Last time, we checked the regularity of the Wick power

$$:\Psi^k: = \lim_{N\to\infty} :\Psi_N^k: , \qquad \Psi_N = P_{\leq N}\Psi_{\text{wave}}$$

for the wave case.

- We only checked (**) for fixed t

but the other conditions in the regularity lemma.

(For the time difference part, simply use the mean value theorem to get $\langle n\rangle^\sigma |h|^\sigma$.

- $:\Psi^k: \in \mathcal{H}_k \Rightarrow$ By the Wiener chaos estimate, we get

$$P\left(\|:\Psi^k:\|_{L^q_T W_x^{-\varepsilon,\infty}} > \lambda \right) \leq C \exp\left(-c \frac{\lambda^{2/k}}{T^{1+2/qk}}\right)$$

$\forall$ $\varepsilon > 0$, finite $q \geq 1$, $T > 0$.

For $q = \infty$, we can use the Garsia-Rodemich-Rumsey ineq. ②

and get

see Friz-Victoir
GKO-Tolomeo
IMRN '21

$$P\left(\| :\Psi^k: \|_{L^\infty([j,j+1]; W_x^{-\varepsilon,\infty}(\mathbb{T}^2))} > \lambda \right)$$

$$\leq C \exp\left(-c \frac{\lambda^{2/k}}{j+1} \right)$$

- 2-d heat: $\quad \Psi = \Psi_{heat} = \int_0^t e^{(t-t')(\Delta-1)} dW(t')$.

We only check (**) in the regularity lemma. (drop N.)

$$\mathbb{E}\left[|\widehat{:\Psi^k:}(t,n)|^2 \right] = \iint_{\mathbb{T}_x^2 \times \mathbb{T}_y^2} \mathbb{E}\left[:\Psi^k(t,x)::\Psi^k(t,y): \right] e_n(y-x)\, dx\, dy$$

$$\mathbb{E}\left[:\Psi^k(t,x)::\Psi^k(t,y): \right] = k! \left\{ \mathbb{E}[\Psi(t,x)\Psi(t,y)] \right\}^k$$

$$\mathbb{E}[\Psi(t,x)\Psi(t,y)] = \sum_m \int_0^t e^{-2(t-t')\langle m \rangle^2} dt'\, e_m(x-y)$$

$$\int_0^t e^{-2(t-t')\langle m \rangle^2} dt' = \boxed{\frac{1 - e^{-2t\langle m \rangle^2}}{\langle m \rangle^2}}$$

$$\lesssim \sum_{n_1+\cdots+n_k=n} \prod_{j=1}^{k} \frac{1}{\langle n_j \rangle^2} \lesssim \langle n \rangle^{-2+\varepsilon}, \quad \forall \varepsilon > 0.$$

as in the wave case.

<u>Prop:</u> $\{ :\Psi_N^k: \}_{N \in \mathbb{N}}$ forms a Cauchy sequence in $L^p(\Omega; C_t C_x^{-\varepsilon}(\mathbb{T}^2))$ and in $C_t C_x^{-\varepsilon}(\mathbb{T}^2)$, a.s.

$C^{-\varepsilon} = B^{-\varepsilon}_{\infty,\infty}$.

· <u>Idea of the proof of the regularity lemma:</u>

Translation invariance (in x)

(+) $\Rightarrow \mathbb{E}\left[\widehat{X}(t,n) \widehat{X}(t,m) \right] = 0$ for $n+m \neq 0$.

(real-valued setting)

$$\mathbb{E}\left[\hat{X}(t,n)\, \hat{X}(t,m) \right]$$

$$= \iint \underbrace{\mathbb{E}\left[X(t,x)\, X(t,y) \right]}_{= F(t,\, x-y)} \underbrace{e_{-n}(x)\, e_{-m}(y)}_{= e_{-(n+m)}(x)\, e_m(x-y)}\, dx\, dy$$

$$= \hat{F}(t,m) \int_{\mathbb{T}^d} e^{-i(n+m)\cdot x}\, dx$$

$$= 0 \quad \text{for } n+m \neq 0.$$

Suppose (**) holds. (suppress t.) Then, for $p \gg 1$,

$$\big\| \|X\|_{W^{s,\infty}} \big\|_{L^p(\Omega)} \overset{\text{Sobolev}}{\lesssim} \big\| \|X\|_{W^{s+\varepsilon,r}} \big\|_{L^p} \qquad r \gg 1.$$

$$\overset{\text{Mink}}{\leq} \Big\| \big\| \langle \nabla \rangle^{s+\varepsilon} X(x) \big\|_{L^p(\Omega)} \Big\|_{L^r_x} \qquad p \geq r$$

Wiener ch. est.

