Lec 7 : 31 / 03 / 21 (Wed)

Again back to ① Ito approach:

Fix $R > 0$. Set

$$\tau_R = \inf\{ t > 0 : \| u(t) \|_{L^2} \geq R \}.$$

$\Leftarrow$ Check τ_R is a stopping time.

Remark: on the construction of global-in-time solns (i.e. a.s. GWP).

Moral: $\boxed{\text{almost a.s. GWP} \Longrightarrow \text{a.s. GWP.}}$

unrelated Given $T > 0$, $\varepsilon > 0$, $\exists\ \Omega_{T,\varepsilon}$ s.t. if $\omega \in \Omega_{T,\varepsilon}$, then the corresponding soln u^ω exists on $[-T, T]$. and $P(\Omega^c_{T,\varepsilon}) < \varepsilon$

① Fix $\varepsilon > 0$ and set

$$\Omega_j = \Omega_{2^j, \varepsilon/2^j}$$

② Set $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_j \quad \Rightarrow P(\Omega_\varepsilon^c) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

<u>Note</u>: if $\omega \in \Omega_\varepsilon$, then u^ω exists globally in time.

③ Now, set

$$\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon \quad \Rightarrow P(\Sigma^c) = \inf_{\varepsilon > 0} P(\Omega_\varepsilon^c) = 0.$$

$\Rightarrow$ a.s. GWP. (or use Borel-Cantelli lemma).

Stochastic damped NLW:

$$\partial_t^2 u + \partial_t u + (1-\Delta) u + u^k = \sqrt{2}\,\xi, \qquad k \in 2\mathbb{N}+1$$

Let $v = \partial_t u$. Then,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} v \\ (-1+\Delta) u - u^k \end{pmatrix}}_{\text{NLW}} + \underbrace{\begin{pmatrix} 0 \\ -v + \sqrt{2}\,\xi \end{pmatrix}}_{\text{Ornstein-Uhlenbeck process.}}$$

Gibbs meas: $d\rho = Z^{-1} \exp\left(-\frac{1}{k+1}\int u^{k+1}\,dx\right) d\mu_1 \otimes d\mu_0(u,v)$

white noise for v.

Note: NLW preserves ρ.

OU on v preserves $\mu_0(dv) \Rightarrow$ preserves ρ.

· Ornstein-Uhlenbeck process (real-valued)

$$\begin{cases} dX = -aX\,dt + b\,dB \\ X(0) = x_0 \end{cases} \qquad a>0,\ b\in\mathbb{R}$$

$$\Rightarrow \quad X(t) = e^{-at}x_0 + b\int_0^t e^{-a(t-t')}\,dB(t')$$

Suppose $x_0 \sim N(0, \frac{b^2}{2a})$, indep from B.

Then, $X(t)$ is a mean 0 Gaussian r.v. with var

$$\mathbb{E}[X^2(t)] = e^{-2at}\,\mathbb{E}[x_0^2] + b^2\int_0^t e^{-2a(t-t')}\,dt'$$

$$= \cancel{e^{-2at}\frac{b^2}{2a}} + \frac{b^2}{2a}(1-\cancel{e^{-2at}})$$

$$= b^2/2a.$$

i.e. $N(0, \frac{b^2}{2a})$ is invariant under OU.

Back to SdNLW: Look at the OU part (for v).

$$(*) \qquad \partial_t v = -v + \sqrt{2}\,\xi$$

$$\overset{\text{F.T. in } x}{\Longrightarrow} \quad \partial_t \widehat{v}(n) = -\widehat{v}(n) + \sqrt{2}\, d\beta_n.$$

[Further separate into $\operatorname{Re}\widehat{v}(n)$, $\operatorname{Im}\widehat{v}(n)$]

$\Rightarrow$ $g_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ is invariant for

$\Rightarrow$ $(*)$ preserves $\mu_0(dv)$ and hence $\rho(du, dv)$

- How to prove a.s. GWP & invariance of ρ under SdNLW?

- Consider the truncated dynamics.

