch}}(x,t)\overline{\Psi_{\text{Sch}}(y,\tau)}] &= \sum_{n\in\mathbb{Z}^d} |\hat{\phi}_n|^2 e_n(x-y) \int_0^t e^{i(t-t')|n|^2} e^{-i(\tau-t')|n|^2} dt' \\ &= \sum_{n\in\mathbb{Z}^d} |\hat{\phi}_n|^2 e_n(x-y) \int_0^t e^{i(t-\tau)|n|^2} dt' \qquad (2.1) \\ &= \sum_{n\in\mathbb{Z}^d} |\hat{\phi}_n|^2 e_n(x-y) t e^{i(t-\tau)|n|^2} \end{aligned}$$

where the first equality comes from the independence of the Brownian motions and the centredness of the Wiener Integral. Consider then:

$$\mathbb{E}\big[\langle\nabla_x\rangle^s \Psi_{\text{Sch}}(x,t)\overline{\langle\nabla_y\rangle^s \Psi_{\text{Sch}}(y,\tau)}\,\big] = \sum_{n\in\mathbb{Z}^d} |\hat{\phi}_n|^2 \langle n\rangle^{2s} e_n(x-y) t e^{i(t-\tau)|n|^2}$$

By setting $x = y$ and $t = \tau$ in above, and recalling that Wiener Integral is a Gaussian random variable, we may get the following bound on its p-th moment:

$$\begin{aligned} \mathbb{E}[|\langle\nabla\rangle^s \Psi_{\text{Sch}}(x,t)|^p] &\leq p^{\frac{p}{2}} \Big(\mathbb{E}\big[|\langle\nabla\rangle^s \Psi(x,t)|^2\big]\Big)^{\frac{1}{2}} \\ &= p^{\frac{p}{2}} t^{\frac{p}{2}} \|\phi\|_{\text{HS}(L^2;H^s)}^p \end{aligned}$$

■ $\boldsymbol{p \geq r}$,

1) $\boldsymbol{r < \infty}$:

From the Definition 1.15 and Minkowski's Integral Inequality (noting that it is in the hypothesis of Minkowski's that we use $p \geq r$ assumption):

$$\big\| \|\Psi_{\text{Sch}}(t)\|_{W_x^{s,r}} \big\|_{L^p(\Omega)} \leq \big\| \|\langle \nabla \rangle^s \Psi_{\text{Sch}}(t)\|_{L^p(\Omega)} \big\|_{L_x^r}$$
$$\lesssim p^{\frac{1}{2}} t^{\frac{1}{2}} \|\phi\|_{\text{HS}(L^2;H^s)}$$

For the second inequality we have used the p-th moment bound derived prior and the fact that $|\mathbb{T}^d| \sim 1$. This proves that, for fixed t, if $\phi \in HS(L^2, H^s)$ then $\Psi_{\text{Sch}}(t) \in W_x^{s,r}$ (a.s.). A consequence of Chebyshev Inequality is the bound:

$$\mathbb{P}(\|\Psi_{\text{Sch}}(t)\|_{W_x^{s,r}} > \lambda) \leq Ce^{-c\lambda^2/t\|\phi\|_{\text{HS}(L^2;H^2)}}$$

2) $\boldsymbol{r = \infty}$: By theorem 1.16, we are able to proceed as before:

$$\big\| \|\Psi_{\text{Sch}}(t)\|_{W_x^{s,\infty}} \big\|_{L^p(\Omega)} \leq \big\| \|\Psi_{\text{Sch}}(t)\|_{W_x^{s+\varepsilon,r}} \big\|_{L^p(\Omega)}$$
$$\lesssim p^{\frac{1}{2}} t^{\frac{1}{2}} \|\phi\|_{\text{HS}(L^2;H^{s+\varepsilon})}$$

We can adjust the ε to conclude that, for a fixed t, $\Psi_{\text{Sch}}(t) \in W_x^{s-\varepsilon,\infty}$ if $\phi \in \text{HS}(L^2, H^s)$. A similar tail bound to the one in the case $r < \infty$ can be deduced here but there will be a loss of regularity.

What we have proven till now, is that if we fix a t then one can find a set of ω with full probability such that the stochastic convolution on it, is in the purported space. However this set of events is dependent on the choice of t and because t comes from an uncountable set, there is no guarantee that the set of events for which the stochastic convolution is in the purported space for any t is of full probability. To remedy this we bound the difference operator on Stochastic Convolution and apply Theorem 1.12.

To this end, define for a given $h \in \mathbb{R}$, $\delta_h \Psi_{\text{Sch}}(x,t) \stackrel{\text{def}}{=} \Psi_{\text{Sch}}(x, t+h) - \Psi_{\text{Sch}}(x,t)$. Then we compute:

$$\begin{aligned}
\mathbb{E}\big[\delta_h \Psi_{\text{Sch}}(x,t)\overline{\delta_h \Psi_{\text{Sch}}(y,t)}\big] &= \mathbb{E}\big[\Psi_{\text{Sch}}(x,t+h)\overline{\Psi_{\text{Sch}}(y,t+h)}\big] \\
&\quad - \mathbb{E}\big[\Psi_{\text{Sch}}(x,t+h)\overline{\Psi_{\text{Sch}}(y,t)}\big] \\
&\quad - \mathbb{E}\big[\Psi_{\text{Sch}}(x,t)\overline{\Psi_{\text{Sch}}(y,t+h)}\big] \\
&\quad + \mathbb{E}\big[\Psi_{\text{Sch}}(x,t)\overline{\Psi_{\text{Sch}}(y,t)}\big]
\end{aligned}$$

If we assume $t, h > 0$, we get:

$$= \sum_{n \in \mathbb{Z}^d} |\hat{\phi}_n|^2 e_n(x-y) \underbrace{\big\{(t+h) - te^{-ih|n|^2} - te^{ih|n|^2} + t\big\}}_{F_n(t,h)}$$

It is easy to see that $F_n(t,h) \xrightarrow{h\to\infty} 0$ and $|F_n(t,h)| \lesssim |t| + |h|$ and also that:

$$
\begin{aligned}
|F_n(t,h)| &\le |h| + t|1-e^{-ih|n|^2}| + t|1-e^{ih|n|^2}| \\
&\lesssim |h| + t\min(1,|h||n|^2) \qquad\qquad (2.2)\\
&\lesssim |h| + t|h|^{\alpha}|n|^{2\alpha} \qquad\qquad (\text{for all } 0\le\alpha\le 1)
\end{aligned}
$$

One may see that the second inequality holds by triangle inequality and mean value theorem. A similar calculation as in the beginning of the proof yields:

$$
\begin{aligned}
\left\|\langle\nabla\rangle^s\delta_h\Psi_{\text{Sch}}(x,t)\right\|_{L^p(\Omega)} &\le p^{\frac{1}{2}}\left\|\langle\nabla\rangle^s\delta_h\Psi_{\text{Sch}}(x,t)\right\|_{L^2(\Omega)} \\
&\lesssim p^{\frac{1}{2}}(1+T)^{\frac{1}{2}}|h|^{\alpha/2}\Big(\sum_n\langle n\rangle^{2(s+\alpha)}|\hat{\phi}_n|^2\Big)^{\frac{1}{2}} \quad \forall t\in[1,T]
\end{aligned}
$$

The second inequality follows from (2.2) because one h is assumed to be small, and $t \le T$ by design. As before:

$$\left\|\|\delta_h\Psi_{\text{Sch}}(t)\|_{W_x^{s,r}}\right\|_{L^p(\Omega)} \lesssim C_T p^{\frac{1}{2}}\|\phi\|_{\text{HS}(L^2;H^{s+\alpha})}|h|^{\frac{\alpha}{2}}$$

Using theorem 1.12 (identifying s with $t+h$, and replacing $s+\alpha$ by s) we can conclude that $\Psi_{\text{Sch}} \in C_t^{\frac{\alpha}{2}-\frac{1}{p}-}W_x^{s-\alpha,r}$ (a.s.) if $\phi \in \text{HS}(L^2;H^s)$. By making p arbitrarily large, we get the required conclusion.

3) $\boldsymbol{r=2}$: The α in the previous conclusion can be traced back to the presence of t (and τ) in $S(t-t')$ ($S(\tau-t')$). While this problem is something one has to accept for the general result, the boundedness of $S(t)$ in L^2, allows us to do better. Infact, when $r=2$, it suffices to study the continuity property of:

$$\tilde{\Psi}_{\text{Sch}}(t) = S(-t)\Psi_{\text{Sch}}(t) = \int_0^t S(-t')\phi dW(t')$$

Doing a similar argument as before, we get:

$$\mathbb{E}[\tilde{\Psi}_{\text{Sch}}(x,t)\overline{\tilde{\Psi}_{\text{Sch}}(y,\tau)}] = \sum_n |\hat{\phi}_n|^2 e_n(x-y)t$$

and that:

$$\left\|\|\delta_h\tilde{\Psi}_{\text{Sch}}(t)\|_{W_x^{s,r}}\right\|_{L^p(\Omega)} \lesssim C_T p^{\frac{1}{2}}\|\phi\|_{\text{HS}(L^2;H^{s+\alpha})}|h|^{\frac{1}{2}}$$

By theorem 1.12, we get:

$$\tilde{\Psi}_{\text{Sch}} \in C_t^{\frac{1}{2}-}H_x^s, \text{ a.s.}$$

The difference defined by: $\Psi_{\text{Sch}}(t+h) - \Psi_{\text{Sch}}(t) = S(t+h)[\tilde{\Psi}_{\text{Sch}}(t+h) - \tilde{\Psi}_{\text{Sch}}(t)] + [S(t+h)-S(t)]\tilde{\Psi}_{\text{Sch}}(t)$ then goes to zero, as does h. We can see this for the first term by recalling that $S(t)$ is unitary, so is safely dropped and then the continuity property of $\tilde{\Psi}_{\text{Sch}}(t)$ just derived forces it to go to zero, with h. For the second term, we recall that $S(\cdot)f$ is continuous as a map in t into H^s. We can conclude then:

$$\Psi_{\text{Sch}} \in C_t H_x^s$$

almost surely, as required.

□

Example 2. *Consider $\phi = \mathrm{Id}$, that is the identity operator. In this case, $\phi \in \mathrm{HS}(L^2; H^s)$ when $s < -\frac{d}{2}$. Recall that $-\frac{d}{2}$ is exactly the spatial regularity of the (space-time) white noise. Then proposition 1.17 says that $\Psi_{\mathrm{Sch}} \in C_t H_x^s$ a.s., when $s < -\frac{d}{2}$. This means that if you start with a white noise, there is no improvement in the regularity of the Stochastic convolution. This makes analysis of the* (SNLS) *difficult.*

We are able to do to similar investigations into the (SNLW) and (SNLW). With ϕ again taken to be the identity operator, the stochastic convolutions (and the equations they solve with initial data zero) in these cases will be:

- $\Psi_{\text{wave}}(t) = \displaystyle\int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} dW(t'), \qquad (\partial_t^2 + 1 - \Delta)\Psi = \xi$

  Here:

$$\left(\frac{\sin(t|\nabla|)}{|\nabla|} f\right)^{\widehat{}}(n) = \begin{cases} \frac{\sin(t|n|)}{|n|}\hat{f}(n) & n \neq 0 \\ t\hat{f}(0) & \text{otherwise} \end{cases}$$

  The behaviour at frequency 0 is strange but easily fixed by replacing $|\nabla|$ by $\langle \nabla \rangle$ in the operator above. The only difference appears at frequency 0 and the same result of well-posedness holds. Now the $\langle \nabla \rangle$ in the denominator has the effect of smoothing, and hence we "gain one derivative".

- $\Psi_{\text{heat}}(t) = \displaystyle\int_0^t e^{(t-t')(\Delta-1)} dW(t'), \qquad (\partial_t + 1 - \Delta)\Psi = \xi$ Here:

$$\left(e^{t(\Delta-1)f}\right)^{\widehat{}}(n) = e^{-t(1+|n|^2)}\hat{f}(n)$$

  The 1 shows up here only for convenience and in terms of wellposedness doesn't change anything. Now if one were to go back and repeat the proof of Proposition 1.17 there is a factor $\frac{1}{\langle n \rangle^2}$ in (2.1) because we are in the real case and t' doesn't get cancelled out. This means that there is a gain of one derivative as in the wave case.

Therefore, by repeating the proof of proposition 1.17, and keeping in mind the discussion above, it is possible to conclude that: $\Psi_{\text{heat}}, \Psi_{\text{wave}} \in C_t W_x^{s,\infty}(\mathbb{T}^d)$ a.s., when $s < -\frac{d}{2} + 1$ with ϕ the identity operator.

2.1. **Chapter 1. One-dimensional case.** We return now to (SNLS) and discuss its local well-posedness. Recall that the general equation is given by:

$$\mathrm{SNLS}: \begin{cases} i\partial_t u - \Delta u + |u|^{k-1}u = \phi\xi, & x \in \mathbb{T}^d \\ u|_{t=0} = u_0 \end{cases}$$

We fix $s > \frac{d}{2}$. Then with $\phi \in \mathrm{HS}(L^2, H^s)$ and $u_0 \in H^s(\mathbb{T}^d)$ the Duhamel formulation is given by:

$$u(t) = S(t)u_0 + i\int_0^t S(t-t')|u|^{k-1}u(t')dt' + \Psi(t) \tag{2.3}$$

We denote the RHS of the above equation by $\Gamma_{u_0,\phi}(u)$. Before proving our result on the LWP of (SNLS), we recall the following property of H^s space:

Theorem 2.1. *For $s > \frac{d}{2}$, H^s is an algebra under pointwise products, which is to say:*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$$

Proof. (*Sketch*) One checks first that the following triangle inequality holds: $\langle n_1 + n_2\rangle^s \leq \langle n_1\rangle^s + \langle n_2\rangle^s$, for $s \geq 0$. Then this inequality along with Young's convolution inequality and Cauchy-Schwarz inequality yields: $\|\hat{f} * \hat{g}\|_{\ell^2_n} \lesssim \|\hat{f}\|_{\ell^2_n}\|\hat{g}\|_{\ell^1_n} \lesssim \|\hat{f}\|_{\ell^2_n}\|g\|_{H^s}$ and the required conclusion is easily recovered. □

Now consider the following bound:

$$\|\Gamma_{u_0,\phi}(u)\|_{C_T H^s_x} \leq \|u_0\|_{H^s} + \underbrace{\left\| \int_0^t \underbrace{\|S(t-t')|u|^{k-1}u(t')\|_{H^s_x}}_{\lesssim \|u(t')\|^k_{H^s_x}} dt' \right\|_{L^\infty_T}}_{\leq C_1 T\|u\|^k_{C_T H^s}} + \|\Psi\|_{C_T H^s_x} \tag{2.4}$$

The first term on the RHS follows from the fact that $S(t)$ is unitary in H^s. For the second term, we have again used the fact that $S(t)$ is unitary coupled with Theorem 2.1 and the fact k is odd. With the integrand bounded, the bound on the integral proper is elementary.

Now set $R = R_\omega = 2(\|u_0\|_{H^s} + \|\Psi\|_{C([0,1];H^s)})$. Note that the norm on the stochastic convolution is independent of T; this is because we will want to chose T depending on R in what follows. Further note that the subscript here is just to signal the fact that this R is random, which is because of the randomness of the Stochastic Convolution. To move towards a contraction argument we need to show that $\Gamma_{u_0,\phi}(u)$ is bounded by R, whenever u is in a ball of radius R around the origin in $C_T H^s$. So for $u \in B_R \subset C_T H^s$ and $T < 1$, we need:

$$\begin{aligned} \|\Gamma_{u_0,\phi}(u)\|_{C_T H^s} &\leq \tfrac{1}{2}R + C_1 T R^k \\ &\leq R \end{aligned}$$

The second inequality can be enforced by choosing a (random) $T = T_\omega$ such that $T \ll R$ and $T \sim R^{-\frac{1}{k}}$. This inequality means that $\Gamma_{u_0,\phi} : B_R \mapsto B_R$.

$$\begin{aligned} \|\Gamma_{u_0,\phi}(u) - \Gamma_{u_0,\phi}(v)\|_{C_T H^s} &\leq C_2 T(\|u\|^{k-1}_{C_T H^s} + \|v\|^{k-1}_{C_T H^s})\|u-v\|_{C_T H^s} \\ &\leq C_2 T R^{k-1}\|u-v\|_{C_T H^s} \\ &\leq \frac{1}{2}\|u-v\|_{C_T H^s} \end{aligned}$$

To see the first inequality. one can write $|u|^{k-1}u - |v|^{k-1}v$ as a telescopic sum and then apply Young's Inequality on every summand in the telescopic sum. The second inequality comes from the fact that $u, v \in B_R$ and finally the third inequality comes from choosing a convenient T. We have thus shown that if T is chosen such that both the above inequalities hold then $\Gamma_{u_0,\phi}$ defines a contraction on $B_R \subset C_T H^s$. The Banach Fixed Point theorem then gives us that $\exists!\ u \in B_{R_\omega}$ such that $u = \Gamma_{u_0,\psi}(u)$, on $[0, T_\omega]$, a.s.. The time of local existence is random here, because it depends on ω.

The solution depends continuously on $u_0 \in H^s$ and $\phi \in \mathrm{HS}(L^2; H^s)$. To see this, suppose that you have two solutions with different initial data and smoothing operators: $u \stackrel{\text{def}}{=} \Gamma_{u_0,\phi_1}(u)$ and $v \stackrel{\text{def}}{=} \Gamma_{v_0,\phi_2}(v)$. Then by considering a small enough T (say less than the minimum of the R_ω given by the two data sets and convolutions):

$$\begin{aligned} \|u - v\|_{C_T H^s_x} &\lesssim \|u_0 - v_0\|_{H^s} + \|\underbrace{\Psi_1 - \Psi_2}_{\int_0^t S(t-t')(\phi_1 - \phi_2) dW t'}\| \\ &\leq \|u_0 - v_0\|_{H^s} + C_\omega \|\phi_1 - \phi_2\|_{\mathrm{HS}(L^2;H^s)} \end{aligned}$$

This proves the continuous (in fact Lipshcitz) dependency asserted. In the above calculation we hid the non-linear part on the LHS. For the second inequality we have used the facts $\mathbb{E}\left[\|\Psi - \Psi\|^2_{C_T H^s_x}\right] \leq C_T \|\phi_1 - \phi_2\|^2_{\mathrm{HS}(L^2;H^s)}$ and Chebyshev's inequality. It should be emphasised that the constant C_ω is random.

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1. Lecture 3 (typed by Aigerim Davletzhanova)

Littlewood-Paley decomposition:

$$f = \sum_{N \geq 1} \mathbf{P}_N f = \sum_{j=0}^{\infty} \mathbf{P}_j f,$$

where $N \in 2^j, j \geq 0$ and $\mathbf{P}_N$ is a L^p projector, i.e. "projection" onto the frequency $\{|n| \sim N = 2^j\}$.
$\phi(\xi)$ is a radial, supported on $\{|\xi| \leq 2\}$ and $\phi(\xi) \equiv 1$ on $\{|\xi| \leq 6/5\}$.

$$\phi_j(\xi) = \phi\left(\frac{\xi}{2^j}\right) - \phi\left(\frac{\xi}{2^{j-1}}\right), j \geq 1.$$

$$\widehat{\mathbf{P}_j f}(\xi) = \phi_j(\xi)\widehat{f}(\xi).$$

ϕ_j needs to be normalised, i.e.

$$\psi_j(\xi) = \frac{\phi_j(\xi)}{\sum_{k=0}^{\infty} \phi_k(\xi)},$$

where $\sum_{k=0}^{\infty} \phi_k(\xi)$ is a finite sum.
Finally we get

$$\mathbf{P}_j f = f^{-1}(\psi_j f).$$

L^p **Theorem**: For $1 < p < \infty$ we have,

$$||f||_{L^p} \sim ||(\sum_{N \geq 1, dyadic} |\mathbf{P}_N f|^2)^{\frac{1}{2}}||_{L^p}$$

and RHS is a square function.
If $p \geq 2$, then

$$||f||_{L^p} \leq || \quad ||\mathbf{P}_N f||_{L^p} \quad ||_{l^2_N, dyadic}$$

and $\mathbf{P}_N f$ here is a simpler object, compared to the previous case.

Sobolev spaces:

$$||f||_{H^s} \sim (\sum_{N \geq 1, dyadic} N^{2s} ||\mathbf{P}_N f||^2_{L^2_x})^{\frac{1}{2}}$$

Besov spaces $B^s_{p,q}$ or $B^{s,p}_q$:

$$||f||_{B^s_{p,q}} = ||N^s||\mathbf{P}_N f||_{L^p_x}||_{l^q_{N \geq 1, dyadic}},$$

where $N^s = 2^{js}$ and $l^q_{N \geq 1, dyadic} = l^s_j(\mathbb{Z} \geq 0)$.
There are 2 main points:

- $H^s = B^s_{2,2}$
- $B^s_{p,1} \subset W^{s,p} \subset B^s_{p,\infty}$, where $B^{s+\epsilon}_{p,\infty} \subset B^s_{p,1}$ and $W^{s,p} = L^p_s$.

Hölder-Besov space $C^s = B^s_{\infty,\infty}$, $s \in \mathbb{R}$:

- Natural extension of the classical Hölder space $C^s, 0 < s < 1$.

$$||f||_{C^s} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s},$$

where $\dot{\Lambda}^s$ is a homogeneous Lipshitz space and $\Lambda^s = \dot{\Lambda}^s \cap L^\infty$.
FACT: $\dot{\Lambda}^s = \dot{B}^s_{\infty,\infty}$, $0 < s < 1$ and $||f||_{\dot{B}^s_{p,q}} = ||2^j||\mathbf{P}_j f||_{L^p_x}||_{l^q_j(\mathbb{Z})}$

- $s > 0$: C^s is an algebra and

$$||fg||_{C^s} \lesssim ||f||_{C^s}||g||_{C^s}.$$

1-d SNLW:

$$(\partial_t^2 + 1 - \Delta)u + u^k = \xi \quad \text{on} \quad \mathbb{T}.$$

$$u(t) = \partial_t S(t)u_0 + S(t)u_1 - \int_0^t S(t-t')u^k(t')dt' + \int_0^t S(t-t')dW(t'),$$

where $s(t) = \frac{sin(t<\Delta>)}{<\Delta>}$ and $\Psi = \int_0^t S(t-t')\xi(dt')$ ia s stochastic convolution.
Let's denote RHS by $\Gamma(u) = \Gamma_{(u_0,u_1),\xi}(u)$
There are 2 main points to outline:

- Recall $\Psi = \Psi_{wave} \in C_t W_x^{1/2-,\infty}(\mathbb{T})$
- $||\partial_t S(t)u_0 + S(t)u_1||_{H^s} \lesssim ||(u_0,u_1)||_{\mathcal{H}^s}$, where $\mathcal{H}^s = H^s \times H^{s-1}$

Taking into consideration above, we get

$$||\Gamma(u)||_{C_T H^s} \lesssim ||(u_0,u_1)||_{\mathcal{H}_s} + \int_0^t ||u^k(t')||_{H^{s-1}}dt' + ||\Psi||_{C_T H^s},$$

where

$$||u^k(t')||_{H^{s-1}} \leq ||u^k(t')||_{L^2} = ||u(t')||^k_{L^{2k}} \lesssim_{Sobolev} ||u(t')||^k_{H^s} \quad \text{for} \quad s \geq \frac{1}{2} - \frac{1}{2k}$$

and

$$||\Psi||_{C_T H^s} \leq C_\omega <_{a.s.} \infty \quad \text{for} \quad s < \frac{1}{2}.$$

By choosing $\frac{1}{2} - \frac{1}{2k} \leq s < \frac{1}{2}$, we have

$$||\Gamma(u)||_{C_T H^S} \lesssim ||(u_0,u_1)||_{\mathcal{H}_s} + T||u||^k_{C_T H^s} + ||\Psi||_{C_T H^s}.$$

Similarly,

$$||\Gamma(u) - \Gamma(v)||_{C_T H^s} \lesssim T(||u||^{k-1}_{C_T H^s} + ||v||^{k-1}_{C_T H^s})||u-v||_{C_T H^s}.$$

$$R = R_\omega \sim ||(u_0,u_1)||_{\mathcal{H}_s} + ||\Psi||_{C([0,1],H^s)},$$

therefore Γ is a contraction on $B_R \subset C_T H^s$, $T = T_\omega = T(R_\omega) << 1$.
Remark: By a similar argument, we can show LWP for

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

It is important for global-in-time study.

