Lec 8 02/04/21 (Fri) ①

law of u

$$u(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x} \qquad \mathcal{L}(u) = \mu_1 = \underbrace{\text{GFF}}_{\text{massive}} \text{ on } \mathbb{T}^2$$

$\{g_n\}_{n\in\mathbb{Z}^2}$ indep std $\mathbb{C}$-valued Gaussian r.v.'s

(In the real-valued setting, impose $g_{-n} = \overline{g_n}$

· Last time: $u \in H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, a.s. $\boxed{s<0}$

What about $L^p(\mathbb{T}^2)$, $p<2$?

$$L^p(\mathbb{T}^2) \supset L^2(\mathbb{T}^2)$$

· $p=1$:

$$u \notin L^1(\mathbb{T}^2), \text{ a.s.}$$

Let $X_n = \frac{g_n}{\langle n \rangle} e^{in\cdot x}$ (for fixed $x \in \mathbb{T}^2$) ②

and $Y_n = X_n \cdot \mathbb{1}_{\{|X_n| \le A\}}$ (for some $A > 0$)

· Kolmogorov three series thm: X_n, indep

Then, $\sum X_n$ converges a.s. iff the following 3 cond. hold: for ANY $A > 0$

(i) $\sum P(|X_n| \ge A)$ converges

(ii) With $Y_n = X_n \cdot \mathbb{1}_{\{|X_n| \le A\}}$,

$\sum \mathbb{E}[Y_n]$ converges.

(iii) $\sum \mathrm{var}(Y_n)$ converges.

For our problem, $\sum \mathrm{var}(Y_n) \ge \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^2} \mathbb{E}\left[|g_n|^2 \cdot \mathbb{1}_{\{|g_n| \le A\}}\right]$ $\{|g_n| \le A\langle n \rangle\}$

$= \infty$

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$\Rightarrow$ By Kolmog. 3-series thm, $\sum X_n$ diverges on a set of positive prob.

$\Rightarrow$ By Kolmog 0-1 law, $\sum X_m(x)$ diverges a.s.

Compute

$$\mathbb{E}\left[\int_{\mathbb{T}^2} |u(x)|\, dx\right] = \int_{\mathbb{T}^2} \overbrace{\mathbb{E}[|u(x)|]}^{=\infty} dx = \infty$$

$\Rightarrow$ u_N does not converge in L^1 with positive prob.

$\Rightarrow$ u_N diverges in $L^1(\mathbb{T}^2)$, a.s.

Back to $u(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in\cdot x}$.

$$\mathbb{E}\big[u_N(x) \overline{u_N(y)} \big] = \sum_{|n| \le N} \frac{1}{\langle n \rangle^2} e_n(x-y)$$

insert $\langle \nabla_x \rangle^{-\varepsilon}$ $\langle \nabla_y \rangle^{-\varepsilon}$

and set $x = y$.

$$\Rightarrow \underline{\mathbb{E}\big[|\langle \nabla \rangle^{-\varepsilon} u_N(x)|^2 \big]}$$

$$\lesssim \sum_{\substack{|n| \le N \\ n \in \mathbb{Z}^2}} \frac{1}{\langle n \rangle^{2+2\varepsilon}} < \infty \quad \text{unif in } N \in \mathbb{N}.$$

Note:

$\mathbb{E}[g_n^k \overline{g_n}^\ell] = k!\, \delta_{k\ell}$

In particular,

$\mathbb{E}[g_n^2] = 0.$

$$\underset{r<\infty}{\Rightarrow} \Big\| \, \|u_N\|_{W^{s,r}(\mathbb{T}^2)} \Big\|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \Big\| \underbrace{\| \langle \nabla \rangle^s u_N(x) \|_{L^p(\Omega)}}_{\le p^{1/2} \| \langle \nabla \rangle^s u_N(x) \|_{L^2(\Omega)}} \Big\|_{L^r_x(\mathbb{T}^2)}$$

$$\lesssim p^{1/2}, \quad \text{unif in } N \in \mathbb{N}$$

for $s < 0$

A similar computation for $u_N - u_M$ holds

$\Rightarrow$ $\{u_N\}_{N\in\mathbb{N}}$ converges in $L^p(\Omega; W_x^{s,r}(\mathbb{T}^2))$

$s<0$ and $r\le\infty$.

Also, we have an exponential tail estimate:

$$P(\|u\|_{W_x^{s,r}} > \lambda) \le C e^{-c\lambda^2}, \quad \forall \lambda>0$$

Since u is NOT a function and is only a distribution, the power u^k is ill-defined.