(SdNLW$_N$) $$\partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N + P_{\le N}\big((P_{\le N} u)^k\big) = \sqrt{2}\,\xi.$$

Note: For high frequencies ($|n| > N$), it is

(**) $$\partial_t^2 P_{>N} u_N + \partial_t P_{>N} u_N + (1-\Delta) P_{>N} u_N = \sqrt{2}\, P_{>N}\xi.$$

$\mu_{1,N}^{\perp} \otimes \mu_{0,N}^{\perp}$ is invariant.

$\mu_{j,N}^{\perp}$ = marginal measure of μ_j on $E_N^{\perp} = \operatorname{span}\{e_n : |n| > N\}$.

and truncated Gibbs measure

marginal on E_N

$$d\rho_N = Z_N^{-1} \exp\Big(-\frac{1}{k+1}\int (\underline{P_{\le N} u})^{k+1} dx\Big)\, d\mu_{1,N} \otimes d\mu_{0,N}\big(P_{\le N}(u_N, \partial_t u_N)\big)$$

$$\otimes \underbrace{\Big(d\mu_{1,N}^{\perp} \otimes d\mu_{0,N}^{\perp}\big(P_{>N}(u_N, \partial_t u_N)\big)\Big)}_{\text{invariant under (**)}}.$$

ρ_N invariant under SdNLW$_N$

$\Longleftrightarrow$ $\mathcal{L}^N$ = generator for the Markov semigp for SdNLW$_N$

Then, $\underline{(\mathcal{L}^N)^* \rho_N = 0}.$

adjoint

$\downarrow$

phase space $\vec{u} = (u, v)$

$\partial_t u$

$\underline{P_t} F(\vec{u}) = \mathbb{E}\big[F(\underset{\uparrow}{\Phi_t} \vec{u})\big]$

soln map

generator = time deriv at $t = 0$.

$\mathcal{L}^N = \mathcal{L}^N_1 + \mathcal{L}^N_2$

$\mathcal{L}^N_1$ = generator for ⋯ for NLW$_N$

$\mathcal{L}^N_2$ = generator for ⋯ for OU process (for v).

(***) $$\boxed{(\mathcal{L}^N)^* \rho_N = (\mathcal{L}^N_1)^* \rho_N + (\mathcal{L}^N_2)^* \rho_N = 0 + 0 = 0.}$$

$\Rightarrow$ ρ_N is invariant under SdNLW$_N$.

⇒ Repeat Bourgain's invariant meas argument.

⇒ a.s GWP of SdNLW & invariance of ρ under SdNLW.

(For $(***)$, also think of Lie-Trotter product formula: $e^{(A+B)t} = \lim_{n\to\infty}\left(e^{A\frac{t}{n}}e^{B\frac{t}{n}}\right)^m$

- Parabolic Φ^{k+1}_1-model:

$$\partial_t u + \underbrace{(1-\Delta)u - u^k}_{\uparrow} = \sqrt{2}\,\xi.$$

Consider $du = -\gamma\nabla_u H(u)\,dt + \sqrt{2}\,\gamma\,dW.$

$$\mathcal{L}(F(u)) = \gamma\,\mathrm{Tr}\,D^2F(u) - \gamma\langle\nabla F(u), \nabla H(u)\rangle$$

$$\begin{aligned}\mathcal{L}^*(\rho) = 0 \iff & \int \mathcal{L}F(u)\,\underline{\underline{\rho}}(du)\\ &= \gamma\int \mathrm{Tr}\,D^2F(u)\,\underline{e^{-H(u)}\,du} + \gamma\int\langle\nabla F(u), \underline{\nabla(e^{-H(u)})}\rangle\,du\\ &= 0 \text{ by IBP.}\end{aligned}$$

$$\partial_t u = (\Delta - 1) u - u^k + \sqrt{2}\,\xi.$$

$$\hat{u}_n = a_n + i b_n. \qquad \mathcal{N}(u) = u^k.$$

$$d a_n = \Big(\underline{-\langle n\rangle^2 a_n} \; \underline{- \mathrm{Re}(\hat{\mathcal{N}}(u)(n))}\Big)\, dt + \sqrt{2}\, d(\mathrm{Re}\,\beta_n)$$