1-d SNLH:

$$(\partial_t + 1 - \Delta)u + u^k = \xi$$

Schauder estimate: For $1 \leq p \leq q \leq \infty$,

$$||\mathcal{D}^\alpha \mathbf{P}(t) f||_{L^q_x} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}} ||f||_{L^p_x} \quad \text{on} \quad \mathbb{R}^d \quad \text{or} \quad \mathbb{T}^d, t > 0, \alpha \geq 0,$$

where $\mathbf{P}(t) = e^{t(\Delta - 1)}$.
Proof: Proof in the detail also can be found in [1] and Grafakos.

- $\mathbb{R}^d$ case

$$e^{t\Delta} f(x) = \int K_t(x-y) f(y) dy$$

$$\widehat{K_t}(\xi) = e^{-t|\xi|^2} = \widehat{K}(t^{\frac{1}{2}}\xi), \quad \text{where} \quad K = K_1.$$

By applying inverse Fourier Transform, we have

$$K_t(x) = \frac{1}{t^{\frac{\alpha}{2}}} K\left(\frac{x}{t^{\frac{1}{2}}}\right),$$

$$||K_t||_{L^r_x} = t^{-\frac{\alpha}{2}} \left|\left| K\left(\frac{x}{t^{\frac{1}{2}}}\right)\right|\right|_{L^r_x} = t^{-\frac{\alpha}{2}} t^{\frac{d}{2r}} C_K \sim \text{(Young's inequality)} \sim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})},$$

therefore

$$||e^{t\Delta} f||_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} ||f||_{L^p(\mathbb{R}^d)}.$$

In case of $\alpha > 0$, we have:

$$\mathcal{D}^\alpha(e^{t\Delta} f) = \mathcal{D}^\alpha(K_t * f) = (\mathcal{D}^\alpha K_t) * f,$$

$$\widehat{\mathcal{D}^\alpha K_t}(\xi) = |\xi|^\alpha e^{-t|\xi|^2} = t^{-\frac{\alpha}{2}} (t^{\frac{1}{2}}|\xi|)^\alpha e^{-t|\xi|^2} = t^{-\frac{\alpha}{2}} \widehat{G_t}(\xi) = t^{-\frac{\alpha}{2}} \widehat{G_1}(t^{\frac{1}{2}}\xi), \quad \text{where} \quad G = G_1.$$

By repeating the same computation, we get

$$||\mathcal{D}^\alpha K_t||_{L^r} = t^{-\frac{\alpha}{2}} ||G_t(x)||_{L_r} \sim t^{-\frac{\alpha}{2}} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

and Young's inequality($\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$) could be applied again.

- $\mathbb{T}^d$ case

$$e^{t\Delta} f = R_t * f, \quad \widehat{R_t}(n) = e^{-t|n|^2}.$$

Scaling argument can't be used, but Poisson summation formula can be:

$$|f(x) + |\widehat{f}(x)| \lesssim < x >^{-d-\epsilon}, \quad f \quad \text{on} \quad \mathbb{R}^d, \quad \text{where}$$

$$\sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{inx} = \sum_{n \in \mathbb{Z}^d} f(x+n)$$

Proof of the Poisson summation formula:

$$F(x) = \sum_{n \in \mathbb{Z}^d} f(x+n)$$

is a periodic function on $\mathbb{T}^d$ and $F(x) = \sum_{n \in \mathbb{Z}^d} \widehat{F}(n) e^{inx}$, therefore

$$\widehat{F}(n) = \int_{\mathbb{T}^d} F(x) e^{-inx} dx = \int_{[-\frac{1}{2},\frac{1}{2})^d} \sum_m f(x+m) e^{-inx} dx = \sum_{m \in \mathbb{Z}^d} \int_{m+\mathbb{T}^d} f(y) e^{-iny} dy = \widehat{f}(n)$$

$\square$

Back to the proof of the Schauder estimate on $\mathbb{T}^d$.

$$\widehat{R_t}(n) = \widehat{K_t}(n) = e^{-t|n|^2} \quad \text{and}$$

$||R_t||_{L^r(\mathbb{T}^d)} = ||\sum \widehat{K_t}(n)e^{inx}||_{L^r(\mathbb{T}^d)} = \textit{Poisson formula} = ||\sum_{n\in\mathbb{Z}^d} K_t(x + n)||_{L^r(\mathbb{T}^d)} \lesssim ||(\sum < n >^{-\beta r'})^{\frac{1}{r'}}|| < n >^{\beta} K_t(x+n)||_{l^r_n}||_{L^r_x(\mathbb{T}^d)} \lesssim || < x >^{\beta} K_t(x)||_{L^r_x(\mathbb{R}^d)} \sim ||K_t(x)||_{L^r_x(\mathbb{R}^d)} + |||x|^{\beta} K_t(x)||_{L^r_x(\mathbb{R}^d)}$.

By repeating the previous argument and knowing that $|x|^{\beta} \lesssim \left|\frac{x}{t^{\frac{1}{2}}}\right|^{\beta} and K_t(x) = \frac{1}{t^{\frac{\alpha}{2}}} K\left(\frac{x}{t^{\frac{1}{2}}}\right) \quad \text{for} \quad 0 < t < 1$, we get Schauder estimate for $e^{t\Delta}$ on $\mathbb{T}^d$, $0 < t \leq 1$.

As for $\mathbf{P}(t) = e^{t(\Delta-1)} = e^{-t}e^{t\Delta}, e^{-t}$ can absorb $t^{\frac{\beta}{2}}$ for any t, therefore we get Schauder estimate for $\mathbf{P}(t)$ on $\mathbb{T}^d$, $t > 0$ $\square$

Note:

$$||\mathbf{P}(t)f||_{C^{s_1}} \lesssim t^{-\frac{s_1-s_2}{2}}||f||_{C^{s_2}}, \quad s_1 \geq s_2$$

Back to SNLH on $\mathbb{T}$:

$$u(t) = \mathbf{P}(t)u_0 - \int_0^t \mathbf{P}(t-t')u^k(t')dt' + \int_0^t \mathbf{P}(t-t')dW(t')$$

and $\Psi = \int_0^t \mathbf{P}(t-t')dW(t')$, $\Psi \in C_t C_x^{\frac{1}{2}-}$, $C^{s+\epsilon} \subset W^{s,\infty}$, where $s = \frac{1}{2} - 2\epsilon$.
$\Gamma(u)$ is a RHS of the Duhamel formulation and

$$||\Gamma(u)(t)||_{C^s_x} \lesssim ||u_0||_{C^s_x} + \int_0^t ||u(t')||^k_{C^s_x} dt' + ||\Psi(t)||_{C^s_x}, \quad 0 < s < \frac{1}{2}$$

$$||\Gamma(u)||_{C_T C^s_x} \lesssim ||u_0||_{C^s_x} + T||u||^k_{C_T C^s_x} + ||\Psi||_{C_T C^s_x}, \quad T \leq 1.$$

Considering above and difference estimate, we get LWP in $C^s(\mathbb{T}^d)$, $\quad 0 < s < \frac{1}{2}$.
Rougher data? $u_0 \in C^s$, $s < 0$,

$$||\mathbf{P}(t)u_0||_{C^{\sigma}} \lesssim t^{-\frac{\sigma-s}{2}}||u_0||_{C^s}, \quad \sigma \geq s.$$

The right hand side blows up as $t \to 0+$.
For the case $\sigma > s$:

$$||u||_{Y^{\sigma}(T)} = \sup_{0\leq t\leq T} t^{\theta}||u(t)||_{C^{\sigma}}$$

$$t^{\theta}||\Gamma(u)(t)||_{C^{\sigma}} \lesssim t^{\theta}t^{-\frac{\sigma-s}{2}}||u_0||_{C^s} + t^{\theta}\int_0^t (t')^{-\theta k}((t')^{\theta}||u(t')||_{C^{\sigma}})^k dt' + t^{\theta}||\Psi(t)||_{C^{\sigma}},$$

where $\frac{\sigma-s}{2} < \theta < \frac{1}{k}$.
Take $\sup_{0\leq t\leq T}$ and get estimate for the $Y^{\sigma}(T)$ norm. For the $s < 0$,

$$||\Gamma(u)||_{C_T C^s_x} \lesssim ||u_0||_{C^s} + ||\int_0^t (t')^{-\theta k}((t')^{\theta}||u(t)||_{C^{\sigma}})^k dt'||_{L^{\infty}_T} + ||\Psi||_{C_T C^s_x}$$

$\theta - \frac{\sigma-s}{2} \geq 0 \Rightarrow s \geq \sigma - 2\theta$, $s > -\frac{2}{k}$, because $\sigma > 0, \quad \theta < \frac{1}{k}$.
Run a contraction argument on a ball in $Y^{\sigma}(T)$ and show $u \in C_T C^s_x \Rightarrow u \in C([0, T_{\omega}]; C^s) \cap C((0, T_{\omega}]; C^{\sigma})$.

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GLOBAL WELL-POSEDNESS OF 1D CUBIC STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION USING ITO APPROACH

PIERRE DE ROUBIN

Contents

1. Local well-posedness of SNLS on $\mathbb{R}$.

Let us first recall the form of the cubic stochastic nonlinear Schrödinger equation in dimension d :

$$i\partial_t u(t,x) - \partial_x^2 u(t,x) + |u|^2 u(t,x) = \phi(t,x)\xi(t,x), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d \tag{1.1}$$

where ξ is a space-time white noise and $\phi \in \mathrm{HS}(L^2; L^2)$ so that, if we denote

$$\Psi(t) = \int_0^t S(t-t')\phi \mathrm{d}W(t'), \qquad \forall t \in [0,T] \tag{1.2}$$

with $S(t)f = e^{-it\Delta}f = \mathcal{F}\left(e^{it|\cdot|^2}\widehat{f}(\cdot)\right)$, then $\Psi \in C_T L_x^2 \cap L_T^q L_x^r$ for any finite $q \geq 1$ and $r \leq 1$ such that

$$\begin{cases} r \leq \frac{2d}{d-2} & \text{if} \quad d \geq 3 \\ r < \infty & \text{if} \quad d = 1,2. \end{cases}$$

Remark 1.1. *If $\phi \in \mathrm{HS}(L^2; H^s)$ with $s \in \mathbb{R}$, then we have $\Psi \in C_T L_x^2 \cap L_T^q W_x^{s,r}$. The proof of this result can be found in* [6] *or* [17].

In order to solve equation 1.1, we need to state some estimates. First, observe that using Plancherel's theorem twice, we get for $f \in L^2$:

$$\|S(t)f\|_{L_x^2} = \|e^{-it|\xi|^2}\widehat{f}(\xi)\|_{L_\xi^2} = \|\widehat{f}(\xi)\|_{L_\xi^2} = \|f\|_{L_x^2}$$

so that $S(t)$ is *unitary* for any $t \in \mathbb{R}$. But this will not be sufficient, we will need the so-called *Strichartz estimates*. To do so, let us first define the following :

Definition 1.2. *A couple of real numbers (q, r) is called* Schrödinger admissible *if it satisfies $2 \leq q, r \leq \infty$, $(q, r, d) \neq (2, \infty, 2)$ (with d the dimension) and the following* scaling condition*:*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{1.3}$$

This allows us to state the following (note that, for any real number $q \geq 1$, we denote $q' \geq 1$ the real number such that $\frac{1}{q} + \frac{1}{q'} = 1$):

Theorem 1.3 (Strichartz estimates on $\mathbb{R}^d$)**.** *Let (q, r) and $(\widetilde{q}, \widetilde{r})$ be Schrödinger admissible. Let us take f a function of the space variable $x \in \mathbb{R}^d$ and F a function of the space-time variable $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. Then, we have:*

(1) *Homogeneous estimate:*

$$\|S(t)f\|_{L_t^q L_x^r\left(\mathbb{R}\times\mathbb{R}^d\right)} \lesssim \|f\|_{L_x^2} \tag{1.4}$$

(2) *Dual homogeneous estimate:*

$$\left\|\int_{\mathbb{R}} S(-t')F(t')\mathrm{d}t'\right\|_{L^2\left(\mathbb{R}^d\right)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \tag{1.5}$$

(3) *Non-homogeneous estimate / Retarded estimate:*

$$\left\|\int_0^t S(t-t')F(t')\mathrm{d}t'\right\|_{L_t^q L_x^r\left(\mathbb{R}\times\mathbb{R}^d\right)} \lesssim \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \tag{1.6}$$

Note that this means, on time averaged sense, that there is a smoothing in terms of integrability (but **NOT** in terms of differentiability). We will give an idea of the proof of Theorem 1.3, but first we need to state two preliminary results.

Theorem 1.4 (Dispersive estimate)**.** *Let $f \in L^1\left(\mathbb{R}^d\right)$, then for any $t > 0$:*

$$\|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1} \tag{1.7}$$

We will not prove this estimate here, but the proof relies on two ideas. First, we need to express $S(t)f$ in the following way:

$$\left[S(t)f\right](x) = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4ti}} f(y)\mathrm{d}y$$

by using the method of the stationnary phase (see other lecture notes) by seeing $S(t)f$ as something of the form:

$$\left[S(t)f\right] = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi)\mathrm{d}\xi.$$

Also, we need the following inequality:

Theorem 1.5 (Hardy-Littlewood-Sobolev inequality)**.** *Let $1 < p, q, r < \infty$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. Then, for any function $g \in L^p\left(\mathbb{R}^d\right)$,*

$$\left\||x|^{-\frac{d}{p}} * g\right\|_{L^r\left(\mathbb{R}^d\right)} \lesssim \|g\|_{L^q\left(\mathbb{R}^d\right)} \tag{1.8}$$

Note that, in Theorem 1.5, the function $x \to |x|^{-\frac{d}{p}}$ does not belong to $L^r\left(\mathbb{R}^d\right)$, but "almost". Hence, Theorem 1.5 can be seen as an endpoint version of Young's inequality.

Idea of proof of Theorem 1.3. First, observe that by interpolating the dispersive estimate (1.7) and the estimate resulting from the unitarity of $S(t)$ in L^2:

$$\|S(t)f\|_{L^2} = \|f\|_{L^2}$$

we get, for any p, p' such that $p \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\|S(t)f\|_{L^p} \lesssim t^{-\left(\frac{d}{2}-\frac{d}{p}\right)} \|f\|_{L^{p'}}. \tag{1.9}$$

Also, we claim that saying $S(t)$ is a bounded operator from L^2_x to $B = L^q_t L^r_x$ is equivalent to say that there is a bounded dual operator $T^* : B' = L^{q'}_t L^{r'}_x \to L^2_x$ defined by :

$$T^*F = \int_{\mathbb{R}} S(-t')F(t')\mathrm{d}t' \tag{1.10}$$

Indeed, observe that :

$$\langle S(t)f, F\rangle_{L^2_{t,x}} = \int_{\mathbb{R}}\int_{\mathbb{R}^d} S(t)f\overline{F(t,x)}\mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}^d} f\overline{S(-t)F(t,x)}\mathrm{d}t\mathrm{d}x = \langle f, T^*F\rangle_{L^2_{t,x}}$$

where the last inequality comes from the definition of the dual operator. Thus, note also that saying $T = S(t)$ is bounded is equuivalent to say that the operator $TT^* : B' \to B$, defined by

$$TT^*F = \int_{\mathbb{R}^d} S(t-t')F(t')\mathrm{d}t'$$

is bounded. We will prove, in fact, this last result. To do so, put TT^*F in $B = L^q_t L^r_x$, with (q, r) Schrödinger admissible, and then, using equation 1.9:

$$\|TT * F\|_{L^q_t L^r_x} = \left\| \int_{\mathbb{R}} \|S(t-t')F(t')\|_{L^r_x} \mathrm{d}t' \right\|_{L^q_t} \lesssim \left\| \int_{\mathbb{R}} (t-t')^{-\left(\frac{d}{2}-\frac{d}{r}\right)} \|F(t')\|_{L^{r'}_x} \mathrm{d}t' \right\|_{L^q_t}$$

Thus, we get a convolution in time in the last term and observe that, since (q, r) is Schrödinger admissible, then we have:

$$\frac{d}{2} - \frac{d}{r} + \frac{1}{q'} = \frac{d}{2} - \frac{d}{r} - \frac{1}{q} + 1 = \frac{1}{q} + 1$$

so that, applying Theorem 1.5 for $t \in \mathbb{R}$ we get:

$$\|TT * F\|^q_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}$$

Thus, equations (1.4) and (1.5) come from the facts that $S(t) : L^2_x \to L^q_t L^r_x$ is bounded and $T^* : L^{q'}_t L^{r'}_x \to L^2_x$ is bounded respectively. There is also two ways to prove equation (1.6). Either we observe that $\int_0^t \mathrm{d}t' = \int_{\mathbb{R}} \chi_{[0,t]}(t')\mathrm{d}t'$ and we prove it by hand, or we can use *Christ-Kiselev lemma* (see [18]).

The endpoint case, with $q = 2$ and $r = \frac{2d}{d-2}$, can be found in [8].

□

Now that we proved Strichartz estimates, we can go back to the cubic SNLS on $\mathbb{R}$, namely the equation (1.1). We have the *Duhamel formulation*:

$$u(t) = S(t)u_0 - \int_0^t S(t-t')\,|u|^2\,u(t')\mathrm{d}t' + \Psi(t) =: \Gamma u(t) \tag{1.11}$$

Thus, as $(\infty, 2)$ and $(8, 4)$ are two Schrödinger admissible couples in dimension 1, if we denote $X(T) \coloneqq C_T L_x^2 \cap L_T^8 L_x^4$ and $\|\cdot\|_{X(T)}$ the associated norm, we have thanks to Theorem 1.3:

$$\|\Gamma u\|_{X(T)} \lesssim \|u_0\|_{L_x^2} + \left\||u|^2 u\right\|_{L_T^{8/7} L_x^{4/3}} + \|\Psi\|_{X(T)}$$

and then, using Hölder's inequality with $\frac{7}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$, we have

$$\left\||u|^2 u\right\|_{L_T^{8/7} L_x^{4/3}} \leq T^{1/2}\,\|u\|^3_{L_T^8 L_x^4} \leq T^{1/2}\,\|u\|^3_{X(T)}$$

which gives

$$\|\Gamma u\|_{X(T)} \lesssim C_1 \left(\|u_0\|_{L_x^2} + \|\Psi\|_{X(T)}\right) + T^{1/2}\,\|u\|^3_{X(T)}$$

for any $T \leq 1$. Also, we have a similar difference estimates using similar arguments. Then, using a fixed point argument, since Γ is a contraction of a ball $B \subset X(T)$ of size $M \sim \left(\|u_0\|_{L_x^2} + \|\Psi\|_{X(T)}\right)$ (recall $\|\Psi\|_{X(T)}$ is almost surely constant **if** $\phi \in \mathrm{HS}\left(L^2; L^2\right)$), we have local well-posedness in $L^2(\mathbb{R})$.

2. Local well-posedness of SNLS on $\mathbb{T}^d$

2.1. **Zygmund's L^4-Strichartz estimates.** Let us focus now on the stochastic nonlinear Schrödinger equation on the d-dimensional torus $\mathbb{T}^d$:

$$i\partial_t u(t,x) - \partial_x^2 u(t,x) + |u|^2 u(t,x) = \phi(t,x)\xi(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^d \tag{2.1}$$

The main issue here comes from the Strichartz estimates on $\mathbb{T}^d$. Indeed, three problems arise form the latter:

(1) They are only local in time,
(2) They are **NOT** as good as the Strichartz estimates on $\mathbb{R}^d$,
(3) The proof is much harder.

For more references on Strichartz estimates on the torus, one can look into [19], [2] (where Bourgain used analytic number theory with the HL circle method), [3], [4] and [13].

Theorem 2.1 (L^4-Strichartz estimate on $\mathbb{T}$ (Zygmund 1974))**.** *For any $u_0 \in L^2(\mathbb{T})$, we have*

$$\left\|\sum_{n\in\mathbb{Z}} e^{inx} e^{itn^2} \widehat{u}_0(n)\right\|_{L^4_{t,x}(\mathbb{T}\times\mathbb{T})} \lesssim \|u_0\|_{L^2(\mathbb{T})} \tag{2.2}$$

Proof. Let us denote $F(t,x) \coloneqq \sum_{n\in\mathbb{Z}} e^{inx} e^{itn^2} \widehat{u}_0(n)$. Then observe that we have

$$\left\|F\overline{F}\right\|_{L^2(\mathbb{T}^2)} = \left\|\sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \widehat{u}_0(n_1)\overline{\widehat{u}_0(n_2)} e^{it(n_1^2 - n_2^2)} e^{i(n_1-n_2)x}\right\|_{L^2(\mathbb{T}^2)}$$

Write this sum as a Fourier series in t and x, $\sum_{(\tau,n)\in\mathbb{Z}^2} a(\tau,n)e^{i(nx+\tau t)}$ where

$$a(\tau,n) = \sum_{(n_1,n_2)\in P(\tau,n)} \widehat{u}_0(n_1)\overline{\widehat{u}_0(n_2)}$$

with

$$P(\tau,n) = \{(n_1,n_2) : \ n_1^2 - n_2^2 = \tau \text{ and } n_1 - n_2 = n\}$$

Now we claim that, given a couple $(\tau,n) \neq (0,0)$, there exists at most one couple (n_1,n_2) such that $(\tau,n) = (n_1^2 - n_2^2, n_1 - n_2)$. Indeed, observe that if (n_1,n_2) satisfies this condition, we have

$$\tau - n^2 = n_1^2 - n_2^2 - n^2 = -2n_2(n_2 - n_1) = 2nn_2$$

so this determines n_2 and then we have $n = n + n_2$. Also, observe that

$$a(0,0) = \sum_{n\in\mathbb{Z}} |\widehat{u}_0(n)|^2 = \|u_0\|_{L^2}^2 .$$

Putting all this together, we have, using Plancherel's equality:

$$\begin{aligned}
\|F\|_{L^4(\mathbb{T}^2)} &= \left(\sum_{\tau,n\in\mathbb{Z}} |a(\tau,n)|^2\right)^{1/4} \\
&\sim \left(\sum_{(\tau,n)\neq(0,0)} |a(\tau,n)|^2\right)^{1/4} + (a(0,0))^{1/2} \\
&\lesssim \left(\sum_{n_1,n_2\in\mathbb{Z}} |\widehat{u}_0(n_1)\overline{\widehat{u}_0(n_2)}|^2\right)^{1/4} + \|u_0\|_{L^2} \\
&\sim \|u_0\|_{L^2(\mathbb{T})}
\end{aligned}$$

where the last inequality comes from the fact that $\sum_{n_1,n_2} |\widehat{u}_0(n_1)\overline{\widehat{u}_0(n_2)}|^2$ is a disjoint sum in n_1 and n_2. This ends the proof.

□

For more results on Strichartz estimates, specially for the Korteweg-de Vries equation, one can look into the work of Kenig, Ponce and Vega ([9], [10], [11] and [12]).

2.2. **Fourier restriction norm method.** Note that Zygmund's L^4-Strichartz estimates (2.2) is **NOT** enough to prove local well-posedness of cubic NLS in $L^2(\mathbb{T})$ (neither is it for any $H^s(\mathbb{T})$, for any $s \leq \frac{1}{2}$). Instead, we are going to follow the *Fourier restriction norm method* approach, developed by Bourgain in [2]. Also, note that this Fourier restriction norm method was also used by Klainerman and Machedon for the wave equation in [14].

Definition 2.2 ($X^{s,b}$ spaces)**.** *We define $X^{s,b}$, with $s \in \mathbb{R}$ and $b > 0$, as the space of all functions such that the following norm is finite*

$$\|u\|_{X^{s,b}} = \left\| \langle n\rangle^s \langle \tau - |n|^2\rangle^b \widehat{u}(\tau,n)\right\|_{\ell^2_n L^2_\tau(\mathbb{Z}^d\times\mathbb{R})}$$

The $X^{s,b}$ spaces are useful for "perturbative" study: the idea behind is to look for something "close to being a linear solution". Yet, a linear solution should be something like $\mathcal{F}^{-1}(\delta(\tau - n^2))$, with δ the Dirac delta operator. Indeed, if we take the linear Schrödinger equation

$$i\partial_t u - \Delta u = 0$$

and we apply the space-time Fourier transform, we get the equation

$$-(\tau - n^2)\widehat{u}(\tau, n) = 0$$

so that $\widehat{u}(\tau, n)$ is a measure supported on $\{\tau = n^2\}$. Thus, something close to a linear solution should become very small when it is away form the line $\tau = n^2$. We measure this "distance from the linear solution" with the weight $\langle \tau - n^2 \rangle^b$, with $b > 0$, that penalizes functions whose Fourier transforms support is away from $\{\tau = n^2\}$.