$\Rightarrow$ We need to introduce a renormalization.

In the following, we restrict our attention to the real-valued setting.

Hermite polynomial:

generating function: $e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$

- $H_k(x;\sigma)$ = Hermite poly of deg k.

  $H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x, \quad H_2(x;\sigma) = x^2 - \sigma$

  $H_3(x;\sigma) = x^3 - 3\sigma x, \quad H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2, \dots$

- Set $H_k(x) = H_k(x;1)$

orthogonality: $\int H_k(x) H_m(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = k!\, \delta_{km}$

FACT: $\left\{ \frac{H_k(x)}{\sqrt{k!}} \right\}_{k \in \mathbb{Z}_{\geq 0}}$ forms an O.N.B. of $L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$

Note: $\{x^k\}_{k \in \mathbb{Z}_{\geq 0}} \xRightarrow{\text{Gram-Schmidt in } L^2(\mathbb{R};\text{Gauss})} \{H_k(x)\}_{k \in \mathbb{Z}_{\geq 0}}$

We also have the following orthogonality:

f, g. jointly Gaussian

f, g, Gaussian r.v.'s (mean 0) with var σ_f and σ_g

(*)
$$\boxed{\mathbb{E}\left[H_k(f;\sigma_f)\, H_m(g;\sigma_g)\right] = \delta_{km}\, k!\, \left\{\mathbb{E}[fg]\right\}^k}$$

$\Leftarrow$ Take the product of the generating functions for (f, σ_f) and (g, σ_g). and express in two different series and compare the coeff of t^k.

- In the complex-valued setting, we instead use the (generalized) Laguerre polynomials: See Oh-Thomann SPDE '18.

• Wick renormalization

Wick power, renormalized power

$$: u_N^k(x) : \; = \; H_k(u_N(x); \sigma_N) \qquad \text{pointwise}$$

where $\sigma_N = \mathbb{E}[u_N^2(x)] = \sum_{|n| \le N} \frac{1}{\langle n \rangle^2} \sim \log N \to \infty$.

(On a mfd, σ_N depends on $x \in \mathcal{M}$.

• $k = 2$. $\quad : u_N^2 : \; = \; u_N^2 - \sigma_N$ $\quad \infty$

$k = 3$ $\quad : u_N^3 : \; = \; u_N^3 - 3 \sigma_N u_N$

• Wick renormalization = orthogonal projection onto the homog Wiener chaos of deg k

- (H, B, μ), abstract Wiener space

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du$$

$$\left(\text{Think of} \quad \begin{array}{l} H = H^1(\mathbb{T}^2) \\ B = H^{-\epsilon}(\mathbb{T}^2) \end{array} \right.$$

- $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ complete orthonormal system of $H^* = H$.

- Consider $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j\rangle)$, $k_j \neq 0$ only for finitely many j's

$\quad\quad \shortparallel$

product of finitely many Hermite poly in different directions.

$\langle \cdot, \cdot \rangle = B - B^*$ dual pairing.

$\mathcal{H}_k$ = homog Wiener chaoses of deg k

$$= \overline{\operatorname{span}\left\{ \prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j\rangle) : \sum_j k_j = k \right\}}^{\|\cdot\|_{L^2(B,\mu)}}$$

Wiener - Ito decomposition:

$$L^2(B,\mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

We set $\quad \mathcal{H}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$

Lemma (Wiener chaos estimate)

Let $k \in \mathbb{N}$. Then, we have

$$\boxed{\| X \|_{L^p(\Omega)} \leq (p-1)^{k/2} \| X \|_{L^2(\Omega)}}$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$

$\Leftarrow$ follows from the hypercontractivity of the OU semigroup

Nelson '65. $\quad e^{tL} \qquad L = \Delta - x \cdot \nabla \qquad (B = \mathbb{R}^d)$

elements in $\mathcal{H}_k$ = eigenfunction of L with e-val $-k$.

Prop: $\{ :u_N^k: \}_{N \in \mathbb{N}}$ forms a Cauchy seq

in $L^p(\Omega; W^{s,r}(\mathbb{T}^2))$, $s<0$, $r \le \infty$.