When $n=0$, $b_0 = 0$.

$$d b_n = \Big(\underline{-\langle n\rangle^2 b_n} \; \underline{- \mathrm{Im}(\hat{\mathcal{N}}(u)(n))}\, dt + \sqrt{2}\, d(\mathrm{Im}\,\beta_n)$$

(should consider the truncated dynamics..)

from the noise

$$L_1^N F(\bar{a}, \bar{b}) = \sum_n \Big(\underline{-\langle n\rangle^2 a_n \partial_{a_n}} + \underline{\tfrac{1}{2}}\, \partial_{a_n}^2\Big) F + \sum_n \Big(\underline{-\langle n\rangle^2 b_n \partial_{b_n}} + \tfrac{1}{2}\, \partial_{b_n}^2\Big) F$$

when $n=0$ $\partial_{a_0}^2 F$

$$L_2^N F(\bar{a}, \bar{b}) = \sum_n \underline{\partial_{a_n} F \,\big(-\mathrm{Re}(\hat{\mathcal{N}}(u))(n)\big)} + \sum_n \underline{\partial_{b_n} F \,\big(-\mathrm{Im}(\hat{\mathcal{N}}(u))(n)\big)}.$$

- Linear case: $\mathcal{N}(u) = 0$

$$d\mu_N = Z_N^{-1} e^{-\frac{1}{2}\|P_{\le N} u\|_{H^1}^2} \, dP_{\le N} u.$$

$$(\mathcal{L}_1^N)^* \mu_N = 0$$

$$\iff \int \mathcal{L}_1^N F(\bar{a}, \bar{b}) \, \underset{\overset{\shortparallel}{Z_N^{-1} e^{-\frac{1}{2}\sum \langle n\rangle^2 a_n^2 + \langle n\rangle^2 b_n^2} \prod da_n db_n}}{d\mu_N} = 0 \; ?$$

$$\int \left(-\langle n\rangle^2 a_n \partial_{a_n} F + \frac{1}{2}\partial_{a_n}^2 F\right) e^{-\Box} \, da_n$$

$$\overset{IBP}{=} \int \underbrace{\langle n\rangle^2 F \partial_{a_n}(a_n e^{-\Box})}_{\cancel{\langle n\rangle^2 F e^{-\Box}} - \cancel{2\langle n\rangle^2 F \langle n\rangle^2 a_n^2 e^{-\Box}}} + \underbrace{\frac{1}{2} F \partial_{a_n}^2 e^{-\Box}} \, da_n = 0.$$

$a_{-n} = a_n$

$$F \partial_{a_n}(-\langle n\rangle^2 a_n e^{-\Box})$$

$$= \cancel{-F\langle n\rangle^2 e^{-\Box}} + \cancel{2F\langle n\rangle^4 a_n^2} e^{-\Box}$$

similar comp holds for the b_n-part.

$$\Rightarrow \quad (\mathcal{L}_1^N)^* \mu_N = 0.$$

Gibbs measure invariance (truncated)

$$\mathcal{L}^N = \mathcal{L}_1^N + \mathcal{L}_2^N$$

$$(\mathcal{L}^N)^* \rho_N = 0 \iff \int \mathcal{L}^N F(\bar{a},\bar{b})\, d\rho_N(\bar{a},\bar{b}) = 0 \ ?$$

$$\swarrow$$

$$\sim \underbrace{e^{-\frac{1}{k+1}\int (P_{\le N} u)^{k+1}}}_{\overset{\shortparallel}{e^{-M(\bar{a},\bar{b})}}} \quad d\mu_N$$

$$\wr$$

$$e^{-\frac{1}{2}\sum \langle n\rangle^2 a_n^2 + \langle n\rangle^2 b_n^2}$$

$$\prod da_n\, db_n$$

$$\partial_{a_n} M(\bar{a},\bar{b}) = \partial_{a_n} \frac{1}{k+1} \sum_{n_1+\cdots+n_{k+1}=0} \hat{u}_{n_1} \cdots \hat{u}_{n_{k+1}}$$