Proposition 2.3 (Basic properties)**.** *Let $s \in \mathbb{R}$ and $b > 0$. Then,*

(1) *If $b > \frac{1}{2}$, then $X^{s,b} \subset C_t H^s_x$. If $b = \frac{1}{2}$, we need some other spaces.*

(2)
$$\|u\|_{X^{s,b}} = \|S(-t)u\|_{H^s_x H^b_t} \coloneqq \left\| \langle \partial_x \rangle^s \langle \partial_t \rangle^b \left(S(-t) u(t) \right) \right\|_{L^2_{t,x}} \tag{2.3}$$

(3) *Let $\eta(t)$ be a smooth cutoff function supported on $[-2, 2]$, with $\eta(t) = 1$ for any $|t| \leq 1$. Then,*
$$\|\eta(t) S(t) f\|_{X^{s,b}} \lesssim \|f\|_{H^s} \tag{2.4}$$

(4) *If $b > \frac{1}{2}$, we have the following estimate on the Duhamel term*
$$\left\| \eta(t) \int_0^t S(t - t') F(t') \mathrm{d}t' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}} \tag{2.5}$$
Also, for $\theta > 0$ small, $b > \frac{1}{2}$, $T \leq 1$ and $t \in [-T, T]$, we get the estimate
$$\left\| \eta\left(\frac{t}{T}\right) \int_0^t S(t - t') F(t') \mathrm{d}t' \right\|_{X^{s,b}} \lesssim T^\theta \|F\|_{X^{s,b-1+\theta}} \tag{2.6}$$
and for $b > 0$, $b \neq \frac{1}{2}$,
$$\left\| \eta\left(\frac{t}{T}\right) S(t) f \mathrm{d}t' \right\|_{X^{s,b}} \lesssim T^{\frac{1}{2}-b} \|f\|_{H^s} \tag{2.7}$$

Remark 2.4. *Note that, in equation* (2.7)*, the factor $T^{\frac{1}{2}-b}$ is bad for $b > \frac{1}{2}$ et $T \ll 1$.*

Proof of equation (2.4). Observe that $\mathcal{F}\left[\eta(t) S(t) f\right](\tau, n) = \widehat{\eta}(\tau - |n|^2)\widehat{f}(n)$ so that, if we set $\widetilde{\tau} = \tau - |n|^2$, we have

$$\|\eta(t) S(t) f\|_{X^{s,b}} = \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^b \widehat{\eta}(\tau - |n|^2) \widehat{f}(n) \right\|_{\ell^2_n L^2_\tau} = \left\| \langle n \rangle^s \langle \widetilde{\tau} \rangle^b \widehat{\eta}(\widetilde{\tau}) \widehat{f}(n) \right\|_{\ell^2_n L^2_\tau} = \|\eta\|_{H^b} \|f\|_{H^s}$$

where $\|\eta\|_{H^b}$ is bounded. □

The proofs of the other properties are available in [18]. From now on, we will consider the new mild formulation

$$u(t) = \eta(t) S(t) u_0 - \eta\left(\frac{t}{T}\right) \int_0^t S(t - t') |u|^2 u(t') \mathrm{d}t' + \eta(t) \int_0^t S(t - t') \phi \mathrm{d}W(t') \tag{2.8}$$

In this context, we have the following result

Theorem 2.5 (L^4-Strichartz (Bourgain '93))**.** *For any u smooth enough, we have*

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,3/8}(\mathbb{R}\times\mathbb{T})} \tag{2.9}$$

The proof of this theorem can be found in Tao's book [18], using Tzvetkov's approach. Note also that this result is better than Theorem 2.1 since, using Transference principle (see Andreia's first project [1]), this theorem only gives us

$$\|\eta(t)u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,b}}$$

for any $b > \frac{1}{2}$. So we gain a bit of regularity in Theorem 2.5. In addition to this theorem, we also have the following result

Theorem 2.6. *Suppose $\phi \in \mathrm{HS}\left(L^2; H^s\right)$ and $b < \frac{1}{2}$, then (recall $\Psi(t) = \int_0^t S(t-t')\phi \mathrm{d}W(t')$)*

$$\chi_{[0,T]}\Psi \in X^{s,b}$$

almost surely in the sense that

$$\left\|\chi_{[0,T]}\Psi \in\right\|_{X^{s,b}} \leq C_\omega < \infty \quad a.\ s.$$

Proof. Observe that, since $\phi \in \mathrm{HS}\left(L^2; H^s\right)$, we can write $\phi = \sum_{n\in\mathbb{Z}^d} \widehat{\phi_n} e_n$, with $e_n(x) = e^{in\cdot x}$, hence we write

$$\Psi(t) = \sum_{n\in\mathbb{Z}^d} \widehat{\phi_n} e_n \int_0^t e^{i(t-t')|n|^2} \mathrm{d}\beta_n(t').$$

Also, using equation (2.3), observe that we really want to study

$$\mathcal{F}_x\left(S(-t)\chi_{[0,T]}(t)\Psi(t)\right)(n) = \chi_{[0,T]}(t)\widehat{\phi_n} \int_0^t e^{-it'|n|^2} \mathrm{d}\beta_n(t').$$

Denote $F(t) = S(-t)\chi_{[0,T]}(t)\Psi(t)$. If we take space-time Fourier transform, we have

$$\begin{aligned}\widehat{F}(\tau, n) &= \widehat{\phi_n} \int_{\mathbb{R}} e^{-it\tau}\chi_{[0,T]}(t) \int_0^t e^{-it'|n|^2} \mathrm{d}\beta_n(t')\mathrm{d}t \\ &= \widehat{\phi_n} \int_0^T e^{-it'|n|^2} \int_{t'}^T e^{-it\tau} \mathrm{d}t\mathrm{d}\beta_n(t')\end{aligned}$$

where the second equality comes from Stochastic Fubini Theorem, with the last integral defined as a Wiener integral (see [7]). Then, observe that

$$\int_{t'}^T e^{-it\tau}\mathrm{d}t \lesssim_T \min\left(1, \frac{1}{|\tau|}\right) \sim \frac{1}{\langle\tau\rangle}.$$

Thus, we get using equation (2.3)

$$\mathbb{E}\left[\left\|\chi_{[0,T]}\Psi\right\|^2_{X^{s,b}}\right] = \left\|\langle n\rangle^s \langle\tau\rangle^b \left\|\widehat{F}(\tau, n)\right\|_{L^2(\Omega)}\right\|^2_{\ell^2_n L^2_\tau}$$

and, using the properties of Wiener integral, we have

$$\left\|\widehat{F}(\tau,n)\right\|_{L^2(\Omega)} \sim |\widehat{\phi_n}| \left(\int_0^T \left|e^{-it'|n|^2}\int_{t'}^T e^{-it\tau}\mathrm{d}t\right|^2 \mathrm{d}t'\right)^{1/2} \lesssim_T |\widehat{\phi_n}|\frac{1}{\langle\tau\rangle}$$

so that, if we inject this in the previous equality,

$$\begin{aligned}\mathbb{E}\left[\left\|\chi_{[0,T]}\Psi\right\|_{X^{s,b}}^2\right] &\lesssim_T \sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}|\widehat{\phi_n}|^2\int_{\mathbb{R}}\langle\tau\rangle^{2(b-1)}\mathrm{d}\tau \\ &\lesssim_T \|\phi\|_{\mathrm{HS}(L^2;H^s)}\end{aligned}$$

where the last inequality comes from the fact that $\int_{\mathbb{R}}\langle\tau\rangle^{2(b-1)}\mathrm{d}\tau$ is finite if and only if $b<\frac{1}{2}$.

Also, we have for $p\geq 2$,

$$\left\|\left\|\chi_{[0,T]}\Psi\right\|_{X^{s,b}}\right\|_{L^p(\Omega)} \lesssim p^{1/2}\|\phi\|_{\mathrm{HS}(L^2;H^s)}$$

hence, using Chebyshev's theorem

$$\mathbf{P}\left(\left\|\chi_{[0,T]}\Psi\right\|_{X^{s,b}}>\lambda\right)\leq Ce^{-c\frac{\lambda^2}{\|\phi\|_{\mathrm{HS}(L^2;H^s)}}}$$

which ends the proof.

□

Now that we proved these two result, we come back to the fixed point problem (2.8). Let us denote

$$\Gamma(v)(t) := \eta(t)S(t)u_0 - \eta\left(\frac{t}{T}\right)\int_0^t S(t-t')\,|v|^2\,v(t')\mathrm{d}t' + \chi_{[0,T]}(t)\int_0^t S(t-t')\phi\mathrm{d}W(t')$$

and we want to find a unique u such that

$$u(t)=\Gamma(u)(t)$$

for any $t\in[0,T]$ and $T\leq 1$. Fix $\frac{3}{8}\leq b\leq\frac{1}{2}$ and suppose $\phi\in\mathrm{HS}\left(L^2;L^2\right)$. Then, using equations (2.4) and (2.6) and Theorem 2.6, we have

$$\|\Gamma u\|_{X^{0,b}} \lesssim \|u_0\|_{L^2} + T^\theta\left\||u|^2u\right\|_{X^{0,-\frac{1}{2}+2\theta}} + C_\omega$$

Then, by a duality argument and Hölder's inequality, we have

$$\begin{aligned}\left\||u|^2u\right\|_{X^{0,-\frac{1}{2}+2\theta}} &= \sup_{\|v\|_{X^{0,\frac{1}{2}-2\theta}}\leq 1}\left|\int_{\mathbb{R}\times\mathbb{T}}|u|^2u\bar{v}\mathrm{d}x\mathrm{d}t\right| \\ &\leq \sup_{\|v\|_{X^{0,\frac{1}{2}-2\theta}}\leq 1}\|u\|_{L^4_{x,t}}^3\|v\|_{L^4_{x,t}} \\ &\lesssim \sup_{\|v\|_{X^{0,\frac{1}{2}-2\theta}}\leq 1}\|u\|_{X^{0,\frac{3}{8}}}^3\|v\|_{X^{0,\frac{3}{8}}}\end{aligned}$$

where the last inequality comes from Theorem 2.5. Thus, we have

$$\|\Gamma u\|_{X^{0,b}} \lesssim \|u_0\|_{L^2} + T^\theta\|u\|_{X^{0,b}}^3 + C_\omega.$$

Using similar arguments, we also get

$$\|\Gamma u - \Gamma v\|_{X^{0,b}} \lesssim T^{\theta} \left(\|u\|_{X^{0,b}}^2 + \|v\|_{X^{0,b}}^2\right) \|u - v\|_{X^{0,b}} .$$

Thus Γ is a contraction on a ball $B_R \subset X^{0,b}$ of radius

$$R = R_\omega \sim \|u_0\|_{L^2} + C_\omega$$

by choosing $T = T(R) \ll 1$. So we constructed a solution $u \in B_R \subset X^{0,\frac{1}{2}-}$ to equation (2.8), but $X^{0,\frac{1}{2}-}$ is not a subset of $C_t L_x^2$. We then need to prove that $u \in C_t L_x^2$. Observe that

(1) $S(t)u_0 \in C_t L_x^2$, so the linear part is continuous.
(2) According to Remark 1.1, $\Psi \in C_t L_x^2$ for $\phi \in \mathrm{HS}\left(L^2; L^2\right)$.
(3) In fact, using equation (2.5), we can prove the nonlinear part is in $X^{0,\frac{1}{2}+} \subset C_t L_x^2$.

so $u \in C\left([0,T]; L_x^2\right)$ and we proved local well-posedness of equation (2.1) in $L^2(\mathbb{T})$.

Remark 2.7. *Similarly, we can prove local well-posedness of equation* (2.1) *in* $H^s(\mathbb{T})$*, for* $s \geq 0$*, assuming* $\phi \in \mathrm{HS}\left(L^2; H^s\right)$*. To do so, observe that for any* $s \geq 0$ *and* $n = n_1 - n_2 + n_3$*, we have*

$$\langle n\rangle^s \lesssim \langle n_1\rangle^s \langle n_2\rangle^s \langle n_3\rangle^s$$

so that, with the same notations,

$$\|u_1 \overline{u_2} u_3\|_{X^{s,-\frac{1}{2}+}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}}$$

and the rest of the proof follows in the same way.

2.3. **Improvement by Moyua and Vega.** Previously, we proved Zygmund's L^4-Strichartz estimates (2.2), but in 2008, Moyua and and Vega proved the improved result (see [15]):

Theorem 2.8. *Let* $I \subset \mathbb{R}$ *be an interval, denote* $L_{I,x}^4 = L_t^4(I) L_x^4(\mathbb{T})$*, then*

$$\|S(t)u_0\|_{L_{I,x}^4} \lesssim |I|^{1/8} \|u_0\|_{L^2(\mathbb{T})} \tag{2.10}$$

Then, by duality and using Hölder's inequality, we have

$$\begin{aligned} \left\|\int_I S(-t')F(t')\mathrm{d}t'\right\|_{L_x^2} &= \sup_{\|f\|_{L^2}=1} \left|\int_I \langle F(t'), S(t')f\rangle_{L_x^2} \mathrm{d}t'\right| \\ &\leq \sup_{\|f\|_{L^2}=1} \|F\|_{L_{I,x}^{4/3}} \left\|S(t')f\right\|_{L_{I,x}^4} \end{aligned}$$

hence, if we apply Theorem 2.8 on $\|S(t')f\|_{L_{I,x}^4}$, we have

$$\left\|\int_I S(-t')F(t')\mathrm{d}t'\right\|_{L_x^2} \lesssim |I|^{1/8} \|F\|_{L_{I,x}^{4/3}} \tag{2.11}$$

Also, since $S(t-t') = S(t)S(-t')$, applying first Theorem 2.8 and next equation (2.11), we have

$$\left\| \int_I S(t-t')F(t')\mathrm{d}t' \right\|_{L^4_{I,x}} \lesssim |I|^{1/4} \|F\|_{L^{4/3}_{I,x}} \tag{2.12}$$

which is a bit alike the following lemma (see [5])

Lemma 2.9 (Christ-Kiselev '01)**.**

$$\left\| \int_0^t S(t-t')F(t')\mathrm{d}t' \right\|_{L^4_{I,x}} \lesssim |I|^{1/4} \|F\|_{L^{4/3}_{I,x}}$$

Once we have this result, we can again prove local well-posedness using a contraction argument in $L^4_{T,x} \cap C_T L^4_x$, just recall the following results:

(1) $\left\| |u|^2 u \right\|_{L^{4/3}_{T,x}} \leq \|u\|^3_{L^4_{T,x}}$,
(2) $\Psi \in L^4_{T,x} \cap C_T L^2_x$.

3. Global well-posedness using Ito calculus approach

Let us first recall the form of the 1 dimensional cubic stochastic nonlinear Schrödinger equation on $\mathbb{T}$:

$$i\partial_t u(t,x) - \partial_x^2 u(t,x) + |u|^2 u(t,x) = \phi(t,x)\xi(t,x)$$

Then, if we set ourselves in deterministic case, namely $\phi = 0$, we have

$$\partial_t \int_{\mathbb{T}} |u|^2 \mathrm{d}x = 2\operatorname{Re} \int_{\mathbb{T}} \partial_t u \cdot \overline{u} \mathrm{d}x = -2\operatorname{Re} i \int_{\mathbb{T}} \partial_x^2 u \cdot \overline{u} \mathrm{d}x + 2\operatorname{Re} i \int_{\mathbb{T}} |u|^2 u \cdot \overline{u} \mathrm{d}x$$

Then, observe $\int_{\mathbb{T}} |u|^2 u \cdot \overline{u}\mathrm{d}x = \int_{\mathbb{T}} |u|^4 \mathrm{d}x$ and, using integration by parts, $\int_{\mathbb{T}} \partial_x^2 u \cdot \overline{u}\mathrm{d}x = \int_{\mathbb{T}} |\partial_x u|^2 \mathrm{d}x$. Thus, we have

$$\partial_t \int_{\mathbb{T}} |u|^2 \mathrm{d}x = 0$$

There is then L^2-conservation and we use this result to prove global well-posedness in $L^2(\mathbb{T})$. However, we cannot use this for stochastic NLS. We will use instead Ito's lemma on the mass

$$M(u) = \int_{\mathbb{T}} |u|^2 \mathrm{d}x = \sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2 = \sum_{n\in\mathbb{Z}} \left(p_n^2 + q_n^2\right) \tag{3.1}$$

where $p_n = \operatorname{Re} u(n)$ and $q_n = \operatorname{Im} u(n)$. Then, if we rewrite equation (2.1) in the following way (let us denote $\widehat{u}_n(t) \coloneqq \widehat{u}(t,n)$)

$$\mathrm{d}\widehat{u}_n = \left(in^2 \widehat{u}_n + i\widehat{|u|^2 u}(n) \right) \mathrm{d}t - i\widehat{\phi}_n \mathrm{d}\beta_n$$

we can split it into

$$\mathrm{d}p_n = \left(-n^2 q_n - \operatorname{Im}\left(\widehat{|u|^2 u}(n) \right) \right) \mathrm{d}t + \operatorname{Im}\left(\widehat{\phi}_n \mathrm{d}\beta_n \right) \tag{3.2}$$

and

$$\mathrm{d}q_n = \left(n^2 p_n + \operatorname{Re}\left(\widehat{|u|^2 u}(n) \right) \right) \mathrm{d}t - \operatorname{Re}\left(\widehat{\phi}_n \mathrm{d}\beta_n \right) \tag{3.3}$$

where we have

$$\begin{cases} \operatorname{Im}\left(\widehat{\phi}_n \mathrm{d}\beta_n\right) & = & \operatorname{Im}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Re}\beta_n\right) + \operatorname{Re}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Im}\beta_n\right) \\ \operatorname{Re}\left(\widehat{\phi}_n \mathrm{d}\beta_n\right) & = & \operatorname{Re}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Re}\beta_n\right) - \operatorname{Im}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Im}\beta_n\right) \end{cases}$$

with $\operatorname{Re}\beta_n$ and $\operatorname{Im}\beta_n$ independent for any $n \in \mathbb{Z}$. Then let us recall Ito's lemma:

Lemma 3.1 (Ito)**.** *Let X be a stochastic process such that*

$$\mathrm{d}X = f\mathrm{d}t + g\mathrm{d}B$$

then, if we consider $F(X)$, with F a function, we have

$$\begin{aligned} \mathrm{d}F &= \partial_x F \mathrm{d}X + \frac{1}{2}\partial_x^2 F (\mathrm{d}X)^2 \\ &= \partial_x F (f\mathrm{d}t + g\mathrm{d}B) + \frac{1}{2}\partial_x^2 F \cdot g^2 \mathrm{d}t \end{aligned}$$

Idea of proof. Note that $\mathrm{d}F$ is to be understood as $\int \mathrm{d}F$, the first line is just like a second order Taylor expansion and the second line follows from the following equalities under an integral sign :

(1) $(\mathrm{d}t)^2 = 0$
(2) $\mathrm{d}t\mathrm{d}B = 0$
(3) $\mathrm{d}B\mathrm{d}t = 0$
(4) $(\mathrm{d}B)^2 = \mathrm{d}t$

□

The idea, from now on, is to use Ito's lemma on the mass $M(u)$. Then we should get

$$\begin{aligned} \mathrm{d}M &= 2\sum_{n\in\mathbb{Z}} (p_n \mathrm{d}p_n + q_n \mathrm{d}q_n) + 2\sum_{n\in\mathbb{Z}} \left((\mathrm{d}p_n)^2 + (\mathrm{d}q_n)^2\right)\mathrm{d}t \\ &= 2\sum_{n\in\mathbb{Z}} \left(p_n \operatorname{Im}\left(\widehat{\phi}_n \mathrm{d}\beta_n\right) - q_n \operatorname{Re}\left(\widehat{\phi}_n \mathrm{d}\beta_n\right)\right) + 2\left\|\phi\right\|_{\mathrm{HS}(L^2;L^2)} \mathrm{d}t \end{aligned}$$

Then, we want to use the Burkholder-Davis-Gundy inequality:

Theorem 3.2 (Burkholder-Davis-Gundy inequality)**.** *Let X be a (local) martingale and $1 \le p < \infty$, then*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] \sim \mathbb{E}\left[\langle X_t\rangle_{[0,T]}^{p/2}\right] \tag{3.4}$$

where, in the previous theorem, $\langle X_t\rangle_{[0,T]}$ is the quadratic variation. In the case of an Ito process X (*i.e.* such that $\mathrm{d}X = f\mathrm{d}t + g\mathrm{d}B$), we have

$$\langle X_t\rangle_{[0,T]} = \int_0^t g^2 \mathrm{d}t'.$$

Let us explain how we can use Theorem 3.2 with the following example:

Example 1. *Let us study* $2\sum_{n\in\mathbb{Z}} p_n \operatorname{Im}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Re}\beta_n\right)$*. To do so, we apply Theorem 3.2 on the following:*

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t 2\sum_{n\in\mathbb{Z}} p_n \operatorname{Im}\widehat{\phi}_n \mathrm{d}\left(\operatorname{Re}\beta_n\right)\mathrm{d}t'\right] &\lesssim \mathbb{E}\left[\left(\int_0^T \sum_{n\in\mathbb{Z}} p_n^2|\widehat{\phi}_n|^2 \mathrm{d}t\right)^{1/2}\right]\\
&\lesssim \mathbb{E}\left[\left(\int_0^T \|u\|_{L^2}^2 \|\phi\|_{\mathrm{HS}(L^2;L^2)}^2 \mathrm{d}t\right)^{1/2}\right]\\
&\lesssim \mathbb{E}\left[\left(\sup_{t\in[0,T]} M(u)(t)\right)^{1/2} T^{1/2}\|\phi\|_{\mathrm{HS}(L^2;L^2)}^{1/2}\right]\\
&\lesssim \varepsilon\mathbb{E}\left[\sup_{t\in[0,T]} M(u)(t)\right] + \frac{1}{\varepsilon}T\|\phi\|_{\mathrm{HS}(L^2;L^2)}
\end{aligned}$$

where the last line comes from Cauchy's inequality.

Then, using the previous example, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(u)(t)\right] \le \|u_0\|_{L^2}^2 + C\left(T, \|\phi\|_{\mathrm{HS}(L^2;L^2)}\right)$$

which gives almost sure existence up to time T, but for **any** finite T, hence the global well-posedness in $L^2(\mathbb{T})$.

Unfortunately, we cannot directly do these computations. Instead, we will decompose our solution u using the frequency cutoffs $P_{\le N}$ defined by :

$$\mathcal{F}\left(P_{\le N} f\right) = \chi_{\{|n|\le N\}}\widehat{f}. \tag{3.5}$$

Remark 3.3. *Note that we could use also smooth cutoffs.*

From now on, we will consider the finite dimensional approximation:

$$i\partial_t u_N - \partial_x^2 u_N + P_{\le N}\left(|u_N|^2 u_N\right) = \phi_n \xi \tag{3.6}$$

with $u_N = P_{\le N} u_N$ and $\phi_N = P_{\le N}\circ\phi$. If we write the equation on the Fourier side, we get

$$i\mathrm{d}\widehat{u}_N = \left(-n^2\widehat{u}_N + P_{\le N}\widehat{(|u_N|^2 u_N)}(n)\right)\mathrm{d}t + \widehat{\phi}_N(n)\mathrm{d}\beta_n \tag{3.7}$$

with $|n| \le N$. This gives us a finite dimensional system of stochastic partial differential equations for $(p_n, q_n)_{|n|\le N}$. We can then apply Ito's lemma 3.1, but we need to check that u_N is adapted first. Since we do not know if u_N exists globally in time, we need to use a *stopping time argument.*

Fix a target time $T \ll 1$ and let τ be a stopping time such that

$$0 < \tau \le \min\left(T, T_{\max}\right)$$

where $T_{\max} = T_{\max}(\omega)$ is the maximal time of existence.

Example 2. *Fix* $R > 0$*, then one can check that*

$$\tau_R = \inf\left\{t > 0\colon\ \|u(t)\|_{L^2} \ge R\right\}$$

is a stopping time, and this can work for our case.

Then insert $\chi_{[0,\tau]}(t)$ to the equation (3.7) and apply Ito's lemma 3.1.Then, by the local well-posedness argument, we have as N tends to infinity, if we denote $\widetilde{T}$ the local existence time,

$$\|u_N - u\|_{X^{0,3/8}_{t\in[0,\widetilde{T}]}\cap C_{\widetilde{T}}L^2_x} \longrightarrow 0$$

Then we can verify Ito's lemma for u and prove global well-posedness. For more details on this part, see [6] or [16].