Denoting the limit by $:u^k:$, we have

$$:u^k: \in W^{s,r}(\mathbb{T}^2), \quad \text{a.s.} \qquad s<0, \ r \le \infty$$

Pf: $\mathbb{E}[:u_N^k(x): \ :u_N^k(y):]$

$$= k! \{ \mathbb{E}[u_N(x) u_N(y)] \}^k$$

$$= k! \sum_{\substack{|n_j| \le N \\ j=1,\dots,k}} \frac{1}{\langle n_1 \rangle^2} e_{n_1}(x-y) \cdots \frac{1}{\langle n_k \rangle^2} e_{n_k}(x-y)$$

$$= k! \sum_{|n_j| \le N} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2} \, e_{n_1 + \cdots + n_k}(x-y)$$

Insert $\langle \nabla_x \rangle^{-\varepsilon}$ and $\langle \nabla_y \rangle^{-\varepsilon}$ and set $x = y$.

$$\Rightarrow \mathbb{E}\left[|\langle \nabla \rangle^{-\varepsilon} : u_N^k(x) : |^2 \right]$$

$$= k! \sum_{|n_j| \le N} \prod_{j=1}^{k} \frac{1}{\langle n_j \rangle^2} \cdot \frac{1}{\langle n_1 + \cdots + n_k \rangle^{2\varepsilon}}$$

$$\lesssim_{\varepsilon, k} 1, \quad \text{unif in } N \in \mathbb{N}.$$

$$\underset{r<\infty}{\Rightarrow} \| \, \| : u_N^k : \|_{W_x^{s,r}} \|_{L^p(\Omega)} \underset{p \ge r}{\overset{\text{Mink}}{\le}} \| \underbrace{\| \langle \nabla \rangle^s : u_N^k(x) : \|_{L^p(\Omega)}}_{\le p^{k/2} \| \cdots \|_{L^2(\Omega)}} \|_{L_x^r}$$

$$\lesssim p^{k/2} \quad \underline{\text{for } s<0}$$

$\uparrow$ Wiener chaos estimate.

- A similar computation holds for $:u_N^k: - :u_M^k:$, $N \ge M \ge 1$.
In this case, we use $\max(|n_j|) \ge M \Rightarrow \frac{1}{\langle n_j \rangle^2} \le \frac{1}{M^\delta} \frac{1}{\langle n_j \rangle^{2-\delta}}$

$\boxtimes$

Note:

$$P\left(\| :u^k: \|_{W^{s,r}} > \lambda \right) \le C e^{-c\lambda^{2/k}} \quad \text{for any } \lambda > 0$$

($\Leftarrow$ Chebyshev $\qquad \Updownarrow$

$$\int e^{c \| :u^k: \|_{W^{s,r}}^{2/k}} d\mu_1 < \infty$$

Tzvetkov Lem 4.5 in PTRF '10

Lemma: Let $\{X_N\}_{N \in \mathbb{N}}$ and X be spatially homogenous stoch processes : $\mathbb{R} \to \mathcal{D}'(\mathbb{T}^d)$

i.e. for any $x_0 \in \mathbb{T}^d$,

$$\{X(t,\cdot)\}_{t \in \mathbb{R}} \text{ and } \{X(t, \cdot + x_0)\}_{t \in \mathbb{R}}$$

have the same law.

· Suppose that $X_N(t)$, $X(t) \in \mathcal{H}_{\le k}$, $\forall t \in \mathbb{R}$

(i) Fix $t \in \mathbb{R}$. If $\exists s_0 \in \mathbb{R}$ s.t.

$$(**) \quad \mathbb{E}\left[|\hat{X}(t,n)|^2 \right] \lesssim \boxed{\langle n \rangle^{-d-2s_0}}, \quad \forall n \in \mathbb{Z}^d.$$

then, $\quad X(t) \in W^{s,\infty}(\mathbb{T}^d), \quad s < s_0, \quad$ a.s.

Moreover, if $\exists \theta > 0$ s.t.

$$\mathbb{E}\left[|\hat{X}_N(t,n) - \hat{X}(t,n)|^2 \right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0}, \quad \forall n \in \mathbb{Z}^d$$

then, $\quad X_N \to X$ in $W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, a.s.

(ii) Fix $T > 0$. Suppose (i) holds on $[-T,T]$.

- If $\exists \sigma \in (0,1)$ s.t.

$$\mathbb{E}\left[|\delta_h \hat{X}(t,n)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma \quad \forall n \in \mathbb{Z}^d, \ \forall |h| \le 1, \ \forall t \in [-T,T]$$

diff op: $\delta_h f(t) = f(t+h) - f(t)$.

then $X \in C([-T,T]; W^{s,\infty}(\mathbb{T}^d))$

for $s < s_0 - \sigma/2$, a.s.