$$\leq p^{k/2} \Big\| \underbrace{\Big\| \sum_n \langle n \rangle^{s+\varepsilon} \hat{X}(n)\, e_n(x) \Big\|_{L^2(\Omega)}}_{\overset{\oplus}{=} \left(\sum_m \langle m \rangle^{2(s+\varepsilon)} \mathbb{E}[|\hat{X}(m)|^2] \right)^{1/2}} \Big\|_{L^r_x}$$

$$\overset{(**)}{\lesssim} p^{k/2} \Big(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2(s+\varepsilon)-d-2s_0} \Big)^{1/2} < \infty$$ 5

for $s < s_0$ (by taking $0<\varepsilon \ll 1$ s.t. $s+\varepsilon < s_0$].

$\square$ (As for (ii), use Kolmogorov's conti criterion.)

(2.1) LWP of SNLW on $\mathbb{T}^2$.

Basic product estimates:

(i) (fractional Leibniz rule): $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j=1,2$

$0 \le s \le 1$

$$\| \langle \nabla\rangle^s (fg) \|_{L^r(\mathbb{T}^d)} \lesssim \Big(\|f\|_{L^{p_1}} \|\langle\nabla\rangle^s g\|_{L^{q_1}} + \|\langle\nabla\rangle^s f\|_{L^{p_2}} \|g\|_{L^{q_2}} \Big).$$

(ii) $1 < p, q, r < \infty$ $\quad \boxed{\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{s}{d}}$

$s > 0$

$$\| \langle\nabla\rangle^{-s} (fg) \|_{L^r(\mathbb{T}^d)} \lesssim \| \langle\nabla\rangle^{-s} f \|_{L^p} \, \| \langle\nabla\rangle^{s} g \|_{L^q}$$

(ii) See GKO. Here, f is merely a distribution but the product fg makes sense as long as the sum of the regularities ≥ 0.

(i) fractional Leibniz rule on $\mathbb{R}^d$ & transference principle. (see GKO).

↑ Coifman-Meyer thm.

Kato-Ponce inequality '88.

$$[D^s, f]g = D^s(fg) - (D^s f)g.$$

- Christ-Weinstein JFA '91.
  ($\Leftarrow$ vector-valued maximal function estt.) see Grafakos.

- Gatto JFA '02.

SNLW: $\begin{cases} (\partial_t^2 + 1 - \Delta) u + u^k = \xi & \text{on } \mathbb{T}^2. \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$

truncated noise $\xi_N = P_{\leq N} \xi$.

Consider

$$(\partial_t^2 + 1 - \Delta) u_N + u_N^k = \xi_N$$

no truncation here.

· Duhamel

$$\Rightarrow \quad u_N = \Psi_N + v_N$$

$$\Rightarrow \quad u_N^k = \sum_{j=0}^{k} \binom{k}{j} \Psi_N^j \, v^{k-j}$$

No limit for $j \geq 2$.
need to renormalize this.

Replace Ψ_N^j by $H_j(\Psi_N ; \sigma_N)$

$\sim \log N$.

$$H_k(x+y;\sigma) = \sum_{j=0}^{k} \binom{k}{j} H_j(x;\sigma)\, y^{k-j}.$$

- Define the Wick power $:u_N^k:$

$$:u_N^k: \;=\; \sum_{j=0}^{k} \binom{k}{j} \underbrace{:\Psi_N^j:}_{\text{limit exists a.s.}} \; \underbrace{v_N^{k-j}}_{\text{postulate that the remainder is smoother}}$$

- defined only for u_N of the form

$$u_N = \Psi_N + v_N$$

- v_N satisfies

$$(\partial_t^2 + 1 - \Delta) v_N + \sum_{j=0}^{k} \binom{k}{j} \underbrace{:\Psi_N^j:}_{\downarrow\; :\Psi^j:} \, v_N^{k-j} = 0.$$

By taking $N \to \infty$, we arrive at

$$(\mathrm{SNLW}_v) \quad \boxed{(\partial_t^2 + 1 - \Delta) v + \sum_{j=0}^{k} \binom{k}{j} :\Psi^j: \, v^{k-j} = 0.}$$

Then, the "solution" u to the renormailzed SNLW:

$$(*) \qquad (\partial_t^2 + 1 - \Delta) u + :u^k: = \xi$$

$(**)$ is given by $\boxed{u = \Psi + v}$

- Here, the eqn $(*)$ for u is just a formal expression. For example, when $k=3$, we have

$$(\partial_t^2 + 1 - \Delta) u + u^3 - 3 \cdot \underbrace{\infty}_{= \lim_{N\to\infty} \sigma_N} \cdot u = \xi$$

By $(*)$, we really mean (SNLW_v) with the first order decomposition $(**)$.