Remark 3.4. *We can also prove global well-posedness of equation* (2.1) *on* $\mathbb{R}$ *if we also inser t a cutoff in size* $\theta\left(\frac{\|u\|}{R}\right)$ *on the nonlinearity in the equation* (3.7)*.*

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LECTURE 5 (PART II), LECTURE 6 & 7

FABIAN GERM

Contents

1. Lecture 5, part II & lecture 6 (17/03/21) & (24/03/21).

1.1. **A pathwise approach for the SNLW.** It is often not possible to obtain global well-posedness via a pathwise approach. However, let us look at case that allows such a method. Consider the (defocusing) SNLW on $\mathbb{T}^3$, given by

$$\partial_t^2 u + (1-\Delta)u + u^3 = \phi\xi, \tag{1.1}$$

where $\phi \in HS(L^2, H^{s-1})$ for $s > 0$. Note that in this case $\Psi \in C_t W_x^{s-\varepsilon,\infty}$.
The Duhamel formulation is then

$$u(t) = \partial_t S(t)u_0 + S(t)u_0 - \int_0^t S(t-t')u^3(t')\,dt' + \Psi(t),$$

where $S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$. Using a method sometimes referred to *Da Prato-Debussche trick*, which relies on a first-order expansion of u it is possible to show local well-posedness for the SNLW. Indeed, we write $u(t) = \Psi(t) + v(t)$ and postulate that the remainder term $v \in H^1$. Note that then v satisfies

$$v(t) = u_{lin}(t) - \int_0^t S(t-t')(v(t') + \Psi(t'))^3\,dt' =: \Gamma v,$$

Date: May 12, 2021.
2010 *Mathematics Subject Classification.* 35L71, 60H15.
Key words and phrases. stochastic nonlinear wave equation; nonlinear wave equation; damped nonlinear wave equation; renormalization; white noise; Gibbs measure.

where u_{lin} is the solution to the linear equation. Let $\mathcal{H}^1 := H^1 \times L^2$. Then we can estimate

$$\begin{aligned}\|(\Gamma v, \partial_t \Gamma v)\|_{C_T \mathcal{H}^1} &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^1} + T\|(v+\Psi)^3\|_{L_T^\infty L_x^2} \\ &= \|(u_0, u_1)\|_{\mathcal{H}^1} + T\|(v+\Psi)\|^3_{L_T^\infty L_x^6} \\ &\lesssim \|(u_0, u_1)\|_{\mathcal{H}^1} + T\big(\|v\|^3_{L_T^\infty L_x^6} + \|\Psi\|^3_{L_T^\infty L_x^6}\big).\end{aligned}$$

Hence, as longs as $\phi \in HS(L^2, H^{-1})$, such that also $\Psi \in C_T L_x^6$ almost surely, we can obtain local well-posedness for sufficiently small T. To prove this, one can employ a fixed-point argument for Γ, where the radius of contraction depends on the norm of the initial data and the (random) norm $C_T L_x^6$-norm of Ψ. For more details on the Da Prato-Debussche trick we refer to [3], one of the initial articles where this method was exploited. For a more detailed analysis of the SNLW we refer to [2] and [5].
We can also use this method to obtain global well-posedness. To do that, consider a target time $T_0 >> 1$. Then for almost all ω there exists a $K = K(\omega)$ such that

$$\|\Psi\|_{C_{T_0} L_x^6} \leq K,$$

and we will obtain a (local) time of existence $T = T(\|(u_0, u_1)\|_{\mathcal{H}^1}, K) > 0$. Then, if we can control $\|(v, \partial_t v)\|_{\mathcal{H}^1}$ on $[0, T]$, we can iterate the above argument on $[jT, (j+1)T]$ to obtain existence on all of $[0, T_0]$, where T_0, initially, was arbitrary. Thus we get global well-posedness.

Theorem 1.1. *The defocusing SNLW* (1.1) *is globally well-posed.*

Proof. We will use an energy estimate to control $\|(v, \partial_t v)\|_{\mathcal{H}^1}$ on $[0, T]$. With our first order expansion $u = \Psi + v$, the remainder v satisfies

$$\partial_t^2 v + (1-\Delta)v + (v+\Psi)^3 = 0,$$

where clearly almost surely

$$(v+\Psi)^3 = v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3.$$

Define the energy

$$E(v, \partial_t v) = \frac{1}{2}\int_{\mathbb{T}^3} |\langle \nabla \rangle v|^2\, dx + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t v)^2\, dx + \frac{1}{4}\int_{\mathbb{T}^3} v^4\, dx.$$

Then almost surely, taking the derivative yields

$$\begin{aligned}\partial_t E(v, \partial_t v) &= \int \partial_t(\partial_t^2 v + (1-\Delta)v + v^3)\, dx \\ &= \int \partial_t(-3v^2\Psi - 3v\Psi^2 - \Psi^3)\, dx \\ &\lesssim C\|\Psi\|_{L_t^\infty L_x^\infty}\left(\int (\partial_t v)^2\, dx\right)^{\frac{1}{2}}\left(\int v^4\, dx\right)^{\frac{1}{2}} + C\|\Psi\|^{\frac{1}{2}}_{L_T^\infty L_x^6}\left(\int (\partial_t v)^2\, dx\right)^{\frac{1}{2}} \\ &\leq C(\Psi, T)\left(1 + E(v, \partial_t v)\right),\end{aligned}$$

where we used that

$$\left(\int (\partial_t v)^2\, dx\right)^{\frac{1}{2}}\left(\int v^4\, dx\right)^{\frac{1}{2}} \leq E(v, \partial_t v),$$

as well as that by Young's inequality

$$v\Psi^2 \leq \frac{1}{2}(v^2\Psi + \Psi^3).$$

Hence, applying Grönwall's inequality yields for some $C = C(\omega, T)$, almost surely

$$\sup_{t\in[0,T]} \|(v(t), \partial_t v(t))\|_{\mathcal{H}^1} \leq C < \infty.$$

□

1.2. **An invariant measure argument: the Gibbs measure.** Up to now we relied on methods stemming from the deterministic conservation laws. The invariant measure approach, though similar in nature, is of a different kind. To give an example, consider the (deterministic, defocusing) NLW

$$\partial_t^2 u + (1-\Delta)u + u^k = 0, \tag{1.2}$$

for $k \in 2\mathbb{N}+1$. We can rewrite this into a Hamiltonian equation of the form

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial E/\partial u \\ \partial E/\partial(\partial_t u) \end{pmatrix}, \tag{1.3}$$

where E is the energy for the wave equation,

$$E(u, \partial_t u) = \frac{1}{2}\int |\langle\nabla\rangle u|^2\, dx + \frac{1}{2}\int (\partial_t u)^2\, dx + \frac{1}{k+1}\int u^{k+1}\, dx,$$

which is conserved under the NLW dynamics. In the finite-dimensional case, the Hamiltonian preserves the volume in the phase-space, $dud(\partial_t u)$. This means we should expect that $dud(\partial_t u)$ remains invariant under the flow as infinite-dimensional Lebesgue measure. However, it is well-known that such a measure cannot exist. Still, following this rationale, we expect the *Gibbs measure*

$$d\rho(u, \partial_t u) = Z^{-1} e^{-E(u,\partial_t u)} dud(\partial_t u),$$

where Z is a normalizing constant, to be invariant. Note that by plugging in the energy E, we can write

$$d\rho(u, \partial_t u) = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}\, dx} e^{-\frac{1}{2}\|u\|_{H^1}^2} du \otimes e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2} d(\partial_t u),$$

where $e^{-\frac{1}{2}\|u\|_{H^1}^2} du$ has the form of a Gaussian free field measure, $e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2} d(\partial_t u)$ is the spatial white noise measure and $Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}\, dx}$ is some weight. Before we continue, let us briefly discuss the above measures. As we mentioned, we cannot immediately consider

$$s\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$$

as Gaussian probability measure. However, we can consider it as the limit of truncated measures of the form

$$d\mu_{s,N} = Z_N^{-1} e^{-\frac{1}{2}\|P_{\leq N} u\|_{H^s}^2} d(P_{\leq N} u),$$

where $P_{\leq N}$ is the sharp frequency cut-off, cutting off frequencies greater than N. As such, it is a measure on

$$E_N = span\{e_n, |n| \leq N\},$$

and by Plancherel's inequality we can write

$$d\mu_{s,N} = Z_N^{-1} \prod_{|n|\leq N} e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}(n)|^2} d\hat{u}(n),$$

with $d\hat{u}(n)$ the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. Defining the complex-valued independent standard Gaussian random variables

$$g_n = \langle n\rangle^s \hat{u}(n), \quad n \in \mathbb{N},$$

we get that under $\mu_{s,N}$, u has the representation

$$u_N(x) := \sum_{|n|\leq N} \frac{g_n}{\langle n\rangle^s} e^{inx}.$$

We would like to take the limit as $N \to \infty$ and hence have to investigate whether u_N converges appropriately. A computation gives

$$\begin{aligned}\mathbb{E}\left(\|u_N - u_M\|_{H^\sigma}^2\right) &= \sum_{M<|n|\leq N} \frac{\langle n\rangle^{2\sigma}\mathbb{E}|g_n|^2}{\langle n\rangle^{2s}} \\ &= \sum_{M<|n|\leq N} \langle n\rangle^{2\sigma-2s} \to 0,\end{aligned}$$

if and only if $2\sigma - 2s < -d$, which is equivalent to $\sigma < s - \frac{d}{2}$, where d is the dimension of the Torus. We follow that for such σ, $\{u_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, H^\sigma(\mathbb{T}^d))$ (or by an analogous computation in $L^p(\Omega, H^\sigma(\mathbb{T}^d))$, for $p < \infty$), with (almost sure) limit

$$u(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^s} e^{inx}.$$

Using this, we can understand μ_s as the pushforward measure $P \circ u^{-1}$. Note that μ_s is not a probability measure on H^s and we needed to enlarge the space to $\mathcal{H}^\sigma$, where σ satisfies $\sigma < s - \frac{d}{2}$. In other words, we lowered the regularity in order to enable this construction. With the same σ, other possible choices of larger spaces include $W^{\sigma,p}$, the Besov-spaces $B^\sigma_{\infty,\infty}$ as well as the Fourier-Lebesgue spaces $FL^{\sigma,p}$ for $\sigma < s - \frac{d}{p}$. The triple (μ_s, H^s, B) is referred to as *abstract Wiener space*, where B is the enlarged space.
For a more detailed treatise on Gaussian measures in Banach spaces, we refer to [4].

1.3. **The one-dimensional NLW..** Let $d = 1$ and consider again the NLW (1.2) above. We can write the Gibbs measure as

$$\begin{aligned}d\rho(u, \partial_t u) &= Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}\,dx} d\mu_1 \otimes d\mu_0(u, \partial_t u), \quad \text{where} \\ \mu_1(du) &= e^{-\frac{1}{2}\|u\|_{H^1}^2} du \\ \mu_2(d(\partial_t u)) &= e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2} d(\partial_t u).\end{aligned}$$

As $u \in H^{\frac{1}{2}-}$, the Sobolev embedding gives that $u \in L^r_x$ almost surely for $r < \infty$. Thus we know that the weight

$$0 < e^{-\frac{1}{k+1}\int u^{k+1}\,dx} \leq 1, \quad \text{(a.s.)},$$

as well as $e^{-\frac{1}{k+1}\int u^{k+1}\,dx} \in L^p(\mu_1)$, for $p \leq \infty$, almost surely. Therefore we know that ρ and $\mu_1 \otimes \mu_0$ are equivalent on $H^{\frac{1}{2}-}(\mathbb{T}) \times H^{-\frac{1}{2}-}(\mathbb{T})$. Using an argument by Bourgain

(94'), see [1], we can establish global well-posedness. We can use the "finite-dimensional approximation" u_N, which satisfies

$$\partial_t^2 u_N + (1-\Delta)u_N + P_{\leq N}((P_{\leq N}u)^k) = 0.$$

In other words, we truncate the initial data and look at a set of lower frequencies, as well as the associated truncated Gibbs measure

$$d\rho_N = Z_N^{-1} e^{-\frac{1}{k+1}\int (P_{\leq N}u)^{k+1}\,dx} d\mu_1 \otimes d\mu_0(u, \partial_t u).$$

We can verify that the finite-dimensional system, i.e. for frequencies up to N, is a Hamiltonian system, for which the truncated Gibbs measure is invariant. Moreover, notice that the high-frequency part of u, i.e. $u - P_{\leq N}u$ is a solution of the linear problem, for which we can consider the Gaussian measure, which is invariant under linear dynamics. Then, on the one hand we can show that u_N converges to u as $N \to \infty$. On the other hand, we can show that the density of ρ_N converges to the one of ρ in L^p, for $2 \leq p > \infty$. Finally we can show that ρ is invariant under the dynamics of u.

Proposition 1.2. *Given $T > 0$ and $\varepsilon > 0$ there exists $\Omega_{N,T,\varepsilon} \subset \Omega$ such that*

(i) *$\rho_N(\Omega^c_{N,T,\varepsilon}) < \varepsilon$, and*
(ii) *for each $\omega \in \Omega_{N,T,\varepsilon}$ the solution $(u_N, \partial_t u_N) = (u_N^\omega, \partial_t u_N^\omega)$ exists on $[-T, T]$ and satisfies*

$$\sup_{-T\leq t\leq T} \|(u_N(t), \partial_t u_N(t))\|_{\mathcal{H}^1} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

Proof. Fix $K >> 1$. Let the solution map

$$\Phi_N(t) := \begin{cases} \mathcal{H}^{\frac{1}{2}-}(\mathbb{T}) & \to \mathcal{H}^{\frac{1}{2}-}(\mathbb{T}) \\ (u_0, u_1) & \mapsto (u(t), \partial_t u(t)) \end{cases}$$

and define

$$\Omega_{N,T,\varepsilon} := \bigcap_{j=-[T/\delta]}^{[t/\delta]} \Phi_N(-j\delta)B_K,$$

where B_K denotes the ball of radius K in $\mathcal{H}^{\frac{1}{2}-}(\mathbb{T})$ and $\delta \sim K^{-\theta}$ is the local time of existence for solutions starting in B_K. To prove the first claim, we can use the σ-additivity of ρ to write

$$\begin{aligned} \rho_N(\Omega^c_{N,T,\varepsilon}) &\leq \sum_{j=-[T/\delta]}^{[T/\delta]} \rho_N(\Phi_N(-j\delta)B_K^c) \\ &= \sum_{j=-[T/\delta]}^{[T/\delta]} \rho_N(B_K^c) \\ &\lesssim \frac{T}{\delta}\rho_N(B_K^c), \end{aligned}$$

where for the second inequality we used that ρ_N is invariant under Φ_N. Then, recalling that ρ_N is a Gaussian measure, we can further estimate,

$$\rho_N(\Omega^c_{N,T,\varepsilon}) \lesssim TK^\theta e^{-cK^2} < \varepsilon,$$

by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}$. Finally, by a local well-posedness argument on the intervals $[j\delta, (j+1)\delta]$, $j = 0, \dots, [\frac{T}{\delta}] - 1$, we have

$$\sup_{-T \le t \le T} \|(u_N(t), \partial_t u_N(t))\|_{\mathcal{H}^1} \le CK \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}},$$

where we in particular note that the last bound is independent of N. □

To summarize, the set $\Omega_{N,T,\varepsilon}$ depends on N, but the constant in our final bound is independent of N. Hence we get the same log bound for the true solution $(u, \partial_t u)$ by a PDE approximation argument. Thus we obtain almost sure global well-posedness.

For the true solution u, we may call the condition, given in in the above proposition *almost a.s. global well-posedness*, meaning that for any $T, \varepsilon > 0$ there exists a set $\Omega_{T,\varepsilon}$ such that if $\omega \in \Omega_{T,\varepsilon}$ the corresponding solution u^ω exists on $[-T, T]$ and $P(\Omega^c_{T,\varepsilon}) < \varepsilon$. From there we can get a.s. global well-posedness.

Theorem 1.3. *The defocusing NLW is a.s. globally well-posed with respect to random initial data.*

Proof. Fix ε and set $\Omega_j = \Omega_{2^j, \varepsilon/2^j}$. Then setting

$$\Omega_\varepsilon := \bigcap_j \Omega_j, \quad \text{we get} \quad P(\Omega_\varepsilon^c) \le \sum_j \frac{\varepsilon}{2^j} = \varepsilon.$$

Now, if $\omega \in \Omega_\varepsilon$, then u^ω exists globally in time. Finally, set

$$\Sigma := \bigcup_{\varepsilon > 0} \Omega_\varepsilon, \quad \Rightarrow \quad P(\Sigma^c) = \inf_\varepsilon P(\Omega_\varepsilon^c) = 0,$$

and thus we get a.s. global well-posedness. □

2. Lecture 7 (31/03/21)

2.1. Stochastic damped NLW.

Consider

$$\partial_t^2 u + \partial_t u + (1 - \Delta) u + u^k = \sqrt{2}\xi,$$

for $k \in 2\mathbb{N} + 1$, where ξ again denotes white noise. We can write this in vectorial form with $v = \partial_t u$, meaning,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (-1 + \Delta) u - u^k \end{pmatrix} + \begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix}.$$

Note that the first term on the right-hand side are the deterministic NLW dynamics, whereas the second term on the right-hand side is an Ornstein-Uhlenbeck (OU) process for v, if we ignore the NLW dynamics term. We recall from the previous section that the NLW preserves the Gibbs measure

$$d\rho(u, v) = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}\,dx} d\mu_1 \otimes d\mu_0(u, v).$$

Moreover, it can be shown that the OU process preserves the spatial white noise measure $\mu_0(dv)$ and therefore also preserves the Gibbs measure.

Let us recall some properties of an OU process X. For parameters $a > 0, b \in \mathbb{R}$ its stochastic differential is given by

$$\begin{cases} dX(t) &= -aX(t)dt + bdB(t), \\ X(0) &= x_0, \end{cases}$$

where we suppose that $x_0 \sim N(0, \frac{b^2}{2a})$ is independent from the Brownian motion B. Then X is given by

$$X(t) = e^{-at}x_0 + b\int_0^t e^{-a(t-t')}dB(t').$$

It is a Gaussian random variable with variance

$$\begin{aligned}\mathbb{E}[X^2(t)] &= e^{-2at}\mathbb{E}[x_0^2] + b^2\int_0^t e^{-2a(t-t')}dt' \\ &= e^{-2at}\frac{b^2}{2a} + \frac{b^2}{2a}(1-e^{-2at}) \\ &= \frac{b^2}{2a},\end{aligned}$$

i.e. $X(t) \sim N(0, \frac{b^2}{2a})$ for all t. With this in mind, let us consider the OU part of our stochastic damped NLW (SdNLW), that is

$$\partial_t v = -v + \sqrt{2}\xi.$$

With the Fourier transform in x this becomes

$$\partial_t \hat{v}(n) = -\hat{v}(n) + \sqrt{2}d\beta_n,$$

which we should further separate into $\operatorname{Re}\hat{v}(n)$ and $\operatorname{Im}\hat{v}(n)$ in order to use the above properties of the OU process. We note that the distribution of $\hat{v}(n)$ at any time t is determined by the complex random variable $g_n \sim N(0,1)$. Moreover, the $\hat{v}(n)$ are independent for different n, since the β_n are independent Brownian motions. Therefore $\hat{v}$ preserves the spatial whie noise measure μ_0 and hence the Gibbs measure ρ.

So we see that both the NLW dynamics and the OU part preserve the Gibbs measure individually. The questions is, how do we check that together they still preserve the Gibbs measure? Consider the truncated dynamics

$$\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N + P_{\leq N}((P_{\leq N}u)^k) = \sqrt{2}\xi. \tag{2.1}$$

Note that we did not truncate the noise term, and further, that for high frequencies $|m| > N$ we have the linear equation

$$\partial_t^2 P_{>N}u_N + \partial_t P_{>N}u_N + (1-\Delta)P_{>N}u_N = \sqrt{2}P_{>N}\xi,$$

for which the Gaussian measure $\mu_{\frac{1}{1,N}} \otimes \mu_{\frac{1}{0,N}}$ is invariant, where by $\mu_{\frac{1}{j,N}}$ we denote the marginal measure of μ_j on $E_{\frac{1}{N}} := span\{e_n : |n| > N\}$, $j = 0, 1$. Let us further consider the truncated Gibbs measure from the previous section, which we now write as

$$\begin{aligned}d\rho_N(u_N, \partial_t u_N) = Z_N^{-1} e^{-\frac{1}{k+1}\int (P_{\leq N}u)^{k+1}\,dx}\, &(d\mu_{1,N} \otimes \mu_{0,N}(P_{\leq N}(u_N, \partial_t u_N))) \\ &\bigotimes \left(\mu_{\frac{1}{1,N}} \otimes \mu_{\frac{1}{0,N}}(P_{>N}(u_N, \partial_t u_N))\right).\end{aligned}$$

By the invariance of the marginal measures we note that ρ_N is invariant under the truncated SdNLW (2.1).

Let L^N be the generator of the Markov semigroup P_t for (2.1). In other words, L^N is the time derivative of P_t at $t = 0$, where for a functional F on the phase space for (u, v), $v = \partial_t u$, P_t satisfies

$$P_t F((u, v)) = \mathbb{E} F(\Phi_t(u, v)),$$

with Φ denoting again the solution map. ρ_N being invariant for (2.1) means that the adjoint $(L^N)^* \rho_N = 0$, or in other words, for any functional F,

$$\int L^N F(u, v) d\rho(u, v) = 0.$$

In our case, the generator can be written as the sum of two generators, i.e.

$$L^N = L_1^N + L_2^N,$$

where L_1^N is the generator for the truncated NLW and L_2^N is the generator for the OU process (for v). As a result, by linearity for the adjoint

$$(L^N)^* \rho_N = (L_1^N)^* \rho_N + (L_2^N)^* \rho_N = 0,$$

meaning that ρ_N is invariant under the stochastic damped NLW dynamics (2.1). For here we can repeat Bourgain's invariant measure argument to obtain a.s. global-wellposedness for (2.1), as well as the invariance of ρ under its dynamics, as we did in the previous section.

2.2. **Parabolic Φ_1^{k+1}-model.** Suppose first we have

$$du(t) = -\gamma \nabla_u H(u) dt + \sqrt{2} \gamma dW(t)$$

where by ∇ we mean the Fréchet derivative. For a functional F we can write

$$L(F(u)) = \gamma Tr(D^2 F(u)) - \gamma \langle \nabla F(u), \nabla H(u) \rangle.$$

$L^*(F(u) = 0$ is equivalent to

$$\begin{aligned} \int LF(u) \rho(du) &= \gamma \int Tr D^2 F(u) e^{-H(u)}\, du + \gamma \int \langle \nabla F(u), \nabla (e^{-H(u)}) \rangle\, du \\ &= 0, \end{aligned}$$

by integration by parts. Though it is an informal calculation, it shows how one can check invariance by hand.

Now, consider

$$\partial_t u + (1 - \Delta) u - u^k = \sqrt{2} \xi.$$

We will compute the Markov generator and prove invariance by a truncation argument. For $\hat{u}_n = a_n + i b_n$ and $\mathcal{N}(u) = u^k$ we get

$$\begin{aligned} da_n &= \big(- \langle n \rangle^2 a_n - \operatorname{Re} \hat{\mathcal{N}}(u) \big) dt + \sqrt{2} d \operatorname{Re} \beta_n \\ db_n &= \big(- \langle n \rangle^2 b_n - \operatorname{Im} \hat{\mathcal{N}}(u) \big) dt + \sqrt{2} d \operatorname{Im} \beta_n. \end{aligned}$$

In fact, note that we consider the truncated dynamics under the projection $P_{\leq N}$, though we omit the subscript N for readability. Now, we identify the function $\hat{u}$ with its Fourier coefficients, which we denote in vector form as $\vec{a}$ and $\vec{b}$. Then, for a test function F,

$$\begin{aligned} L_1^N F(\vec{a},\vec{b}) &= \sum_n \left(-\langle n\rangle^2 a_n \partial_{a_n} + \frac{1}{2}\partial_{a_n}^2 \right) F \\ &\quad + \sum_n \left(-\langle n\rangle^2 b_n \partial_{b_n} + \frac{1}{2}\partial_{b_n}^2 \right) F, \\ L_2^N F(\vec{a},\vec{b}) &= \sum_n \partial_{a_n} F(-\operatorname{Re}(\hat{\mathcal{N}}(u))(n)) \\ &\quad + \sum_n \partial_{b_n} F(-\operatorname{Im}(\hat{\mathcal{N}}(u))(n)), \end{aligned}$$

where L_1^N is the generator corresponding to the linear part and L_2^N corresponds to the nonlinear part.