Furthermore, if $\exists\, \theta > 0$ s.t.

$$\mathbb{E}\left[|\delta_h \widehat{X}_N(t,n) - \delta_h \widehat{X}(t,n)|^2 \right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0+\sigma} |h|^{\sigma}$$

$\forall\, n \in \mathbb{Z}^d$, $|h| \le 1$, $t \in [-T,T]$

then,

X_N converges to X in $C([-T,T]; W^{s,\infty}(\mathbb{T}^d))$

for $s < s_0 - \frac{\sigma}{2}$, a.s.

<u>Note:</u> $W^{s,\infty}(\mathbb{T}^d)$ can be replaced by $C^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$

Mourrat-Weber-Xu '17. Oh-Okamoto-Tzvetkov

· <u>2-d SNLW</u>:

$$\Psi(t,x) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} dW(t') \in C_t W_x^{s,\infty}(\mathbb{T}^2), \text{ a.s.}$$

$s<0$.

$\boxed{\Psi_N = P_{\le N}\Psi}$ We set

$$:\Psi_N^k(t,x): = H_k(\Psi_N(t,x);\sigma_N(t))$$

where $\sigma_N(t) = \mathbb{E}[|\Psi_N(t,x)|^2] \sim t \log N$

Gubinelli-Koch-Oh TAMS'18

<u>Prop</u>: $\{:\Psi_N^k:\}_{N\in\mathbb{N}}$ forms a Cauchy seq in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$, $p<\infty$.

and also almost surely.

Pf: We only verify (**) in Lemma
regularity lemma

$$\mathbb{E}\left[\,\left|\widehat{:\Psi_N^k:}(t,n)\right|^2\,\right]$$

$$= \iint_{\mathbb{T}^2_x\times\mathbb{T}^2_y} \underbrace{\mathbb{E}\left[:\Psi_N^k(t,x):\;:\Psi_N^k(t,y):\right]}_{} e_n(y-x)\,dx\,dy$$

$$\overset{\text{Lemma}}{=} k!\,\Big\{\underbrace{\mathbb{E}\left[\Psi_N(t,x)\Psi_N(t,y)\right]}_{=\sum_{|m|\le N}\int_0^t\left(\frac{\sin(t-t')\langle m\rangle}{\langle m\rangle}\right)^2 dt'\, e_m(x-y)}\Big\}^k$$

$$\lesssim \sum_{n_1+\cdots+n_k=n}\prod_{j=1}^{k}\frac{1}{\langle n_j\rangle^2} \lesssim \langle n\rangle^{-2+\varepsilon} \quad \left(-2+\varepsilon = -2s_0\right)$$

$$\Rightarrow \Psi_N(t)\in W^{s,\infty}(\mathbb{T}^d),\ s<0,\ \text{a.s.}$$

$\boxtimes$

A word on the summation:

$$\sum_{\substack{n_1+\cdots+n_k=n \\ n_j\in\mathbb{Z}^2}} \prod_{j=1}^{k} \frac{1}{\langle n_j\rangle^2}$$

$k-1$ summations

- Divide the sum into regions I_ℓ

$$I_\ell = \{ |n_\ell| \gtrsim \max_{j=1,\cdots,k} |n_j| \}$$

- Under $n_1+\cdots+n_k = n$, we have $\max_{j=1,\cdots,k} |n_j| \gtrsim |n|$

$$\sum_{I_\ell} \prod_{j=1}^{k} \frac{1}{\langle n_j\rangle^2} \lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}} \sum_{I_\ell} \prod_{\substack{j=1\\ j\neq \ell}}^{k} \frac{1}{\langle n_j\rangle^{2+\frac{\varepsilon}{k-1}}} \lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}}.$$

- One can also use the following lemma (Part (i)) iteratively.

$\boxed{\text{Lemma}}$: (i) $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha+\beta > d$ and $\alpha, \beta < d$

Then,
$$\sum_{\substack{n=n_1+n_2\\ n_j\in\mathbb{Z}^d}} \frac{1}{\langle n_1\rangle^\alpha} \frac{1}{\langle n_2\rangle^\beta} \lesssim \langle n\rangle^{d-\alpha-\beta}$$

resonant case

(ii) $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha + \beta > d$.

Then,
$$\sum_{\substack{n = n_1 + n_2 \\ |n_1| \sim |n_2| \\ n_j \in \mathbb{Z}^d}} \frac{1}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\beta}} \lesssim \langle n \rangle^{d-\alpha-\beta}.$$

This lemma follows from elementary computation.

See Mourrat - Weber - Xu '17 (Also, see Ginibre - Tsutsumi - Velo JFA '97 Lemma 4.2.