$$= \underbrace{\partial_{\hat{u}_n} M}_{\substack{\downarrow \\ \underline{\sum_{n_1+\cdots+n_k=-n} \hat{u}_{n_1}\cdots\hat{u}_{n_k}}}} \cdot \frac{\partial \hat{u}_n}{\partial a_n} + \partial_{\hat{u}_{-n}} M \cdot \frac{\partial \hat{u}_{-n}}{\partial a_n}$$

$$= \widehat{N}(u)(-n) + \widehat{N}(u)(n) = 2\,\mathrm{Re}\,\widehat{N}(u)(n)$$

Similarly,

$$\partial_{b_n} M(\bar{a},\bar{b}) = i\,\widehat{N}(u)(-n) - i\,\widehat{N}(u)(n)$$

$$= 2\,\mathrm{Im}\,\widehat{N}(u)(n)$$

$$(\mathcal{L}_1^N + \mathcal{L}_2^N)\, F(\bar a, \bar b)$$

$\mathcal{L}_2^N$

includes M.

$$\int \underbrace{\left(-\langle n\rangle^2 a_n - \mathrm{Re}\, \widehat{\mathcal{N}(u)}(n)\right)}_{= \frac{1}{2}\partial_{a_n} \square} \underline{\partial_{a_n} F \; e^{-\square}} \; d\bar a\, d\bar b$$

$$\frac{1}{2}\partial_{a_n}^2 F\, e^{-\square} \underset{\text{IBP}}{=} \underline{-\frac{1}{2}\partial_{a_n} F\, \partial_{a_n} \square\, e^{-\square}}$$

2nd term in $\mathcal{L}_1^N$

$$\Longrightarrow \quad (\mathcal{L}^N)^* \rho_N = 0.$$

Chap 2 : 2-d case

Consider $u = \underline{\sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in \cdot x}}$ $\overset{\text{GFF}}{\sim} d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}^2} du.$

$$u_N = \sum_{|n| \leq N} \frac{g_n}{\langle n \rangle} e^{in \cdot x}$$

For fixed $x \in \mathbb{T}^2$, $u_N(x)$ is a mean 0 Gaussian r.v.

with var $\sigma_N = \mathbb{E}\left[|u_N(x)|^2\right] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \sim \log N \to \infty.$

⇒ The limit u is NOT a function but a distribution.

$$\mathbb{E}\left[\|u\|_{H^s}^2\right] = \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^{2(1-s)}} < \infty \quad \text{iff} \quad \underline{s < 0}$$

For $s \geq 0$, $u \notin H^s(\mathbb{T}^2)$ with positive probability.

$\Rightarrow$ By Kolmogorov's 0-1 law,

$u \notin L^2(\mathbb{T}^2)$, a.s.

In particular, the power u^k is ill-defined and thus we need to introduce a <u>renormalization</u>

- A word on the generator L of a Markov semigroup

Given a stochastic process X, satisfying

$$dX = f\,dt + g\,dB, \quad X(0) = x_0,$$

its Markov semigroup P_t is given by

$$P_t F(x_0) = \mathbb{E}[F(X_t)].$$

The generator L is then given by

$$\boxed{LF(x_0) = \lim_{t\to 0} \frac{P_t F(x_0) - F(x_0)}{t} = \lim_{t\to 0} \frac{\mathbb{E}[F(X_t)] - F(x_0)}{t}.}$$

- By Ito's lemma, we have

vanishes under expectation

$$F(X_t) - F(x_0) = \int_0^t F'(X_s) f(s)\,ds + \cancel{\int_0^t F'(X_s)\,g(s)\,dB(s)} + \frac{1}{2}\int_0^t F''(X_s)\,g^2(s)\,ds$$

$$\Rightarrow LF(x_0) = F'(x_0) f(0) + \frac{1}{2} F''(x_0)\, g^2(0)$$

ex) $X = B \Rightarrow L = \frac{1}{2}\partial_x^2$.