As an example, let us first consider the linear case, i.e. $\mathcal{N} = 0$. Then the Gaussian measure is

$$\begin{aligned} d\mu_N &= Z_N^{-1} e^{-\frac{1}{2}\|P_{\leq N} u\|_{H^1}^2} dP_{\leq N} u \\ &= Z_N^{-1} e^{-\frac{1}{2}\sum_n \langle n\rangle^2 (a_n^2 + b_n^2)} \Pi_n da_n db_n. \end{aligned}$$

To see the invariance, consider for simplicity only the a_n-part for a fixed n. Recall first that we impose $a_n = a_{-n}$, whereby we lose the factor $\frac{1}{2}$ in the following exponential functions. Then, integration by parts gives

$$\begin{aligned} &\int (-\langle n\rangle^2 a_n \partial_{a_n} F + \frac{1}{2}\partial_{a_n}^2 F) e^{-\langle n\rangle^2 a_n^2}\, da_n \\ &\qquad = \int \langle n\rangle^2 F \partial_{a_n}(a_n e^{-\langle n\rangle^2 a_n^2}) + \frac{1}{2} F \partial_{a_n}^2 e^{-\langle n\rangle^2 a_n^2}\, da_n = 0, \end{aligned}$$

where we used that

$$\begin{aligned} \partial_{a_n}(a_n e^{-\langle n\rangle^2 a_n^2}) &= e^{-\langle n\rangle^2 a_n^2} - 2\langle n\rangle^2 a_n^2 e^{-\langle n\rangle^2 a_n^2} \\ \frac{1}{2}\partial_{a_n}^2 e^{-\langle n\rangle^2 a_n^2} &= \partial_{a_n}(-\langle n\rangle^2 a_n e^{-\langle n\rangle^2 a_n^2}) \\ &= -\langle n\rangle^2 e^{-\langle n\rangle^2 a_n^2} + 2\langle n\rangle^4 a_n^2 e^{-\langle n\rangle^2 a_n^2}. \end{aligned}$$

An similar computation also holds for the b_n part. Hence we get that $(L^N)^*\rho = 0$.
Now let us consider the truncated Gibbs measure with the nonlinear part, where we again write $L^N = L_1^N + L_2^N$. Again we want to check that for test functions F on the Fourier coefficients,

$$(L^N)^*\rho_N = 0 \quad \Leftrightarrow \quad \int L^N F(\vec{a},\vec{b}) d\rho_N(\vec{a},\vec{b}) = 0.$$

We will rewrite the weight to

$$\rho_N = e^{-\frac{1}{k+1}\int (P_{\leq N} u)^{k+1}}\, d\mu_N = e^{-M(\vec{a},\vec{b})} d\mu_N,$$

where μ_N is the same Gaussian measure as above. Then, by the convolution theorem and the chain rule

$$\begin{aligned}\partial_{a_n} M(\vec{a},\vec{b}) &= \partial_{a_n}\frac{1}{k+1}\sum_{n_1+\cdots+n_{k+1}=0}\hat{u}_{n_1}\cdots\hat{u}_{n_{k+1}} \\ &= \partial_{\hat{u}_n} M\frac{\partial\hat{u}}{\partial_{a_n}} + \partial_{\hat{u}_{-n}} M\frac{\partial\hat{u}_{-n}}{\partial_{a_n}} \\ &= \hat{\mathcal{N}}(u)(-n) + \hat{\mathcal{N}}(u)(n) \\ &= 2\,\mathrm{Re}\,\hat{\mathcal{N}}(u)(n),\end{aligned}$$

where we used that

$$\partial_{\hat{u}_n} M = \frac{1}{k+1}\sum_{n_1+\cdots+n_{k+1}=-n}\hat{u}_{n_1}\cdots\hat{u}_{n_k},$$

and similarly for $-n$. Analogously,

$$\begin{aligned}\partial_{b_n} M(\vec{a},\vec{b}) &= i\hat{\mathcal{N}}(u)(-n) - i\hat{\mathcal{N}}(u)(n) \\ &= 2\,\mathrm{Im}\,\hat{\mathcal{N}}(u)(n).\end{aligned}$$

To check that

$$(L_1^N + L_1^N)^*\rho_N = 0,$$

let us fix again an n and focus on the real part. Then, notice that

$$\big(\langle n\rangle^2 a_n + \mathrm{Re}\,\hat{\mathcal{N}}(u)(n)\big)\partial_{a_n} e^{-M(\vec{a},\vec{b})}e^{-\langle n\rangle^2 a_n^2} = -\frac{1}{2}\partial_{a_n}^2 e^{-M(\vec{a},\vec{b})}e^{-\langle n\rangle^2 a_n^2}.$$

Therefore we have by integration by parts, omitting the imaginary part,

$$\begin{aligned}\int (L_1^N + L_2^N)F\,d\rho &= \prod_n\int F\big(\langle n\rangle^2 a_n\partial_{a_n} + \mathrm{Re}\,\hat{\mathcal{N}}(u)(n)\partial_{a_n} + \frac{1}{2}\partial_{a_n}^2\big)e^{-M(\vec{a},\vec{b})}e^{-\langle n\rangle^2 a_n^2}da_n \\ &= 0.\end{aligned}$$

A similar computation also holds for the b_n part, and hence we have the invariance of the truncated Gibbs measure.

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Singular Stochastic Partial Differential Equations

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May 2021

Abstract

The lecture notes on singular SPDEs are based on a course taught by professor: TadaHiro Oh. In particular, we focus on: Lecture 7 (pp 14-15), Lecture 8, and Lecture 9 (pp 1-5) of the hand written course notes.

1 Introduction

We consider the parabolic Φ_1^{k+1}-model of a two dimensional **Stochatic Non-Linear Heat** equations (SNLH) on $\mathbb{T}^2$ satisfying the system:

$$\begin{cases} \partial_t \mathbf{u} + (1-\nabla)\mathbf{u} - \mathbf{u}^k = \sqrt{2}\,\xi, \\ (\mathbf{u})|_{t=0} = \mathbf{u}_0 \in H^s \end{cases} \tag{1}$$

where $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field; t is time and $x \in \mathbb{T}^2$ (periodic domain). Here, ξ denotes the additive space-time white noise. Since the space-time white noise is more irregular in two dimension than in one dimension, the SNLH system (1) is not well-posed. To circumvent this particular problem we introduce renormalisation.

2 The need for renormalisation

In this section we argue why one needs a renormalisation technique to study the two dimensional SNLH system. In particular we consider

$$\mathbf{u} = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{in\cdot x},$$

and a Gaussian free field (GFF) with a density

$$\mathrm{d}\mu = Z^{-1} e^{-\frac{1}{2}\|\mathbf{u}\|_{H^1}^2} \mathrm{d}u.$$

We start with a truncated version of $\mathbf{u}$ given by

$$\mathbf{u}_N = \sum_{|n|\le N} \frac{g_n}{\langle n\rangle} e^{in\cdot x},$$

for a fixed $x \in \mathrm{T}^2$, and $N \in \mathbb{N}$. Then $\mathbf{u}_N(x)$ is mean zero Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\left[|\mathbf{u}_N(x)|^2\right] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N,$$

where $\log N \to \infty$ as $n \to \infty$, and the limit $\mathbf{u}$ is **not** a function but a distribution. furthermore we observe that

$$\mathbb{E}[\|\mathbf{u}_N\|_{H^s}^2] = \sum_{|n|\leq N} \frac{1}{\langle n \rangle^{2(1-s)}} < \infty,$$

if and only if $s < 0$. If $s \geq 0$, then $\mathbf{u} \notin H^s$ with positive probability. And as consequence of Kolmogorov 0-1 law, $\mathbf{u} \notin L^2(\mathbb{T}^2)$ a.s (diverges almost sure). Hence, the SNLH system is ill-posed since $\mathbf{u}$ is a distribution, and in general we cannot define a product of distributions unless we introduce some structure . In particular, the power $\mathbf{u}^k$ is ill-defined and thus we need to introduce a renormalisation.

Remarks

Before, introducing renormalisation structure, we make the following remarks. As defined before, let

$$\mathbf{u} = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x},$$

where $\{g_n(\omega)\}$ is the independent standard $\mathbb{C}$-valued Gaussian random variable.

In the motivation of why the SNLH system (1) is ill-posed, we deduced that

$$\mathbf{u} \in H^s(\mathbb{T}^2) \backslash L^2(\mathbb{T}^2)$$

a.s. for $s < 0$. Now, a natural phenomenon to investigate at this stage will be the case where $p < 2$ in

$$L^p(\mathbb{T}^2) \supset L^2(\mathbb{T}^2).$$

The Case when p=1:

For $p = 1$: we proceed as follows

Let

$$X_n = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x},$$

for fixed $x \in \mathbb{T}^2$, and

$$Y_n = X_n \cdot \mathbb{1}_{\{\{X_n\}\leq A\}}$$

for some $A \geq 0$. In view of this formulation, we recall the following theorem.

Theorem 1 (Kolomogorov three series:). *Let X_N be a sequence of independent random variable. The random series $\sum_{n=1} X_n$ converges a.s. if and only if the following conditions hold for some $A > 0$:*

(i).

$$\sum_{n=1} \mathbb{P}(|X_n| \geq A) \text{ converges.}$$

(ii) Let $Y_n = X_n \cdot \mathbb{1}_{\{\{X_n\} \leq A\}}$, then $\sum_{n=1} \mathbb{E}[Y_n]$, the series of expected values of Y_n, converges.

(iii)

$$\sum_{n=1} \operatorname{var}(Y_n) \ \textit{converges.}$$

In our case we obtain

$$\sum_{n\in\mathbb{Z}^2} \operatorname{var}(Y_n) \geq \sum_{n\in\mathbb{Z}^2} \frac{1}{\langle n\rangle^2}\mathbb{E}|g_n|\cdot\mathbb{1}_{|g_n|\leq A\langle n\rangle} \geq \sum_{n\in\mathbb{Z}^2} \frac{1}{\langle n\rangle^2}\mathbb{E}|g_n|\cdot\mathbb{1}_{|g_n|\leq A} = \infty.$$

Therefore in view of Theorem 1 (Kolomogorov three series), the series $\sum_{n\in\mathbb{Z}^2} X_n$ diverges on a set of positive probability. As a consequence of Kolomogorov 0-1 Law, the the series $\sum_{n\in\mathbb{Z}^2} X_n$ diverges a.s. Now taking the expectation of $\mathbf{u} \in L^1(\mathbb{T}^2)$ with some cut-off (not relabelled) we observe that

$$\mathbb{E}\left(\int_{\mathbb{T}^2} |\mathbf{u}(x)|\,\mathrm{d}x\right) = \int_{\mathbb{T}^2} \underbrace{\mathbb{E}(|\mathbf{u}(x)|)}_{I=\infty}\,\mathrm{d}x = \infty,$$

for each fixed x as the limit diverges in N. Hence, taking the expectation of the truncated solution i.e. $\mathbf{u}_N$ yields the following results: $\mathbf{u}_N$ does not converge in $L^1(\mathbb{T}^2)$ with positive probability, by Kolomogorov 0-1 Law, $\mathbf{u}_N$ diverges in $L^1(\mathbb{T}^2)$ a.s.

On the Regularity of $\mathbf{u}_N$:

Since

$$\mathbf{u}_N = \sum_{|n|\leq N} \frac{g_n}{\langle n\rangle} e^{in\cdot x},$$

Then,

$$\mathbb{E}\Big[\mathbf{u}_N(x)\overline{\mathbf{u}_N}(y)\Big] = \sum_{|n|\leq N} \frac{1}{\langle n\rangle^2} e_n(x-y),$$

inserting $\langle\nabla_x\rangle^{-\varepsilon}$ and$\langle\nabla_y\rangle^{-\varepsilon}$ on the left side of equation and setting $x = y$, for $n \in \mathbb{Z}^2$ we get

$$\mathbb{E}[|\langle\nabla\rangle^{-\varepsilon}\mathbf{u}_N(x)|^2] \lesssim \sum_{|n|\leq N} \frac{1}{\langle n\rangle^{2+2\varepsilon}} < \infty,$$

uniformly in $N \in \mathbb{N}$. Note, the estimate still holds for $\mathbf{u}_N - \mathbf{u}_M$ by similar computation. It is also worth noting that in the real-setting we have

$$\mathbb{E}[g_n^k \overline{g}_n^l] = k!\,\delta_{kl},$$

and in particular, $\mathbb{E}[g_n^2] = 0$. Now computing, the Sobolev norm, and applying Minkoski's inequality for $r < \infty$ and $p \geq r$ we infer

$$\Big\|\|\mathbf{u}_N(x)\|_{W^{s,r}}\Big\|_{L^p(\Omega)} \leq \Big\|\underbrace{\|\langle\nabla\rangle^s\mathbf{u}_N(x)\|_{L^p(\Omega)}}_{=p^{\frac{1}{2}}\|\cdots\|_{L^2(\Omega)}}\Big\|_{L^r_x(\mathbb{T}^2)} \lesssim p^{\frac{1}{2}},$$

uniformly in $N \in \mathbb{N}$, for $s < 0$. Note, a similar computation for $\mathbf{u}_N - M$ holds. And as such, $\{\mathbf{u}_N\}_{N\in\mathbb{N}}$ converges in $L^p(\Omega, W_x^{s,r}(\mathbb{T}^2)$, for $s < 0$ and $r \leq \infty$. As a consequence of Chebyshev's inequality, we have an exponential tail estimate:

$$\mathbb{P}\left(\|\mathbf{u}\|_{W_x^{s,r}} > \lambda\right) \leq Ce^{-c\lambda^2}.$$

For higher order powers, we get $\frac{\lambda^2}{k}$ on the left hand side of the estimate.

3 Renormalisation

Since $\mathbf{u}$ is not a function but only defined in the sense of a distribution, the power term $\mathbf{u}^k$ is ill-defined. As such, to make sense of the SNLH system (1) we need to introduce a renormalisation. In the following presentation, we restrict our attention to the real-valued setting. Our main object will be a Hermite polynomial.

Hermite Polynomial:

For this we consider a generating function

$$e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x,\sigma), \tag{2}$$

where the term on the right follows from a series expansion, and we note that $H_k(x,\sigma)$ is the Hermite polynomial of degree k. In our case we use the probabilistic k^{th} Hermite polynomial given by

$$H_k(x,\sigma) = (-\sigma)^n e^{\frac{x^2}{2\sigma}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-\frac{x^2}{2\sigma}},$$

with variance σ. We can easily compute the Hermite polynomials using the following recurrence relation

$$H_{k+1}(x,\sigma) = xH_k(x,\sigma) - \sigma \frac{\mathrm{d}}{\mathrm{d}x} H_k(x,\sigma),$$

to deduce

$$\begin{aligned}
H_0(x,\sigma) &= 1, \\
H_1(x,\sigma) &= x, \\
H_2(x,\sigma) &= x^2 - \sigma, \\
H_3(x,\sigma) &= x^3 - 3\sigma x, \\
H_4(x,\sigma) &= x^4 - 6\sigma x^2 + 3\sigma.
\end{aligned}$$

The importance of these polynomials follows from their orthogonality property, thus setting $H_k(x) = H_k(x,1)$, we observe that for $k,m \in \mathbb{N}$, the L^2 product inner of these polynomials with respect to Gaussian measure on $\mathbb{R}$ is

$$\int H_k(x) H_m(x) \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \, \mathrm{d}x = k! \, \delta_{km}.$$

In fact, the Hermite polynomial $\left\{ \frac{H_k(x)}{\sqrt{k!}} \right\}_{k \in \mathbb{Z}_{\geq 0}}$ forms an orthonormal basis of $L^2\left(\mathbb{R}, \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \mathrm{d}x\right)$, see ([1], Lemma 1.1.2) for details.

In view of the generating function (2), we also have another important orthogonality property given by the following lemma.

Lemma 1. *Let f and g be jointly Gaussian random variables with* ***mean zero****, let σ_f and σ_g be variance of f and g respectively. Then for any $n, m \in \mathbb{N}$, we have*

$$\mathbb{E}\Big[H_k(f,\sigma_f)H_m(g,\sigma_g)\Big] = \delta_{km}k!\Big\{\mathbb{E}[fg]\Big\}^k. \tag{3}$$

Proof. Taking the product of the generating functions for (f, σ_f) and (g, σ_g) yields:

$$\mathbb{E}\Big[e^{tf-\frac{1}{2}\sigma_f t^2}e^{sg-\frac{1}{2}\sigma_g s^2}\Big] = e^{ts\mathbb{E}[fg]}$$

Using (2) and expanding both sides we obtain

$$\sum_{k,m=0}^{\infty}\frac{t^k s^m}{k!\,m!}\mathbb{E}\Big[H_k(f,\sigma_f)H_m(g,\sigma_g)\Big] = \sum_{k=0}\frac{(ts)^k}{k!}\Big\{\mathbb{E}[fg]\Big\}^k.$$

Identifying the coefficients of the power series yields (3). □

On the complex-valued setting, we instead use the (generalised) Laguerre polynomials see [2] for more details.

3.1 Wick renormalisation

In these lecture notes, we define wick renormalisation as the orthogonal projection onto the Wiener homogeneous chaoses of degree k by using the Hermite polynomials discussed above. We define the ordered monomial $:\mathbf{u}_N^k:$ by

$$:\mathbf{u}_N^k(x) := H_k(\mathbf{u}_N, \sigma_N),$$

pointwise where $H_k(\mathbf{u}_N, \sigma_N)$ is the Hermite polynomial polynomial of degree k as defined in (2). We note that, for each fixed $x \in \mathbb{T}^2$, the random variable $\mathbf{u}_N$ is a mean-zero real valued Gaussian with variance

$$\sigma_N = \mathbb{E}\left[|\mathbf{u}_N(x)|^2\right] = \sum_{|n|\le N}\frac{1}{\langle n\rangle^2} \sim \log N. \tag{4}$$

Here, our σ_N does not depend on $x \in \mathbb{T}^2$. And it is also essential to note that, on a manifold $\mathcal{M}$, σ_N depends on $x \in \mathcal{M}$. Using the Hermite polynomials computations shown above we see that

$$\text{for } k=2, \quad :\mathbf{u}_N^2 := \mathbf{u}_N^2 - \sigma_N,$$

$$\text{for } k=3, \quad :\mathbf{u}_N^3 := \mathbf{u}_N^3 - 3\sigma_N\mathbf{u}_N.$$

Infinite dimension Case:

Let (H, B, μ) be an abstract wiener space. For our two-dimensional case, we consider the Gaussian measure given by

$$\mathrm{d}\mu = Z^{-1}e^{\frac{1}{2}\|\mathbf{u}\|_H^2}\mathrm{d}u,$$

and set $H = H^1(\mathbb{T}^2)$, and $B = H^{-\varepsilon}(\mathbb{T}^2)$. Let $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ be a complete orthonormal system of $H^* = H$. We consider the product of finitely many Hermite polynomials in different directions given by

$$\prod_{j=1}^{\infty} H_{kj}(\langle x, e_j\rangle),$$

where $kj \neq 0$ for finitely many $j's$. And $\langle\cdot,\cdot\rangle = B - B^*$ is the dual pairing (in our case its just a product in H). Let $\mathcal{H}_k$ denote the collection of homogeneous wiener chaoses of degree k under $\|\cdot\|_{L^2(B,\mu)}$ so that

$$\mathcal{H}_k = \text{span}\left\{\prod_{j=1}^{\infty} H_{kj}(\langle x, e_j\rangle) : \sum_j kj = k\right\}.$$

Wiener-Ito decomposition:

Thus, we have

$$L^2(B,\mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

and we set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{H}_j,$$

to be a polynomial in Gaussian of degree k.

Lemma 2 (Wiener Chaos Estimate)**.** *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.

For more details on the proof see previous lecture notes on hypercontraxtivity (taught by professor Tadahiro Oh).

Proposition 1. $\{:\mathbf{u}_N^k:\}_{N\in\mathbb{N}}$ *forms a Cauchy sequence in $L^p(\Omega; W^{s,r}(\mathbb{T}^2)), s < 0, r \leq \infty$. Denoting the limit by $: u^k :$, we have*

$$:\mathbf{u}^k :\in W^{s,r}(\mathbb{T}^2),\ s<0,\ r\leq\infty.$$

Proof. Let $\{:\mathbf{u}_N^k(x):\}_{N\in\mathbb{N}}$ and $\{:\mathbf{u}_N^k(y):\}_{N\in\mathbb{N}}$ be sequences in $L^p(\Omega; W^{s,r}(\mathbb{T}^2))$ so that in view of (3) the following holds:

$$\begin{aligned}
\mathbb{E}[:\mathbf{u}_N^k(x)::\mathbf{u}_N^k(y):] &= k!\left\{\mathbb{E}[\mathbf{u}_N(x)\mathbf{u}_N(y)]\right\}^k \\
&= k! \sum_{|n|\leq N} \frac{1}{\langle n_1\rangle} e_{n_1}(x-y)\cdots\frac{1}{\langle n_k\rangle} e_{n_k}(x-y) \\
&= k! \sum_{|n_j|\leq N} \prod_{j=1}^{k} \frac{1}{\langle n_j\rangle^2} e_{n_1+\cdots+n_k}(x-y)
\end{aligned}$$

Inserting $\langle \nabla_x \rangle^{-\varepsilon}$ and $\langle \nabla_y \rangle^{-\varepsilon}$, and setting $x = y$ we deduce

$$\begin{aligned}\mathbb{E}[|\langle \nabla \rangle^{-\varepsilon} : \mathbf{u}_N^k : |^2] &= k! \sum_{|n_j| \le N} \prod_{j=1}^{k} \frac{1}{\langle n_j \rangle^2} \cdot \frac{1}{\langle n_1 + \cdots + n_k \rangle^{2\varepsilon}} \\ &\lesssim_{\varepsilon,k} 1,\end{aligned}$$

uniformly in $N \in \mathbb{N}$. Computing the Sobolev norm and using the Minkowski inequality for $p \ge r$, we obtain

$$\Big\| \|: \mathbf{u}_N^k : \|_{W^{s,r}} \Big\|_{L^p(\Omega)} \le \Big\| \| \langle \nabla \rangle^s : u_N^k(x) : \|_{L^p(\Omega)} \Big\|_{L^r_x}$$

In view of Wiener Chaos estimate i.e. Lemma (2), $\| \langle \nabla \rangle^s : u_N^k(x) : \|_{L^p(\Omega)} \le p^{\frac{k}{2}} \| \cdots \|_{L^2(\Omega}$ so that

$$\Big\| \|: \mathbf{u}_N^k : \|_{W^{s,r}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}}$$

for $s < 0$.

$\square$

Note that, a similar computation holds for (different projections) $: \mathbf{u}_N^k : - : \mathbf{u}_M^k :, N \ge M \ge 1$. In this case, we use $\max(|n_j| \ge M$ which implies $\frac{1}{\langle n_j \rangle^2} \le \frac{1}{M^\delta} \frac{1}{\langle n_j \rangle^{2-\delta}}$ for some $\delta < \varepsilon$.

Next we consider the lemma which allows us to study the regularity of stochastic terms.

Lemma 3 (Regularity)**.** *Let $\{X_N\}_{N \in \mathbb{N}}$ and X be spatially homogenous stochastic process such that $X_N, X : \mathbb{R} \to \mathcal{D}'(\mathbb{T}^d)$ i.e. for any $x_0 \in \mathbb{T}^d$, $\{X(t, \cdot)\}_{t \in \mathbb{R}}$ and $\{X(t, \cdot + x_0)\}_{t \in \mathbb{R}}$ have the same law. Suppose that $X_N(t), X(t) \in \mathcal{H}_{\le k}$, for all $t \in \mathbb{R}$*

(i) Fix $t \in \mathbb{R}$, if there exists $s_0 \in \mathbb{R}$ such that

$$\mathbb{E}\Big[\hat{X}(t,n)|^2 \Big] \lesssim \langle n \rangle^{-d-2s_0} \tag{5}$$

for all $n \in \mathbb{Z}^d$. Then,

$$X(t) \in W^{s,\infty}(\mathbb{T}^d), s < s_0, \ a.s.$$

Moreover, if there exists $\theta \ge 0$ such that

$$\mathbb{E}\Big[|\hat{X}_N(t,n) - \hat{X}(t,n)|^2 \Big] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0} \tag{6}$$

for all $n \in \mathbb{Z}^d$. Then,

$$X_N \to X \text{ in } W^{s,\infty}(\mathbb{T}^d), s < s_0, \ a.s.$$

(ii) Fix $T > 0$. Suppose (i) holds on $[-T, T]$ and consider the difference operator $\delta_n f(t) = f(t+h) - f(t)$. If there exists $\sigma \in (0,1)$ such that

$$\mathbb{E}\left[|\delta_n \hat{X}(t,n)|^2\right] \lesssim \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

for all $n \in \mathbb{Z}^d, |h| \leq 1$, and for all $t \in [-T, T]$. Then,

$$X \in C([-T,T]; W^{s,\infty}(\mathbb{T}^d))$$

for $s < s_0 - \frac{-\sigma}{2}$, a.s. Furthermore, if there exists $\theta > 0$ such that

$$\mathbb{E}\left[|\delta_n \hat{X}(t,n) - \delta_n \hat{X}(t,n)|^2\right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

for all $n \in \mathbb{Z}^d, |h| \leq 1$, and for all $t \in [-T, T]$. Then,

$$X_N \to X \text{ in } C([-T,T]; W^{s,\infty}(\mathbb{T}^d)), s < s_0, \text{ a.s.}$$

Note that, $W^{s,\infty}(\mathbb{T}^d)$ can be replaced $C^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$.

4 Stochastic Wave Equation

In this part of the lecture notes we show a renormalisation of the two-dimensional stochastic wave equations (SNLW) on $\mathbb{T}^2$ with an additive space-time white noise forcing

$$\begin{cases} (\partial_t^2 - \Delta)\mathbf{u} + \mathbf{u}^k = \xi, \\ (\mathbf{u}, \partial_t \mathbf{u})|_{t=0} = (\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2), \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R} \end{cases} \tag{7}$$

where $k \geq 2$ is an integer and $\xi(x,t)$ denotes a Gaussian space-time white-noise on $\mathbb{T}^2 \times \mathbb{R}$. In fact, the current formulation of SNLW system (7) is ill-posed since solutions are expected to be distributions in space variable, and as such, $\mathbf{u}^k$ is ill-defined. In addition, it can be shown that the stochastic convolution

$$\Psi(t,x) = \int_0^t \frac{\sin(t-t')\langle \nabla \rangle}{\langle \nabla \rangle} \, \mathrm{d}W(t'), \tag{8}$$

where $\Psi(t,x) \in C_t W_x^{s,\infty}(\mathbb{T}^2) \backslash L^2(\mathbb{T}^2)$ is also ill-posed for $s < 0$ a.s. For more details on this see [2], and references therein. Similarly to the SNLH system, this problem is cured by introducing a renormalisation.

4.1 Renormalised SNLW

We define the ordered monomial $: \Psi_N^k(t,x) :$ of the truncated stochastic convolution by

$$: \Psi_N^k(t,x) := H_k\Bigg(\Psi_N(t,x), \sigma_N(t)\Bigg)$$

where $H_k\Bigg(\Psi_N(t,x), \sigma_N(t)\Bigg)$ is the Hermite polynomial of degree k. Following the presentation in [2], $\Psi_N(t,x)$ is a Gaussian random variable with variance

$$\sigma_N(t) = \mathbb{E}[|\Psi_N(t,x)|^2] \sim t \log N.$$

Next, we look at an example where we can apply regularity Lemma 3 in $: \Psi_N^k(t,x) :$.

Proposition 2. *$\{: \Psi_N^k(t,x) :\}_{N \in \mathbb{N}}$ forms a Cauchy sequence in $L^P(\Omega; W_x^{-\varepsilon,\infty})$, $p < \infty$ a.s.*

Proof. Here, we only verify (5) in the Regularity Lemma 3. Now in view of (3), we deduce that

$$\begin{aligned} \mathbb{E}\left[|: \widehat{\Psi(t,x)}_N^k :(t,n)|^2\right] &= \int_{\mathbb{T}_y^2} \int_{\mathbb{T}_x^2} \mathbb{E}\left[: \Psi(t,x)_N^k(t,x) :: \Psi(t,x)_N^k(t,y) : \right] e_n(y-x) \, \mathrm{d}x \mathrm{d}y \\ &= k! \left\{ \mathbb{E}[\Psi_N(t,x)\Psi_N(t,y)] \right\}^k. \end{aligned}$$

Using Ito isometry, we have

$$\mathbb{E}[\Psi_N(t,x)\Psi_N(t,y)] = \sum_{|m| \leq N} \int_0^t \left(\frac{\sin(t-t')\langle m \rangle}{\langle m \rangle} \right)^2 \mathrm{d}t \, e_m(x-y)$$

therefore,

$$\mathbb{E}\left[|: \widehat{\Psi(t,x)}_N^k :(t,n)|^2\right] \lesssim \sum_{n_1 + \cdots n_k = n} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2} \lesssim \langle n \rangle^{-2+\varepsilon},$$

where $\varepsilon = -2s_0$ in regularity Lemma. Thus $\Psi_N(t,x) \in W^{s,\infty}$, $s < 0$ a.s.

□

Lecture (9):

So far, we checked the regularity of the wick power (same as wick monimial)

$$: \Psi^k(t,x) := \lim_N : \Psi_N^k(t,x) :, \qquad \Psi_N(t,x) = P_{\leq}\Psi_{\text{wave}}(t,x)$$

of the stochastic convolution in 2-d SNLW case using (5) for a fix t in part (i) of the regularity Lemma 3. Then, for part (ii) of Lemma 3 i.e. the difference part, we simply use the mean value Theorem to get $\langle n\rangle^\sigma |h|^\sigma$.

Note that (dropping N), by construction

$$: \Psi^k :\in \mathcal{H}_k.$$

Then, in view of the Wiener chaos estimate in Lemma 2 and Chebyshev's inequality, we get

$$\mathbb{P}\Big(\|: \Psi^k : \|_{L_T^q W_x^{-\varepsilon,\infty}} > \lambda \Big) \leq C \exp\left(-c \frac{\lambda^{\frac{2}{k}}}{T^{1+\frac{2}{qk}}} \right)$$

for all $\varepsilon > 0$, finite $q \geq 1, T > 0$. For $q = \infty$, we can use the Garsia-Rodemich-Rumsey inequality and get

$$\mathbb{P}\Big(\|: \Psi^k : \|_{L^\infty([j,j+1]; W_x^{-\varepsilon,\infty}(\mathbb{T}^2))} > \lambda \Big) \leq \exp\left(-c \frac{\lambda^{\frac{2}{k}}}{j+1} \right).$$

For more details see [3] and references therein.

4.2 2-d Heat convolution

In the following computations, we use Lemma 3 part (i) to show regularity of stochastic convolution in the **Heat case**. For 2-d SNLH system(1), the stochastic convolution is given by

$$\Psi = \int_0^t e^{(t-t')(\Delta - 1)}\, \mathrm{d}W(t'). \tag{9}$$

Now, using Lemma 3 we infer the following Proposition holds a.s.

Proposition 3. *$\{: \Psi_N^k(t,x) :\}_{N\in\mathbb{N}}$ forms a Cauchy sequence in $L^p(\Omega; C_t C_x^{-\varepsilon}(\mathbb{T}^2))$ and in $C_t C_x^{-\varepsilon}(\mathbb{T}^2)$, and $C^{-\varepsilon} = B^{\varepsilon}_{\infty,\infty}$*

Proof. (dropping N):

We only verify (5) in the Regularity Lemma 3. Now in view of (3), we deduce that

$$\begin{aligned} \mathbb{E}\Big[|: \widehat{\Psi(t,x)^k} :(t,n)|^2 \Big] &= \int_{\mathbb{T}_y^2} \int_{\mathbb{T}_x^2} \mathbb{E}\Big[: \Psi(t,x)^k(t,x) :: \Psi(t,x)^k(t,y) : \Big] e_n(y-x)\, \mathrm{d}x\mathrm{d}y \\ &= k! \Big\{ \mathbb{E}[\Psi(t,x)\Psi(t,y)] \Big\}^k. \end{aligned}$$

Using Ito isometry, we have

$$\mathbb{E}[\Psi(t,x)\Psi(t,y)] = \sum_m \int_0^t e^{-2(t-t')\langle m\rangle^2}\,\mathrm{d}t\, e_m(x-y) = \frac{1-e^{-2\langle m\rangle^2}}{\langle m\rangle^2}$$

therefore,

$$\mathbb{E}\Big[|\widehat{:\Psi(t,x)^k_N:}(t,n)|^2\Big] \lesssim \sum_{n_1+\cdots n_k=n}\prod_{j=1}^k \frac{1}{\langle n_j\rangle^2} \lesssim \langle n\rangle^{-2+\varepsilon}$$

for all $\varepsilon>0$ in regularity Lemma. Thus $:\Psi^k:(t,x)\in C_tC_x^{-\varepsilon}(\mathbb{T}^2)$ a.s. □

Idea of the proof of the regularity Lemma 3:

Assuming that all stochastic process are spatial homogeneous i.e. Translation invariance (in x) we get

$$\mathbb{E}[\widehat{X}(t,n)\widehat{X}(t,m)] = 0$$

for $n+m\neq 0$, in the real-value setting. Noting the difference frequencies we get

$$\mathbb{E}[\widehat{X}(t,n)\widehat{X}(t,m)] \;=\; \int\!\!\int \underbrace{\mathbb{E}[X(t,x)X(t,y)]}_{II}\ \underbrace{e_{-n}(x)e_{-m}(y)}_{=e_{-(n+m)}(x)e_m(x-y)}\ \mathrm{d}x\mathrm{d}y,$$

where $II=\widehat{F}(t,x-y)$ for a fixed t. Then,

$$\mathbb{E}\Big[\widehat{X}(t,n)\widehat{X}(t,m)\Big] = \widehat{F}(t,m)\int_{\mathbb{T}^d} e^{-i(n+m)\cdot x}\,\mathrm{d}x = 0$$

for $n+m\neq 0$. Now suppose (5) holds for a fixed t. Then, for $p>>1$,$r>>1$. The Sobolev inequality yields

$$\Big\|\|X\|_{W^{s,\infty}}\Big\|_{L^p(\Omega)} \lesssim \Big\|\|X\|_{W^{s+\varepsilon,\infty}}\Big\|_{L^p(\Omega)}$$

Using Minkowski inequality for $p\geq r$, we get

$$\Big\|\|X\|_{W^{s+\varepsilon,\infty}}\Big\|_{L^p(\Omega)} \leq \Big\|\|\langle\nabla\rangle^{s+\varepsilon}X\|_{L^p(\Omega)}\Big\|_{L^r}.$$

Finally, using Wiener chaos estimate Lemma (2)

$$\Big\|\|\langle\nabla\rangle^{s+\varepsilon}X\|_{L^p(\Omega)}\Big\|_{L^r} \leq p^{\frac{k}{2}}\Big\|\underbrace{\|\sum_n \langle n\rangle^{s+\varepsilon}\widehat{X}(n)e_n(x)\|_{L^2(\Omega)}}_{\mathcal{G}}\Big\|_{L^r},$$

where

$$\mathcal{G} = \oplus\left(\sum_n \langle n\rangle^{2(s+\varepsilon)}\mathbb{E}[|\widehat{X}(n)|^2]\right)^{\frac{1}{2}}.$$

Using (5), we get

$$\left\| \|\langle \nabla \rangle^{s+\varepsilon} X\|_{L^p(\Omega)} \right\|_{L^r} \lesssim p^{\frac{k}{2}} \left(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2(s+\varepsilon)-d-2s_0} \right)^{\frac{1}{2}} < \infty,$$

for $s < s_0$ (by taking $0 < \varepsilon << 1$ such that $s + \varepsilon < s_0$). To prove part (ii) of Lemma 3, we use Kolomogorov continuous criterion.

References

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LECTURE 9 AND LECTURE 10.

GUOPENG LI

1. Lecture 9 (Continues from the start of Section 2.1 of hand-written lecture notes)

1.1. LWP of SNLW on $\mathbb{T}^2$.

Lemma 1.1. *Basic product estimates*

(i) *fractional Leibniz rule:*
Let $0 \le s \le 1$, $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, *and* $j = 1, 2$. *We have*

$$\|\langle\nabla\rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|f\|_{L^{p_1}} \|\langle\nabla\rangle^s g\|_{L^{q_1}} + \|g\|_{L^{q_2}} \|\langle\nabla\rangle^s f\|_{L^{p_2}}$$

(ii) *Let* $s > 0$, $1 < p, q, r < \infty$, *and* $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{s}{d}$. *We have*

$$\|\langle\nabla\rangle^{-s} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle\nabla\rangle^{-s} f\|_{L^p} \|\langle\nabla\rangle^s g\|_{L^q}$$

Proof. The proof of (i), we use the fractional Leibniz rule on $\mathbb{R}^d$[1] and transference principle [3]. The proof of (ii), see [3]. Here, f is merely a distribution but the product fg makes sense as long as the sum of the regularities is greater or equal to 0. □

In the following we study the following stochastic nonlinear wave equations (SNLW) on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ with an additive space-time white noise forcing:

$$\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^k = \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \qquad (x, t) \in \mathbb{T}^2 \times \mathbb{R}_+, \tag{1.1}$$

where $\xi(x, t)$ denotes a (Gaussian) space-time white noise on $\mathbb{T}^2 \times \mathbb{R}_+$. Next, we truncate the noise by writing it as $\xi_N = P_{\le N}\xi$. Then, we consider the truncated SNLW:

$$(\partial_t^2 + 1 - \Delta)u_N + u_N^k = \xi_N,$$

where u_N^k has no truncation. In particular, we write the Duhamel form of u_N as following decomposition: (stochastic convolution + reminder part)

$$u_N = \Psi_N + v_N.$$

Then, by the binomial theorem we see that u_N^k satisfies:

$$u_N^k = \sum_{j=0}^{k} \binom{k}{j} \Psi_N^j v_N^{k-j}.$$

[1] See also Coifman-Meyer theorem and Kato-Ponce inequality.

We note here Ψ_N^j has no limit for $j \geq 2$. Hence, we need to renormalize it by using the Wick renormalization. Namely, we replace Ψ_N^j by its Wick ordered counterpart:

$$:\Psi_N^j(x,t): \stackrel{\text{def}}{=} H_j(\Psi_N(x,t);\sigma_N(t)),$$

where $H_\ell(x;\sigma)$ is the Hermite polynomial of degree j with variance parameter σ and $\sigma_N(t) \sim t\log N$. Then, for each $j \in \mathbb{N}$, the limit of Wick power $:\Psi_N^j:$ exists a.s.. Now, we define the renormalized nonlinearity $:u_N^k:$ is interpreted as

$$:u_N^k: = :(\Psi_N + v_N)^k: = \sum_{j=0}^{k} \binom{k}{j} :\Psi_N^j: v_N^{k-j}.$$

For the above definition, it is only defined for u_N of the form $u_N = \Psi_N + v_N$. Here, the residual term v_N satisfies

$$(\partial_t^2 + 1 - \Delta)v_N + \sum_{j=0}^{k} \binom{k}{j} :\Psi_N^j: v_N^{k-j} = 0.$$

By taking a limit as $N \to \infty$, we then obtain the limiting equation:

$$(\partial_t^2 + 1 - \Delta)v + \sum_{j=0}^{k} \binom{k}{j} :\Psi^j: v^{k-j} = 0. \tag{1.2}$$

Then, the "solution" u to the renormalized SNLW:

$$(\partial_t^2 + 1 - \Delta)u + :u^k: = \xi \tag{1.3}$$

is given by

$$u = \Psi + v \tag{1.4}$$

Remark 1.2. Here, the equation (1.3) for u is just a fomal expression. For example, when $k = 3$, we have

$$(\partial_t^2 + 1 - \Delta)u + u^3 - 3 \underbrace{\infty}_{=\lim_{N\to\infty}\sigma} u = \xi$$

By (1.3), we really mean (1.2) with the first order decomposition of (1.4)

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We consider the following deterministic NLW:

$$\begin{cases} (\partial_t^2 + 1 - \Delta)v + \sum_{j=0}^{k} \binom{k}{j} \Xi_\ell \, v^{k-j} = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{2.1}$$

for given initial data (v_0, v_1) and a source $(\Xi, \dots, \Xi_k)$ with the understanding that $\Xi_0 \equiv 1$. We call the following expression enhanced data set:

$$\boldsymbol{\Xi} = (u_0, u_1, \Xi_1, \Xi_2, \dots, \Xi_k). \tag{2.2}$$

We define $\mathcal{X}^s(\mathbb{T}^2)$ by

$$\mathcal{X}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} \mathcal{H}^s(\mathbb{T}^2) \times \big(L^\infty([0,1]; W^{-\varepsilon,\infty}(\mathbb{T}^2))\big)^{\otimes k}$$

and set

$$\|\Xi\|_{\mathcal{X}^s} = \|(v_0, v_1)\|_{\mathcal{H}^s} + \sum_{j=1}^{k} \|\Xi_j\|_{L^\infty([0,1];W^{-\varepsilon,\infty})}$$

for $\Xi = (u_0, u_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathcal{X}^s(\mathbb{T}^2)$. Then, we have the following local well-posedness result for (2.1).

Proposition 2.1. *There exists $\varepsilon_k > 0$ small such that, for $0 \le \varepsilon < \varepsilon_k$, (2.1) is locally well-posed. More precisely, given an enhanced data set:*

$$\Xi = (v_0, v_1, \Xi_1, \Xi_2, \dots, \Xi_k) \in \mathcal{X}^s(\mathbb{T}^2),$$

there exist $T = T(\|\Xi\|_{\mathcal{X}^s}) > 0$ and a unique solution v to (2.1) in the class:

$$C([0,T]; H^{1-\varepsilon}(\mathbb{T}^2)). \tag{2.3}$$

In particular, the uniqueness of v holds in the entire class (2.3). Furthermore, the solution map: $\Xi \mapsto v \in C([0,T; H^{1-\varepsilon}(\mathbb{T}^2))$ is continuous.

Proof. **Case 1:** $s = 1 - \varepsilon$.

By writing (2.1) in the Duhamel formulation, we have

$$\begin{aligned} v(t) = \Gamma(v) &\overset{\text{def}}{=} \partial_t S(t) u_0 + S(t) u_1 \\ &\quad + \sum_{j=0}^{k} \binom{k}{j} \int_0^t S(t-t')\big(\Xi_j\, v^{k-j}\big)(t')dt', \end{aligned} \tag{2.4}$$

where the map $\Gamma = \Gamma_\Xi$ depends on the enhanced data set Ξ in (2.2) and $S(t) = \frac{\sin t\langle\nabla\rangle}{\langle\nabla\rangle}$. Fix $0 < T < 1$.

We first treat the case $j = 0$. From (2.4) and applying Sobolev's inequality twice, we obtain

$$\begin{aligned} \bigg\| \int_0^t S(t-t') v^k(t') dt' \bigg\|_{C_T H_x^{1-\varepsilon}} &\lesssim T\|v^k\|_{C_T H_x^{-\varepsilon}} \lesssim T\|v^k\|_{C_T L_x^{\frac{2}{1+\varepsilon}}} \lesssim T\|v\|^k_{C_T L_x^{\frac{2k}{1+\varepsilon}}} \\ &\lesssim T\|v\|^k_{C_T H_x^{1-\varepsilon}}, \end{aligned}$$

provided that

$$0 \le \varepsilon \le \frac{1}{k-1}.$$

Case 2: $1 \le \ell \le k-1$.

It follows from Lemma 1.1 (ii) and then (i) followed by Sobolev's inequality that

$$\begin{aligned} \bigg\| \int_0^t S(t-t')\big(\Xi_j\, v^{k-j}\big)(t')dt' \bigg\|_{C_T H_x^{1-\varepsilon}} &\lesssim T\|\Xi_j\, v^{k-j}\|_{L_T^\infty H_x^{-\varepsilon}} \\ &\lesssim T\|\langle\nabla\rangle^{-\varepsilon}\Xi_j\|_{L_T^\infty L_x^{\frac{2}{\varepsilon}}} \|\langle\nabla\rangle^{\varepsilon} v^{k-j}\|_{L_T^\infty L_x^2} \\ &\lesssim T\|\Xi\|_{\mathcal{X}^{1-\varepsilon}} \|\langle\nabla\rangle^{\varepsilon} v\|^{k-j}_{L_T^\infty L_x^{2(k-j)}} \\ &\lesssim T\|\Xi\|_{\mathcal{X}^{1-\varepsilon}} \|v\|^{k-j}_{C_T H_x^{1-\varepsilon}}, \end{aligned}$$

provided that

$$0 \le \varepsilon \le \frac{1}{2(k-1)}.$$

Case 3: $j = k$.
Lastly, we have

$$\Big\| \int_0^t S(t-t')\Xi_k(t')dt' \Big\|_{C_T H_x^{1-\varepsilon}} \lesssim T\|\Xi_k\|_{C_T H_x^{-\varepsilon}} \le T\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}.$$

Putting Case 1, 2 and 3 together, we have

$$\begin{aligned}\|\Gamma(v)\|_{C_T H_x^{1-\varepsilon}} \le C_1\|(u_0,u_1)\|_{\mathcal{H}^{1-\varepsilon}} + C_2 T\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}\big(1+\|v\|_{C_T H_x^{1-\varepsilon}}\big)^{k-1} \\ + C_3 T\|v\|^k_{C_T H_x^{1-\varepsilon}}.\end{aligned}$$

A similar estimate hols for the difference $\Gamma(v_1) - \Gamma(v_2)$. Therefore, by choosing $T = T(\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}) > 0$ sufficiently small, we conclude that Γ is a contraction in the ball of radius

$$R \sim \|(u_0,u_1)\|_{\mathcal{H}^{1-\varepsilon}} + \|\Xi_k\|_{C([0,1];W_x^{-\varepsilon,\infty})}.$$

□

Summary:
Know: $2-d$ NLW is ill-posed in negative Sobolev spaces, see [1, 4, 2]. In particular, if we consider the $2-d$ SNLW such that

$$(u_0, u_1, \xi) \to u$$

is ill-posed. The idea is that we decompose the ill-defined solution map into two steps:

(i) use stochastic analysis (only in this step) to construct an enhanced data set.

(ii) use deterministic analysis to prove LWP for the remainder term

$$v = u - \Psi.$$

$$(u_0,u_1,\xi) \overset{\text{(i)}}{\longmapsto} \Xi = (u_0,u_1,\Psi,:\Psi^2:,\cdots,:\Psi^k:) \overset{\text{(ii)}}{\longmapsto} v \longmapsto u = \Psi + v$$

where the step (i) is not continuous but the step (ii) is a continuous map.

Theorem 2.2. *There exists $\varepsilon_k > 0$ small such that for $0 < \varepsilon < \varepsilon_k$, the renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $\mathcal{H}^{1-\varepsilon}(\mathbb{T}^2)$. Moreover, the solution u_N to the truncated renormalized SNLW:*

$$(\partial_t^2 + 1 - \Delta)u_N + u^k := \xi_N$$

converges to

$$u = \Psi + v \in C([0,T_\omega]; H^{-\varepsilon}(\mathbb{T}^2)),$$

a.s..

Remark 2.3. (i) First order expansion $u = \Psi + v$

- trick to solve the equation
- also gives a description. For example, in small scales, u "behaves like" Ψ.

(ii) Without renormalization, no non-reivial limit exists ($u_N \to 0$ or linear solution). This is called triviality [5].
(iii) Regularization on ξ:

- by $P_{\leq N}$
- by mollification $\xi_\delta = \eta_\delta * \xi$ (spatial/ space-time),

which give a same limit.

Q: Rougher initial data? We consider the NLW on $\mathbb{R}^d$:

$$(\partial_t^2 - \Delta)u + u^k = 0$$

- Scaling invariance: scaling critical regularity s_c

$$s_c = \frac{d}{2} - \frac{2}{k-1}$$

- Lorentz invariance (conformal):

$$s_{\text{conf}} = \frac{d+1}{4} - \frac{1}{k-1}$$

Then, we have

$$s_{\text{crit}} = \max(s_c, s_{\text{conf}}, 0).$$

Hence, when $d = 2$

$$s_{\text{crit}} = \max(1 - \tfrac{2}{k-1}, \tfrac{3}{4} - \tfrac{1}{k-1}, 0).$$

Theorem 2.4. [3] *(i) when $k = 2, 3$ and let $s > s_{\text{crit}}$.*
(ii) when $k \geq 4$ and let $s > s_{\text{crit}}$.
Then, the renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$.

Proposition 2.5. *Strichartz estimates*
Let $0 \leq s \leq 1$. We say (q, r) s-admissible and $(\widetilde{q}, \widetilde{r})$ dual s-admissible if

$$1 \leq \widetilde{q} \leq 2 \leq q \leq \infty \qquad 1 \leq \widetilde{r} \leq 2 \leq r \leq \infty$$

$$(q, r, d) \neq (2, \infty, 3) \qquad (\widetilde{q}, \widetilde{r}, d) \neq (2, 1, 3).$$

If the following conditions hold:

$$\textit{scaling}: \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\widetilde{q}} + \frac{d}{\widetilde{r}} = 2$$

$$\textit{admissibility}: \quad \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}, \qquad \frac{1}{\widetilde{q}'} + \frac{d-1}{2\widetilde{r}'} \leq \frac{d-1}{4}.$$

Note here, $(\widetilde{q}, \widetilde{r})$ dual s-admissible if and only if $(\widetilde{q}', \widetilde{r}')$ is $(1-s)$-admissible.
Then, for the SNLW on $\mathbb{T}^d$:

$$\begin{cases} (\partial_t^2 + 1 - \Delta)u + = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

we have the following estimates:

$$\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}_x^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(u_0, u_1)\|_{\mathcal{H}_x^s} + \|F\|_{L_T^{\widetilde{q}} L_x^{\widetilde{r}}}.$$

Also,

$$\|(u,\partial_t u)\|_{L^\infty_T\mathcal{H}^s_x}+\|u\|_{L^q_TL^r_x}\lesssim\|(u_0,u_1)\|_{\mathcal{H}^s_x}+\|F\|_{L^1_TH^{s-1}_x},$$

for $0\le T\le 1$.

Proof. Follows from Strichartz estimates on $\mathbb{R}^d$, see [7]. □

Proof of the theorem: We only consider the case when $k=3$ and the critical regularity $s_{\text{crit}}=\frac{1}{4}$. For the general case see [3, 6].

Let

$$(q,r)=(\tfrac{12}{1+4\delta},\tfrac{3}{1-2\delta}),\quad (\tfrac14+\delta)-\text{admissible}.$$
$$(\widetilde{q},\widetilde{r})=(\tfrac{12}{9+4\delta},\tfrac{3}{3-2\delta}),\quad (\tfrac14+\delta)-\text{admissible}.$$

The nonlinearity is

$$\Xi_j v^{3-j}\in L^{\widetilde{q}}_TL^{\widetilde{r}}_x+L^1_TH^{s-1}_x.$$

For example,

$$\|v^3\|_{L^{\widetilde{q}}_TL^{\widetilde{r}}_x}=\|v\|^3_{L_T^{\frac{36}{9+4\delta}}L_x^{\frac{9}{3-2\delta}}}\lesssim T^\theta\|v\|^3_{L^q_TL^r_x}$$

where $\frac{36}{9+4\delta}<q<\frac{12}{1+4\delta}$ and $\frac{9}{3-2\delta}<r<\frac{3}{1-2\delta}$. Now, if $s-1\le-\varepsilon$

$$\|\Xi_3\|_{L^1_T\mathcal{H}^{s-1}_x}\le T\|\Xi_3\|_{L^\infty_TW^{-\varepsilon,\infty}_x}\le T\|\Xi\|_{\mathcal{X}^s}.$$

Lastly, for $s-1=-\frac34+\delta$ we use Sobolev's inequality

$$\begin{aligned}\|\Xi_jv^{3-j}\|_{L^1_T\mathcal{H}^{s-1}_x}&\le\|\langle\nabla\rangle^{-\delta}(\Xi_jv^{3-j}\|_{L^1_TL^2_x}\\&\lesssim\|\langle\nabla\rangle^{-\delta}\Xi_j\|_{L^\infty_TL^\infty_x}\|\langle\nabla\rangle^{\delta}v^{3-j}\|_{L^1_TL_x^{\frac{8}{7-8\delta}}}.\end{aligned}$$

By using Lemma 1.1, we see that when $j=2$:

$$\|\langle\nabla\rangle^\delta v^{3-j}\|_{L^1_TL_x^{\frac{8}{7-8\delta}}}\lesssim T\|v\|_{L^\infty_TH^\delta_x}\le T\|v\|_{L^\infty_TH^s_x},$$

and when $j=1$:

$$\|\langle\nabla\rangle^\delta v^{3-j}\|_{L^1_TL_x^{\frac{8}{7-8\delta}}}\lesssim T\|\langle\nabla\rangle^\delta v^2\|_{L^\infty_TL_x^{\frac{16}{7-8\delta}}}\lesssim T\|v\|^2_{L^\infty_TH_x^{\frac{1+16\delta}{8}}}.$$

Let us define the $Y^s(T)$-norm to be

$$\|\vec{u}\|_{Y^s(T)}=\|(u,\partial_tu)\|_{L^\infty_T\mathcal{H}^s_x}+\|u\|_{L^q_TL^r_x}.$$

Then, we have

$$\begin{aligned}\|\vec{\Gamma}(v)\|_{Y^s(T)}&=\|(\Gamma(v),\partial_t\Gamma(v))\|_{Y^s(T)}\\&\le C_1\|(u_0,u_1)\|_{\mathcal{H}^s}+C_2(\|v\|_{Y^s(T)}+\|\Xi\|_{Z^{-\delta}})^3\end{aligned}$$

where $\|\Xi\|_{Z^{-\delta}}=\sum_{j=1}^3\|\Xi_j\|_{L^\infty([0,1];W^{-\delta,\infty}_x)}$. A similar estimate holds for the difference.

□

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SINGULAR STOCHASTIC DISPERSIVE PDES

FOIVOS KATSETSIADIS

1. Lecture 11

1.1. **2-d. Stochastic NLH.**

We study the $P(\Phi)_2$-model (or Φ_2^{k+1}-model) where P denotes a polynomial. We want to prove local well-posedness for the equation

$$\partial_t u + (1-\Delta)u + u^k = \sqrt{2}\xi. \tag{1.1}$$

The basic space on which we will be looking for solutions is $C^s(\mathbb{T}^2) = B^s_{\infty,\infty}(\mathbb{T}^2)$

We introduce the stochastic convolution, which is given by

$$\Psi = \Psi_{heat} = \int_0^t P(t-t')dW(t') \quad , \qquad P(t) = e^{t(\Delta-1)} \tag{1.2}$$

and belongs to the space $C_t C^s_x(\mathbb{T}^2)$ for all $s<0$, a.s. (as well as in $L^p(\Omega)$).

As in the case of the 2 dimensional stochastinc non-linear wave equation, we use the first order expansion:

$$u = \Psi + v \tag{1.3}$$

(see [5]) and solve the equation for $v = u - \Psi$:

$$\partial_t v + (1-\Delta)v + (v+\Psi)^k = 0 \tag{1.4}$$

Since Ψ is not a function (i.e. it is only a distribution-valued function), Ψ^j does not make sense for $j \geq 2$ and thus we consider the renormalized version:

$$\partial_t v + (1-\Delta)v + \sum_{j=0}^{k} \binom{N}{k} :\Psi^j: v^{k-j} = 0 \tag{1.5}$$

where $\Psi^j = \lim_{N\to\infty} :(P_{\leq N}\Psi)^j:$

On the following, we only consider $(SNLH_v)$, but we can also show that $v_N \to v$ where v_N satisfies

$$\partial_t v_N + (1-\Delta)v_N + \sum_{j=0}^{k} \binom{k}{j} :\Psi_N^j: v_N^{k-j} = 0 \tag{1.6}$$

2010 *Mathematics Subject Classification.* 35L71, 60H15.

Key words and phrases. stochastic nonlinear wave equation; nonlinear wave equation; damped nonlinear wave equation; renormalization; white noise; Gibbs measure.

where $\Psi_N = P_{\leq N}\Psi$ and hence $u_N = \Psi_N + v_N \to u = \Psi + v$

Our main tool is the paraproduct decomposition (see [1]). We let $f, g \in \mathbb{T}^d$ (or $\mathbb{R}^d$) of regularities s_1 and s_2. Then,

$$fg = f \varolessthan g + f \circledequal g + f \varogreaterthan g = \sum_{j<k-2} P_j(f)P_k(g) + \sum_{|j-k|\leq 2} P_j(f)P_k(g) + \sum_{k<j-2} P_j(f)P_k(g) \tag{1.7}$$

- $P_j = L^p$ projection onto $\{|n| \sim 2^j\}$ (for $j = 0$ this is just $\{|n| \leq 1\}$)
- $f \varolessthan g$ = Paraproduct of g by f

  - Always makes sense (for any $s_1, s_2 \in \mathbb{R}$) as a distribution with regularity $min(s_1, s_1 + s_2)$
- The same for $f \varogreaterthan g \sim min(s_1, s_1 + s_2)$.
- $f \circledequal g$: Resonant product of f and g.

  - May not make sense as a distribution. In general, if $s_1 + s_2 > 0$, then $f \circledequal g \sim s_1 + s_2$ makes sense

  - Recall the product estimate (ii) from GKO, where $f \circledequal g$ makes sense for $s_1 + s_2 = 0$ in terms of Sobolev spaces.

In studying nonlinear PDEs, the main task is to make sense of (or give a meaning to) the nonlinearity u^k. Since the paraproduct $f \varolessthan g$ and $f \varogreaterthan g$ always make sense, the main job is to make sense of the resonant product $f \circledequal g$ (more in the parabolic thinking). When there is an issue in making sense of u^k, we overcome this issue by imposing a structure on u:

$$u = \Psi + v \quad , \qquad u \in X^{s,b} \tag{1.8}$$

Lemma 1.1. *(Paraproduct estimates) Let $s_1, s_2 \in \mathbb{R}$ and let $1 \leq p, p_1, p_2, q \leq \infty$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*

1. $||f \varolessthan g||_{B^{s_2}_{p,q}} \lesssim ||f||_{L^{p_1}} ||g||^{s_2}_{B_{p_2,q}}$
2. *If $s_1 < 0$ then $||f \varolessthan g||_{B^{s_1+s_2}_{p,q}} \lesssim ||f||^{s_1}_{B_{p_1,q}} ||g||^{s_2}_{B_{p_2,q}}$*
3. *If $s_1 + s_2 > 0$ then $||f \circledequal g||^{s_1+s_2}_{B_{p_1,q}} \lesssim ||f||^{s_1}_{B_{p_1,q}} ||g||^{s_2}_{B_{p_2,q}}$*

Note that in (1)*, f is a function if $s_1 \geq 0$ and in* (1) *and* (2) *one has $f \varolessthan g \sim min(s_2, s_1 + s_2)$*

Proof. We begin by proving (1). We have

$$||f \varolessthan g||_{B^{s_2}_{p,q}} \sim ||2^{s_2 k}||P_k(f \varolessthan g)||_{L^p_x}||_{l^q_k(\mathbb{Z}_{\geq 0})} \tag{1.9}$$

where P_k projects to frequencies $\sim 2^k$. We have

$$P_k(f \otimes g) = \sum_{i=-2}^{2} \sum_{j<k+i-2} P_j(f) P_{k+i}(g) \tag{1.10}$$

Heuristically, one has that freq. of $f \otimes g \sim$ freq. of g. Now, let $S_k(f) = \sum_{j<k} P_j(f)$ which projects f onto the set of frequencies $\{|n| \lesssim 2^k\}$. Therefore,

$$P_k(f \otimes g) = \sum_{i=-2}^{2} S_{k+i-2}(f) P_{k+i}(g) \tag{1.11}$$

Now, we take the L^p_x-norm and apply Hölder's inequality:

$$||P_k(f \otimes g)||_{L^p_x} = \sum_{i=-2}^{2} ||S_{k+i-2}(f)||_{L^{p_1}} ||P_{k+i}(g)||_{L^{p_2}} \tag{1.12}$$

Taking into account that $||S_{k+i-2}(f)||_{L^{p_1}} \lesssim ||f||_{L^{p^1}}$ uniformly in k and i, we then multiply by $2^{s_2 k}$ and take the l^q_k norm on both sides to obtain

$$||f \otimes g||_{B^{S_2}_{p,q}} \lesssim ||f||_{L^{p_1}} ||g||_{B^{s_2}_{p,q}} \tag{1.13}$$

We now proceed to prove (2). We have

$$P_k(f \otimes g) \sim \sum_{i=-2}^{2} 2^{-s_1 j} 2^{s_1 k} \sum_{0 \le j < k+i-2} P_j(f) 2^{s_2 k} P_{k+i}(g) \tag{1.14}$$

$$\lesssim \sum_i \sum_{0 \le j \le k+i-2} 2^{s_1(k-j)} (2^{s_1 j} ||P_j(f)||_{L^{p_1}}) (2^{s_2(k+i)} ||P_{k+i}(g)||_{L^{p_2}}) \tag{1.15}$$

where we took the L^p_x-norm and applied Hölders inequality. We then take l^q_k-norms. This concludes the proof of (2).

Finally, we prove (3). One has

$$2^{(s_1+s_2)k} P_k(f \ominus g) = \sum_{i=-2}^{2} 2^{(s_1+s_2)(k-j)} \sum_{j \ge k-10} P_k(2^{s_1 j} P_j(f) 2^{s_2 j} P_{j+i}(g)) \tag{1.16}$$

We then take l^q_k-norms and viewing the terms summed over i on the right hand side as convolutions of $2^{(s_1+s_2)j} \mathbf{1}_{j \le 0}$ (recall that $s_1 + s_2 > 0$) and $P_j(f) P_{j+i}(g) \mathbf{1}_{j \le 0}$ (which are both in $l^q_j(\mathbb{Z})$) we use Young's inequality to obtain

$$||f \ominus g||_{B^{s_1+s_2}_{p,q}} \lesssim ||f||_{B^{s_1}_{p_1,q}} ||g||_{B^{s_2}_{p_2,\infty}} \tag{1.17}$$

This completes the proof.

□

Corollary 1.2. *Let $s > 0$. Then, $C^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$ is an algebra with $||fg||_{C^s} \lesssim ||f||_{C^s}||g||_{C^s}$*

We now go back to $(SNLH_V)$. The Duhamel formulation reads

$$v(t) = \Gamma v(t) = P(t)u_0 - \sum_{j=0}^{k} \binom{k}{j} \int_0^t P(t-t') : \Psi^j : v^{k-j}(t')dt' \tag{1.18}$$

where $: \Psi^j : \in C_t C_x^{-\epsilon}$.
Let $s > 0$. We write $s = 2\epsilon$. Then,

$$||\Gamma v||_{C_T C_x^s} \lesssim ||u_0||_{C^s} + \sum_{j=0}^{k} \binom{k}{j} \int_0^t (t-t')^{-\frac{3}{2}\epsilon} || : \Psi^j : v^{k-j}(t')||_{C^{-\epsilon}} dt' \tag{1.19}$$

and we also have the bound

$$|| : \Psi : v^{k-j}|| \lesssim || : \Psi^j : ||_{C^{-\epsilon}} ||v||^{k-j}_{C^{2\epsilon}} \tag{1.20}$$

The moral of this is that in order to make sense of the product, the sum of the regularities must be > 0. However, the resulting regularity of the product is given by one of the paraproducts.

Example 1. *Let $s_1 < 0 < s_2$. We need $s_1 + s_2 > 0$, but $fg \sim s$, coming from $f \circleddash g$.*

Hence, we have

$$||\Gamma v||_{C_T C_x^s} \lesssim ||u_0||_{C^s} + \sum_{j=0}^{k} T^\theta \cdot || : \Psi^j : ||_{C_T C_x^{-\epsilon}} ||v||^{k-j}_{C_T C^s} \tag{1.21}$$

for $s > 0$ (s.t. $s + \epsilon > 0$). We thus obtain local well posedness of $SNLH_v$ in $C^s(\mathbb{T}^d)$
We will now consider rougher initial data, i.e. the case of $u_0 \in C^s(\mathbb{T}^2)$ for $s < 0$. We have

$$||u||_{Y(T)} = \sup_{0<t<T} t^\theta ||u(t)||_{C_x^\sigma} \tag{1.22}$$

where $\sigma, \theta > 0$. Hence,

$$t^\theta ||\Gamma v(t)||_{C_x^\sigma} \lesssim t^\theta t^{\frac{s-\sigma}{2}} ||u_0||_{C^s} + \sum_{j=0}^{k} \binom{k}{j} t^\theta \int_0^t (t-t')^{-\frac{3}{2}\epsilon} || : \Psi^j : v^{k-j}(t')||_{C_x^{-\epsilon}} dt' \tag{1.23}$$

where we set $\theta = \frac{\sigma - s}{2}$ and $\sigma = 2\epsilon$.
We also have the bound

$$|| : \Psi : v^{k-j}(t')||_{C_x^{-\epsilon}} \lesssim || : \Psi^j : ||_{C_T C_x^{-\epsilon}} ((t')^\theta ||v(t')||_{C_x^{2\epsilon}})^{k-j} (t')^{-(k-j)\theta} \tag{1.24}$$

where we note that $((t')^\theta ||v(t')||_{C_x^{2\epsilon}})^{k-j} \leq ||v||_{Y(T)}^{k-j}$. We need to control

$$t^\theta \int_0^t (t-t')^{-\frac{3}{2}\epsilon}(t')^{-(k-j)\theta} dt' \tag{1.25}$$

for $t \leq T$. To this end, we recall some facts about the Beta funciton:

$$B(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt \qquad Rex, Rey > 0 \tag{1.26}$$

and

$$t^{a_1} \int_0^t (t-t')^{\alpha_2}(t')^{\alpha_3} dt' = B(a_2+1, a_3+1) < \infty \tag{1.27}$$

where $a_1 + a_2 + a_3 = -1$ and $a_2, a_3 > -1$

In order to bound 1.25 we need $\theta + (-\frac{3}{2}\epsilon) + (-k\theta) \geq -1$, therefore we get $s > -\frac{2}{k-1}$. Since we also need the corresponding $a_2, a_3 > -1$ to apply the formula that involves the Beta function given above, we additionally get $\epsilon < \frac{2}{3}$ and $\theta = \epsilon - \frac{s}{2} < 1/k$. Therefore, in these cases, the expression 1.25 is finite. Hence, by a contraction argument, we obtain $v \in Y(T)$ that satisfies the equation in the Duhammel formulation. This implies that $v \in C([0,T]; C_x^\sigma(\mathbb{T}^2))$, with $\sigma = 2\epsilon > 0$. A posteriori we can show that $v \in C([0,T]; C_x^s(\mathbb{T}^2))$, with $s < 0$. We thus obtain local well-posedness for $(SNLH_v)$ in $C^s(\mathbb{T}^2)$ for $s > -\frac{2}{k}$.

2. Lecture 12

2.1. Global-in-time aspects.

2.1.1. *Parabolic $P(\Phi)_2$-model.*

$$(\partial_t + 1 - \Delta)u + u^k = \sqrt{2}\xi \qquad on\ \mathbb{T}^2, \quad k \in 2\mathbb{N}+1 \tag{2.1}$$

We renormalize the nonlinearity $: u^{k+1} :$

Pathwise approach: Control the L^p - norm (p $= p(k)$) of $v = u - \Psi$. Why is this enough? One has

$$(\partial_t + 1 - \Delta)v + \sum_{j=0}^{k} \binom{N}{k} : \Psi_N^j : v_N^{k-j} = 0 \tag{2.2}$$

with the initial condition $v|_{t=0} = u_0$.

As in Lec. 11, we work with the $Y(T)$-norm:

$$||u||_{Y(T)} = \sup_{0<t<T} t^\theta ||u(t)||_{C^\sigma} \tag{2.3}$$

where $\sigma = 2\epsilon > 0$, $\theta = \epsilon - \frac{s}{2}$ and $2\frac{2}{k} < s < 0$.

$$t^\theta ||\Gamma(v)(t)||_{C^\sigma} \lesssim t^\theta t^{-\frac{\sigma}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{\infty})} sup_m ||P_m u_0||_{L^p} + \ Duhamel\ term \tag{2.4}$$

We then need $\theta - \frac{\sigma}{2} - \frac{1}{p} \geq 0$ i.e. $\frac{1}{p} \leq -\frac{s}{2}$ which is equivalent to $p \geq -\frac{s}{2} >> 1$. when $s \to 0^-$ i.e. $k \to \infty$.

Then, we obtain local well-posedness of $(SNLH_v)$ in $L^p(\mathbb{T}^2)$ and the local existence time $\sim (||u_0||_{L^p(\mathbb{T}^2)} + Wick\ powers\ of\ \Psi)^{-\sigma}$. For fixed $T >> 1$, the part with the Wick powers inside the parenthesis may be large, but finite a.s.

$$\sum_{j=1}^{k} || : \Psi^j : ||_{C([0,T];C^{-\epsilon})} < C_\epsilon < \infty \tag{2.5}$$

Therefore, as long as we control $sup_{0<t<T}||v(t)||_{L^p}$ (for each $T >> 1$), we obtain global well-posedness. To do this, we just compute $\partial_t ||v(t)||^p_{L^p}$ and use the equation and the cotrol on the stochastic terms to get a bound on $sup_{0<t<T}||v(t)||_{L^p}$

- On $\mathbb{T}^2$: See [Trenberth '19] on the stochastic complex Ginzburg-Landau (SCGL) equation.

$$\partial_t u = (a_1 + ia_2)(\Delta - 1)u - (c_1 + ic_2)|u|^{k-1}u + \sqrt{2}\xi. \tag{2.6}$$

  with $r = |\frac{a_1}{a_2}|$, then one can obtain global well-posedness for $r \geq C(k)$.
- On $\mathbb{R}^2$: In [8] global well-posedness of the parabolic $P(\Phi)_2$-model on $\mathbb{R}^2$ is obtained using weighted Besov spaces.

2.2. **Invariant measure argument.** The Gibbs measure on $\mathbb{T}^2$ is given by

$$dp = z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1} dz} d\mu_1 \qquad k \in 2\mathbb{N} + 1 \tag{2.7}$$

where the measure $d\mu_1 = e^{-||u||^2_{H^1}} du$. A typical "function" u under μ_1 is not a function. as then one has $\int_{\mathbb{T}^2} u^{k+1} dx = \infty$ a.s. hence the need to renormalize the potential energy. We use the following renormalization procedure:

$$\int : u^{k+1} : dx = \lim_{N\to\infty} \int_{\mathbb{T}^2} : (P_{\leq N} u)^{k+1} : dx. \tag{2.8}$$

where we recall that $: (P_{\leq N} u)^{k+1} : \to : u^{k+1} :$ in $W^{-\epsilon,\infty}(\mathbb{T}^2)$ a.s. or in $C^{-\epsilon}(\mathbb{T}^2)$ in the $L^p(\Omega)$ sense.

- In the 70's we have the development of Euclidean Quanty Field Theory and that

$$e^{\frac{1}{k+1}} \int_{\mathbb{T}^2} : u^{k+1} : dx \in L^p(d\mu_1) \qquad \forall p < \infty \tag{2.9}$$

- Hypercontractivity of Ornstein-Uhlenbeck process/Wiener chaos estimate due to [9].
- Nelson's estimate
- See also the course Probabilistic Perspectives in Nonlinear Dispersive PDEs from 2017 by Tadahiro Oh
- Also see [11, 12] and [7], [15], [6].

We then use Bourgain's invariant measure argument (see [2], [10] for SNLW on a 2-d compact Riemannian manifold without boundary).

2.3. **2-d SNLW/SdNLW.**

2.3.1. *Pathwise approach.* : Known only for $k = 3$ (GKOT). For $k = 5$ the problem is still open.

With $v = U - \Psi$, we have

$$(\partial_t^2 + 1 - \Delta)v + v^3 + \underbrace{3v^2\Psi + 3v : \Psi^2 : + : \Psi^3 :}_{rough\ perturbation} = 0 \tag{2.10}$$

This presents two difficulties:

(1) $v(t) \in H^{1-\epsilon}(\mathbb{T}^2) - H^1(\mathbb{T}^2)$ Hence, can not use the energy

$$E(\vec{v}) = \frac{1}{2}\int | < \nabla > |^2 dx + \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{4}\int v^4 dx \tag{2.11}$$

We want to smooth out v. We therefore use the I-method.

(2) Even if v were in H^1, v does not satisfy the deterministic NLW. Hence, $E(\vec{v})$ is not conserved.

- If the noise is a bit smoother, $\Psi_2 = (\partial_t^2 + 1 - \Delta)^{-1} < \Delta_x > \xi \in \mathbb{C}_t L_x^\infty$. We therefore obtain global well-posedness by use of the Gronwall argument (due to [3]). This gives:

$$\partial_t E(\vec{v}) \leq C(\Psi) E(t) \tag{2.12}$$

- I-method (= method of almost conservation law), see [4] (after Bourgain's high-low method '98):
  For $N \in \mathbb{N}$ and $0 < s < 1$, let

$$m_N(n) = \begin{cases} 1, & |n| \leq N \\ \frac{N^{1-s}}{|n|^{1-s}}, & |n| \geq 2N \end{cases}$$

  By L^p theory, we have

$$||If||_{W^{\alpha+\sigma,p}} \lesssim N^\sigma ||f||_{W^{\alpha,p}} \quad \forall 0 \geq \sigma \geq 1-s \ \ \forall 1 < p < \infty \tag{2.13}$$

$$||f||_{H^s} \lesssim ||If||_{H^1} \lesssim N^{1-s} ||f||_{H^s} \tag{2.14}$$

  Hence, we now study the $I - SNLW_v$:

$$(\partial_t^2 + 1 - \Delta)Iv + I(v^3) + 3I(v^2\Psi) + 3I(v : \Psi^2 :) : +I(: \Psi^3 :) = 0 \tag{2.15}$$

  Thus, $E(\vec{Iv})$ is not conserved for two reasons:
  (a) $I(v^3) \neq (Iv)^3$ so we need a commutator estimate for $I(v^3) - (Iv)^3$
  (b) The perturbation terms for rough $: \Psi^j :$

Lemma 2.1. *Let $p < \infty$. Then*

$$|| \, ||I\Psi||_{L^p_{T,x}} ||_{L^p(\Omega)} \lesssim p^{1/2} T^{1/2+1/p} (logN)^{1/2} \tag{2.16}$$

Furthermore, one has

$$I(v^2\Psi) \to (Iv)^2 \cdot I\Psi + error \tag{2.17}$$

Hence, at the end, we obtain

$$E(\vec{Iv}) \lesssim C + \int_0^t E(\vec{Iv}) log E(\vec{Iv})(t')dt' \tag{2.18}$$

Therefore, we get a double exponential bound.

2.4. **Invariant measure argument for SdNLW.**

$$(\partial_t^2 + \partial_t + 1 - \Delta)u + u^k = \xi \qquad k \in 2\mathbb{N} + 1 \tag{2.19}$$

The Gibbs measure is given by: $\rho(du, \vec{d}(\partial_t u)) + \rho(du) \otimes \mu_0(d(\partial_t u))$
where ρ denotes the Φ_2^{k+1}-measure and μ_0 denotes the law of the white noise process.
We set $v = u - \Psi$ to obtain the Duhamel formulation:

$$v(t) = \partial_t D(t) u_0 + D(t)(u_0 + u_1) - \sum_{j=0}^{k} \binom{k}{j} \int_0^t D(t-t')(: \Psi^j : v^{k-j})(t')dt' \tag{2.20}$$

where $D(t) = e^{-t/2} \frac{min(t, \sqrt{\frac{3}{4} - \Delta})}{\frac{3}{4} - \Delta}$ (one degree of smoothing).

The same local well-posedness argument (by Sobolev) as in SNLW works. By virtue of this, we obtain local well-posedness and therefore a.s. global well-posedness and invariance of $\vec{\rho}$.

Remark 2.2. *For damped NLW, the same Strichartz estimates hold locally in time on $\mathbb{T}^2$.*

Remark 2.3.
- *Parabolic Φ_2^{k+1}-model = (parabolic) stochastic quantization equation for Φ_2^{k+1}-measure (see* [13]*).*
- *Hyperbolic Φ_2^{k+1}-model = canonical SQE (see* [14]*). With $w = \partial_t u$,*

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E}{\partial u} \\ \frac{\partial E}{\partial W} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{\partial E}{\partial W} + \xi \end{pmatrix}$$

(Langevin equation)

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THE PARABOLIC Φ^4_3 MODEL

YOUNES ZINE

Contents

1. A first look at the model and the need for a paracontrolled approach

We look at the parabolic Φ^4_3 model

$$\begin{cases}(\partial_t + 1 - \Delta)u + u^3 = \xi \\ u|_{t=0} = u_0\end{cases} \qquad (x,t) \in \mathbb{T}^3 \times \mathbb{R}_+, \tag{1.1}$$

where u_0 belong to some $C^s(\mathbb{T}^3)$, $s \in \mathbb{R}$. See [1, 2, 3] for a local theory for (1.1) and [4] for a construction of the related stochastic objects.

Recall that on $\mathbb{T}^3$, we have $\Psi := (\partial_t + 1 - \Delta)^{-1}\xi \sim -\frac{1}{2}$, so that Ψ^3 and hence u^3 does not make sense. As in the two dimensional case it seems tempting to try to attack the problem through a first order expansion $u = v + \uparrow$, where $\uparrow$ is the stochastic convolution Ψ - here we use the tree notation as presented in [4] for instance. Under this decomposition, v solves the equation

$$\begin{aligned}(\partial_t + 1 - \Delta)v &= -(v + \uparrow)^3 \\ &= -v^3 - 3v^2\uparrow - 3v\uparrow^2 - \uparrow^3\end{aligned}$$

Since $\uparrow$ has negative regularity, its powers $\uparrow^2$ and $\uparrow^3$ are ill-defined. By adding counterterms to remove divergences we will replace them by $\vee$ and ψ respectively. The equation for v now reads

$$(\partial_t + 1 - \Delta)v = -v^3 - 3v^2\uparrow - 3v\vee - \psi$$

Let us take a look at the formal regularities of the different terms in the above equation. By "parabolic couting", we have $\vee \sim -1-$ and $\psi \sim -\frac{3}{2}-$. Hence, thanks to Schauder's estimate $v \sim (-\frac{3}{2}-) + (2-) = \frac{1}{2}-$ and the product $v\vee \sim (-\frac{1}{2}-) + (-1-) = -\frac{1}{2} < 0$ is ill-defined.

We now try a second order expansion $u = v + \uparrow - \Upsilon$, where Υ solves

$$(\partial_t + 1 - \Delta)\Upsilon = \psi$$

Thus v solves - again, replacing the powers of $\uparrow$ by their renormalized counterparts,

$$(\partial_t + 1 - \Delta)v = -(v - \Upsilon)^3 - 3(v - \Upsilon)^2\uparrow - 3(v - \Upsilon)\uparrow^2. \tag{1.2}$$

The worst term is $(v - \Upsilon) \ogreaterthan \vee \sim -1-$. Hence $v \sim (-1-) + (2-) = 1-$ and $v\vee \sim (1-) + (-1-) < 0$ is ill-defined.

One could continue to use higher order expansions but this would not help since the worst term would always involve the unknown - i.e. the term $v\vee$. The idea is to impose a structure on v.

Let us denote by $\circledcirc$ the operator $\olessthan + \odot$. Since the worst term in the right-hand side of (1.2) is $v \ogreaterthan \vee$ we make the ansatz $v = X + Y$ where

$$\begin{cases}(\partial_t + 1 - \Delta)X = -3(X + Y - \Upsilon) \ogreaterthan \vee \\ (\partial_t + 1 - \Delta)Y = -(X + Y)^3 - 3(X + Y - \Upsilon) \circledcirc \vee + Q(X + Y),\end{cases} \tag{1.3}$$

where Q is the polynomial defined by

$$Q(v) = b_0 + b_1 v + b_2 v^2, \tag{1.4}$$

and

$$\begin{cases} b_0 = (\Psi)^3 - 3\uparrow(\Psi)^2 \\ b_1 = 6\uparrow\Psi - 3(\Psi)^2 \\ b_2 = -3\uparrow + 3\Psi \end{cases} \tag{1.5}$$

Let us analyze the system (1.3). By "parabolic counting" one gets $X \sim 1-$ and $Y \sim \frac{3}{2}-$. One still needs to make sense of the resonant product $(X + Y - \Psi) \circleddash \vee$. We have $Y \circleddash \vee \sim (\frac{3}{2}-) + (-1-) = \frac{1}{2}- > 0$. Hence this term is not an issue. After a renormalization procedure one can make sense of $\Psi \circleddash \vee \rightsquigarrow \Psi^{\vee}_{\circleddash}$. We now have a look at $X \circleddash \vee$. As is, $X \sim 1-$ and $\vee \sim -1-$ and hence one cannot directly make sense of the resonant product. However, thanks to (1.3) and the inherited structure of X one can make sense of this product. This idea is the essence of the paracontrolled approach. We denote by $(P(t))$ the semi-group associated to $\partial_t + 1 - \Delta$. We write for $t > 0$

$$X(t) = P(t)X_0 - 3\int_0^t P(t-t')\big((X + Y - \Psi) \olessthan \vee\big)(t')dt' \tag{1.6}$$

By the smoothing properties of the heat semi-group, one can always make sense of the product $P(t)X_0 \circleddash \vee$. Let us recall the following rule of thumbs, if f and g are two spatial functions and D is a Fourier multiplier then one expects $D(f \olessthan g) \approx f \olessthan D(g)$. Thus we expect,

$$\int_0^t P(t-t')\big((X + Y - \Psi) \olessthan \vee\big)(t')dt' \approx (X + Y - \Psi) \olessthan \curlyvee(t)$$

It is then natural to introduce

$$\begin{aligned} \texttt{com}_1(X, Y) := P(t)X_0 - 3\int_0^t P(t-t')\big((X + Y - \Psi) \olessthan \vee\big)(t')dt' \\ + 3(X + Y - \Psi) \olessthan \curlyvee. \end{aligned} \tag{1.7}$$

We will show that the term $\texttt{com}_1(X, Y)$ enjoys some smoothing and has actually spatial regularity $1 + \varepsilon$ for some $\varepsilon > 0$. Hence,

$$X \circleddash \vee = -3\big((X + Y - \Psi) \olessthan \curlyvee\big) \circleddash \vee + \texttt{com}_1(X, Y) \circleddash \vee.$$

Because of the discussion above, $\texttt{com}_1(X, Y) \circleddash \vee \sim (1+) + (-1-) > 0$ is well defined. Moreover, since we expect the high frequency regime of $(X+Y-\Psi)\olessthan\curlyvee$ to be governed by that of $\curlyvee$ we expect $\big((X+Y-\Psi)\olessthan\curlyvee\big)\circleddash\vee \approx (X + Y - \Psi) \olessthan (\curlyvee \circleddash \vee)$. To this end we introduce the following notation for three functions f, g, h:

$$[\olessthan, \circleddash](f, g, h) = (f \olessthan g) \circleddash h - f(g \circleddash h).$$

and set

$$\mathtt{com}_2(X+Y) := [\olessthan, \circledequal]\big(-3(X+Y-\Psi), \Upsilon, \vee\big) \tag{1.8}$$

After another renormalization procedure we replace $\Upsilon \circledequal \vee$ by $\Upsilon_{\circ}\vee$. Then we can write

$$X \circledequal \vee = -3(X+Y-\Psi)\Upsilon_{\circ}\vee + \mathtt{com}_2(X+Y) + \mathtt{com}_1(X,Y) \circledequal \vee.$$

2. Local well-posedness for Φ^4_3

We want to solve the following system

$$\begin{cases} (\partial_t + 1 - \Delta)X = -3(X+Y-\Psi) \olessthan \vee \\ (\partial_t + 1 - \Delta)Y = -(X+Y)^3 - 3Y \circledequal \vee + 3\Psi_{\circ}\vee \\ \qquad\qquad -3(X+Y-\Psi) \ogreaterthan \vee - 3\mathtt{com}_1(X+Y) \circledequal \vee \\ \qquad\qquad -3\mathtt{com}_2(X+Y) + Q(X+Y) \end{cases} \tag{2.1}$$

where Q is given by (1.4) and (1.5) and $\mathtt{com}_1(X,Y)$ and $\mathtt{com}_2(X+Y)$ by (1.7) and (1.8) respectively.

Remark 2.1. *If X and Y solve a (frequency truncated version of) (2.1) then $u = \uparrow - \Psi + X + Y$ solves (a truncated version of) the following equation*

$$(\partial_t + 1 - \Delta)u + u^3 - Cu = \xi$$

for some constant $C > 0$.

2.1. On the stochastic objects.

Here we give the regularities of the relevant stochastic objects in the scale of $C_T C^s_x(\mathbb{T}^3)$ spaces, for any $T > 0$ without proof. See [4] for details.

	$\uparrow$	$\vee$	Ψ	$\Psi_{\circ}\uparrow$	$\Psi_{\circ}\vee$	$\Upsilon_{\circ}\vee$
s	$-\frac{1}{2}-\varepsilon$	$-1-\varepsilon$	$\frac{1}{2}-\varepsilon$	$-\varepsilon$	$-\frac{1}{2}-\epsilon$	$-\varepsilon$

Table 1. The list of relevant stochastic terms with their regularities

In what follows we denote by K the constant given by

$$\begin{aligned} &\max_{(\Xi,s)} \sup_{0\le t\le 1} \|\Xi(t)\|_{C^s_x} \le K \\ &\sup_{0\le t_1<t_2\le 1} \frac{\|\Psi(t_1)-\Psi(t_2)\|_{C^{\frac{1}{4}-\varepsilon}}}{|t_1-t_2|^{\frac{1}{8}}} \le K. \end{aligned} \tag{2.2}$$

where the maximum is taken over the couples (Ξ, s) in Table 1.

2.2. **Some deterministic estimates.** For a space-time function f and two times $t, t' > 0$ we denote by $\delta_{t,t'}(f)$ the quantity $\delta_{t,t'}(f) := f(t) - f(t')$.

Proposition 2.2. *Let $\varepsilon > 0$, $\beta \in (4\varepsilon, 1+2\varepsilon]$, $p \in [1,\infty]$, $T > 0$. Then,*

$$\begin{aligned}\big\|\mathtt{com}_1(X,Y) - P(t)X_0\big\|_{C^{1+2\varepsilon}} &\lesssim K^2 + \int_0^t \frac{K}{(t-t')^{1+2\varepsilon-\frac{\beta}{2}}} \|(X,Y)(t')\|_{B^\beta_{p,\infty}\times B^\beta_{p,\infty}} dt' \\ &+ \int_0^t \frac{K}{(t-t')^{1+2\varepsilon}} \|\delta_{t,t'}(X+Y)\|_{L^p} dt',\end{aligned}$$

where K is as in (2.2).

Proof. See Proposition 2.2 in [3]. □

Proposition 2.3. *Let $\alpha < 1$, $\beta, \gamma \in \mathbb{R}$, $1 \le p, p_1, p_2, p_3 \le \infty$ such that*

$$\beta + \gamma < 0, \quad \alpha + \beta + \gamma > 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Then,

$$[\oslash, \ominus] : (f, g, h) \mapsto (f \oslash g) \ominus h - f(g \ominus h),$$

extends to a continuous trilinear map:

$$B^\alpha_{p_1,\infty} \times B^\beta_{p_2,\infty} \times B^\gamma_{p_3,\infty} \to B^{\alpha+\beta+\gamma}_{p,\infty}$$

Proof. See Proposition A.9 in [4]. □

We now record some basic facts on the heat semi-group $(P(t))$. We shall start with the following which is proven by showing the anologue in the euclidian case $\mathbb{R}^3$ using integration by parts and then deducing the result for the periodic domain $\mathbb{T}^3$ by the Poisson formula.

Lemma 2.4. *Let φ be a function supported on the annulus $\{|\xi| \sim 1\}$. Then we have*

$$\|\mathcal{F}^{-1}\big(\varphi(\frac{\cdot}{2^j})e^{-t|\cdot|^2}\big)\|_{L^1_x} \lesssim e^{-ct2^{2j}},$$

for some $c > 0$.

Lemma 2.5. *Let $p, q \in \mathbb{R}$. We have*

$$\|(1-P(t))f\|_{B^\alpha_{p,q}} \lesssim t^{\frac{\beta-\alpha}{2}} \|f\|_{B^\beta_{p,q}},$$

for $0 \le \beta - \alpha \le 2$.

Proof. We claim that if $\widehat{f}$ is supported on $\{|n|\sim 2^j\}$ then

$$\|(1-P(t))f\|_{L^p}\lesssim (t2^{2j}\wedge 1)\|f\|_{L^p}. \tag{2.3}$$

Assuming (2.3) we have for any fixed j, denoting by P_j the smooth frequency projector onto frequencies or order 2^j,

$$\begin{aligned}2^{\alpha j}\|(1-P(t))P_jf\|_{L^p}&\lesssim 2^{\alpha j}(t2^{2j}\wedge 1)\|P_jf\|_{L^p}\\&\lesssim t^{\frac{\beta-\alpha}{2}}\cdot (t2^{2j})^{\frac{\alpha-\beta}{2}}(t2^{2j}\wedge 1)\cdot 2^{\beta j}\|P_jf\|_{L^p}\\&\lesssim t^{\frac{\beta-\alpha}{2}}\cdot 2^{\beta j}\|P_jf\|_{L^p}.\end{aligned}$$

Summing over j proves the lemma.

We now prove the claim. First, by Lemma 2.4, we have from Young's inequality,

$$\|P(t)P_jf\|_{L^p}\lesssim e^{-ct2^{2j}}\|P_jf\|_{L^p},$$

which shows the claim for $t2^{2j}\geq 1$. Thus, it suffices to prove

$$\|(1-P(t))P_jf\|_{L^p}\lesssim t2^{2j}\|f\|_{L^p}, \tag{2.4}$$

for $t2^{2j}\ll 1$. We prove (2.4) on $\mathbb{R}^3$ as the result follows in the periodic domain upon using the Poisson formula.

By Young's inequality it suffices to bound the L^1 norm of

$$g_j(t,x):=\int_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(\frac{\xi}{2^j})(1-e^{-t\langle\xi\rangle^2})d\xi,$$

for some smooth function φ supported on $\{|\xi|\sim 1\}$. We compute

$$\|g_j(t,x)\|_{L^1_x}=\Big\|\int_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(\xi)(1-e^{-t\lambda^2|\xi|^2-t})d\xi\Big\|_{L^1_x(\mathbb{R}^3)}.$$

Let $Q=\{|x|\lesssim 1\}$. Replacing the L^1_x norm L^∞_x norm on the bounded domain Q, we have by Hausdorff-Young's inequality and the mean value theorem (recalling $|\xi|\sim 1$ in the integral),

$$\begin{aligned}\Big\|\int_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(\xi)(1-e^{-t\lambda^2|\xi|^2-t})d\xi\Big\|_{L^1_x(Q)}&\lesssim\|\varphi(\xi)(1-e^{-t2^{2j}|\xi|^2-t})\|_{L^1_\xi}\\&\lesssim t2^{2j}\end{aligned}$$

Further, since $x\mapsto |x|^{-2\times 3}$ is integrable on Q^c, it suffices to bound the L^∞_x contribution of

$$\begin{aligned}&|x|^{-2\times 3}\int_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(\xi)(1-e^{-t\lambda^2|\xi|^2-t})d\xi\\&\qquad=\int_{\mathbb{R}^3}(-\Delta_\xi)^3(e^{ix\cdot\xi})\varphi(\xi)(1-e^{-t\lambda^2|\xi|^2-t})d\xi\\&\qquad=\int_{\mathbb{R}^3}e^{ix\cdot\xi}(-\Delta_\xi)^3\big(\varphi(\xi)(1-e^{-t\lambda^2|\xi|^2-t})\big)d\xi,\end{aligned}$$

by an integration by parts. The last term may be decomposed into a sum of terms where (i) all the derivatives fall on $\varphi(\xi)$ and (ii) at least one derivatives hits $(1-e^{-t\lambda^2|\xi|^2-t})$. For the former we use the mean value theorem as before and bound its contribution by $t2^{2j}$. For the latter, we get a factor of order at most $t2^{2j}|\xi|^2 \lesssim t2^{2j}$ from the derivatives that falls on $(1-e^{-t\lambda^2|\xi|^2-t})$ which makes this contribution acceptable. □

2.3. **The fixed point argument.** We now show the fixed point argument showing the existence and uniqueness of a solution to (2.1) in some well-chosen space. We introduce the operator Γ^X defined by

$$\Gamma^X(X,Y)(t) := P(t)X_0 - 3\int_0^t P(t-t')\big((X+Y-\Psi)\varolessthan \vee\big)(t')dt'$$

We also define Γ^Y in a similar fashion. In the following we show that for $0<T\le 1$, (Γ^X,Γ^Y) is a contraction mapping from $C_TC_x^{\frac{1}{2}+2\varepsilon}\cap C_T^{\frac{1}{8}}L_x^\infty \times C_TC^{1+2\varepsilon}\cap C_T^{\frac{1}{8}}L_x^\infty =: Z(T)$ to itself for $\varepsilon \ll 1$ assuming that the initial datum (X_0,Y_0) lies in $C_x^{\frac{1}{2}+2\varepsilon}\times C_x^{1+2\varepsilon}$. Let $B_R\subset Z(T)$ be the ball of center 0 and radius $R>0$ in $Z(T)$.

We estimate using the standard heat, paraproducts estimates and the results of Section 2.1,

$$\begin{aligned}
\|\Gamma^X(X,Y)\|_{C_TC_x^{1+2\varepsilon}} &\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + T^\theta\|(X+Y-\Psi)\varolessthan \vee\|_{C_TC_x^{-1-\varepsilon}}\\
&\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + T^\theta\|X+Y-\Psi\|_{C_TL_x^\infty}\|\vee\|_{C_TC_x^{-1-\varepsilon}}\\
&\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + T^\theta K(K+R),
\end{aligned}\tag{2.5}$$

for $(X,Y)\in B_R$. Further, we have for $t_1<t_2$,

$$\begin{aligned}
&\Gamma^X(X,Y)(t_2)-\Gamma^X(X,Y)(t_1)\\
&\quad = (P(t_2-t_1)-\mathrm{Id})P(t_1)X_0\\
&\qquad + (P(t_2-t_1)-\mathrm{Id})\int_0^{t_1} P(t_1-t')\big((X+Y-\Psi)\varolessthan \vee\big)(t')dt'\\
&\qquad + \int_{t_1}^{t_2} P(t_2-t')\big((X+Y-\Psi)\varolessthan \vee\big)(t')dt' =: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

We have from Lemma 2.5,

$$\begin{aligned}
\|\mathrm{I}\|_{L_x^\infty} &\lesssim \|\mathrm{I}\|_{C_x^\varepsilon}\\
&\lesssim (t_2-t_1)^{\frac{1}{8}}\cdot\|X_0\|_{C_x^{\frac{1}{4}+\varepsilon}}.
\end{aligned}\tag{2.6}$$

Similarly

$$\begin{aligned}\|\text{II}\|_{L^\infty_x} &\lesssim (t_2-t_1)^{\frac{1}{8}}\Big\|\int_0^{t_1} P(t_1-t')\big((X+Y-\Psi)\mathbin{\textcircled{<}}\vee\big)(t')dt'\Big\|_{C_x^{\frac{1}{4}+\varepsilon}}\\ &\lesssim (t_2-t_1)^{\frac{1}{8}}\cdot T^\theta K(K+R). \end{aligned}\tag{2.7}$$

We also get

$$\|\text{III}\|_{L^\infty_x} \lesssim (t_2-t_1)^{\frac{1}{8}} T^\theta K(K+R). \tag{2.8}$$

Collecting (2.6), (2.7) and (2.8), we deduce:

$$\|\Gamma(X,Y)\|_{C_T^{\frac{1}{8}}L^\infty_x} \lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + T^\theta K(K+R). \tag{2.9}$$

We now look at the operator Γ^Y. By Proposition 2.4, we estimate for $(X,Y)\in B_R$ and $0<t\le T$:

$$\begin{aligned}&\Big\|\int_0^t P(t-t')\mathtt{com}_1(X+Y)\mathbin{\textcircled{=}}\vee(t')dt'\Big\|_{C_x^{1+2\varepsilon}}\\ &\quad\lesssim \int_0^t \frac{1}{(t-t')^{\frac{1+\varepsilon}{2}}}\|\mathtt{com}_1(X+Y)\mathbin{\textcircled{=}}\vee\|_{C_TC_x^{\varepsilon}}\\ &\quad\lesssim T^\theta\|\mathtt{com}_1(X+Y)\|_{C_TC_x^{1+2\varepsilon}}\cdot\|\vee\|_{C_TC_x^{-1-\varepsilon}}\\ &\quad\lesssim T^\theta K\Big(\|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + R^2 + \int_0^t \frac{K}{(t-t')^{\frac{7}{8}+2\varepsilon}}\cdot\frac{\|\delta_{t,t'}(X+Y)\|_{L^\infty_x}}{(t-t')^{\frac{1}{8}}}dt'\Big)\\ &\quad\lesssim T^\theta K(\|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + R^2 + KR).\end{aligned}$$

We also have for $0<t\le T$ and Proposition 2.5,

$$\begin{aligned}&\Big\|\int_0^t P(t-t')\mathtt{com}_2(X+Y)(t')dt'\Big\|_{C_x^{1+2\varepsilon}}\\ &\quad\lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+\frac{5\varepsilon}{2}}}\|\mathtt{com}_2(X+Y)\|_{C_TC_x^{\frac{1}{2}-3\varepsilon}}\\ &\quad\lesssim T^\theta\|X+Y-\Psi\|_{C_TC_x^{\frac{1}{2}-\varepsilon}}\|\Upsilon\|_{C_TC_x^{1-\varepsilon}}\|\vee\|_{C_TC_x^{-1-\varepsilon}}\\ &\quad\lesssim T^\theta K^2(K+R).\end{aligned}$$

Estimating the other terms in the equation for Y in (2.1) using standard arguments and the regularities of the stochastic objects in Subsection 2.1, we get for $(X,Y)\in B_R$,

$$\|\Gamma^Y(X,Y)\|_{C_TC_x^{1+2\varepsilon}} \lesssim \|Y_0\|_{C_x^{1+2\varepsilon}} + T^\theta(K+R)^3.$$

By arguments similar to those for Γ^X, we obtain acceptable estimates for $\|\Gamma^Y(X,Y)\|_{C_T^{\frac{1}{8}}L^\infty_x}$ and some different estimates for $\Gamma(X_1,Y_1)-\Gamma(X_2,Y_2)$ where $\Gamma=(\Gamma^X,\Gamma^Y)$ and $(X_1,Y_1),(X_2,Y_2)\in B_R$. Choosing $R\sim\|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}}+\|Y_0\|_{C_x^{1+2\varepsilon}}$ and T small enough this shows that Γ is a contraction mapping from $Z(T)$ to itself.

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