Lec 6 : 24 / 03 / 21 (Wed)

Back to ① Ito approach

$$i\partial_t u - \partial_x^2 u + |u|^2 u = \phi \xi. \quad \text{on } \mathbb{T}$$

$P_{\le N} = \underset{\text{spatial}}{\wedge}$ freq cutoff onto freq $\{|n| \le N\}$

(sharp cutoff, smooth cutoff)

We consider the finite dim'l approximation:

$$i\partial_t u_N - \partial_x^2 u_N + P_{\le N}\left(|u_N|^2 u_N\right) = \phi_N \xi$$

- $u_N = P_{\le N} u_N$, $\phi_N = P_{\le N} \circ \phi$

$\Rightarrow$ Write on the Fourier side

$$i\, d\hat{u}_N(n) = \left(-n^2 \hat{u}_N(n) + \widehat{P_{\le N}(|u_N|^2 u_N)}(n)\right) dt + \hat{\Phi}_N(n)\, d\beta_n.$$

$|n| \le N$

$\Leftarrow$ a finite dim'l system of SPDEs for $(p_n, q_n)_{|n| \le N}$

$\Rightarrow$ Apply Ito's lemma. (should check u_N is adapted.)

· Since we do not know if u_N exists globally in time, we need to use a stopping time argument.

Fix a target time $T \gg 1$.

Then, let τ be a stopping time s.t.

$$0 < \tau \le \min(T, T_{max}), \quad \text{a.s.}$$

where $T_{max} = T_{max}(\omega)$ is the maximal time of existence.

$\Rightarrow$ insert $\mathbb{1}_{[0,\tau]}(t)$ to the $d\hat{u}_N(n)$ equation and apply Ito's lemma.

$\Big($ · On $\mathbb{R}^d$, also insert a cutoff in size on the nonlinearity ③

$$\theta\left(\frac{\|u\|}{R}\right).$$

· By the LWP argument, we have

$$\|u_N - u\|_{X^{0,3/8}_{\text{local in time}} \cap C_{T_{\text{local}}} L^2_x} \longrightarrow 0$$

$$T_{\text{local}} = T_{\text{local}}(\omega) = \text{local existence time}.$$

$\Rightarrow$ verify Ito's lemma for u (with a stopping time.)

(de Bouard – Debussche '03
Oh - Okamoto '20

Back to (2): 3-d defocusing cubic SNLW on $\mathbb{T}^3$

$$\partial_t^2 u + (1-\Delta) u + u^3 = \phi \xi$$

$$\boxed{\phi \in HS(L^2; H^{s-1}) \quad s>0}$$

Write $u = \Psi + v$.

$$\Rightarrow \boxed{\partial_t^2 v + (1-\Delta) v + \underbrace{(v+\Psi)^3}_{= v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3} = 0.} \qquad \Psi \in L^\infty_T L^\infty_x$$

· If $\Psi \equiv 0$, then the energy for $(v, \partial_t v)$

$$E(v, \partial_t v) = \frac{1}{2}\int |\langle\nabla\rangle v|^2 dx + \frac{1}{2}\int (\partial_t v)^2 dx + \frac{1}{4}\int v^4 dx$$

is conserved.

$\partial_t E(v, \partial_t v)$ Fix good ω.

$$= \int \partial_t v \, (\underbrace{\partial_t^2 v + (1-\Delta) v}_{\text{use eqn}} + \cancel{v^3}) \, dx$$

$$= -\cancel{v^3} - 3v^2\Psi - 3v\Psi^2 - \Psi^3$$

$$\leq C \, \|\Psi\|_{L^\infty_T L^\infty_x} \underbrace{\left(\int (\partial_t v)^2 dx\right)^{1/2} \left(\int v^4 dx\right)^{1/2}}_{\lesssim E(v, \partial_t v)}$$

$$+ C \, \|\Psi\|^{1/2}_{L^\infty_T L^6_x} \left(\int (\partial_t v)^2\right)^{1/2}$$

$$\leq C(\Psi, T) \, (1 + E(v, \partial_t v)).$$

$\Rightarrow$ Apply Gronwall's inequality.

$$\Rightarrow \sup_{t \in [0,T]} \|(v(t), \partial_t v(t))\|_{\mathcal{H}^1} \leq C(\omega, T) < \infty, \text{ a.s.}$$

③ Invariant measure argument.

Consider NLW (defocusing, $k \in 2\mathbb{N}+1$)

(NLW) $$\partial_t^2 u + (1-\Delta)u + u^k = 0$$

- NLW is a Hamiltonian equation.

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}} \begin{pmatrix} \partial E / \partial u \\ \partial E / \partial(\partial_t u) \end{pmatrix}$$

$$\left(\quad \partial p_n = \frac{\partial H}{\partial q_n}, \quad \partial q_n = -\frac{\partial H}{\partial p_n} \Longleftrightarrow \partial_t \begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \partial H / \partial p_n \\ \partial H / \partial q_n \end{pmatrix} \right)$$

$$E = E(u, \partial_t u) = \frac{1}{2}\int |\langle\nabla\rangle u|^2 + \frac{1}{2}\int |\partial_t u|^2 + \frac{1}{k+1}\int u^{k+1}$$

$\Rightarrow$ "$\underline{du\, d(\partial_t u)}$" is invariant under the flow.

↑
does <u>not</u> exist.

· Also, $E(u, \partial_t u)$ is conserved under the NLW dynamics.

⇒ Gibbs measure

$$d\rho(u,\partial_t u) = Z^{-1} e^{-E(u,\partial_t u)}\, du\, d(\partial_t u)$$

should be invariant

$$\underbrace{e^{-\frac{1}{k+1}\int u^{k+1}dx}}_{\text{weight.}}\ \overbrace{\boxed{e^{-\frac{1}{2}\|u\|_{H^1}^2}\, du}}^{\text{massive Gaussian free field}}$$

$$\otimes\ \underbrace{e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2}\, d(\partial_t u)}_{\text{spatial white noise (measure)}}$$

· Gaussian measures on $\mathcal{D}'(\mathbb{T}^d)$

$$d\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$$

sharp freq cut-off

$$\Leftarrow \quad d\mu_{s,N} = Z_N^{-1} e^{-\frac{1}{2}\|P_{\le N} u\|_{H^s}^2} d(P_{\le N} u)$$

$$= Z_N^{-1} \prod_{|n|\le N} e^{-\frac{1}{2} \underbrace{\langle n\rangle^{2s} |\hat{u}(n)|^2}_{|g_n|^2}} \underbrace{d\hat{u}(n)}_{\text{Lebesgue on } \mathbb{C} \cong \mathbb{R}^2}$$

measure on $E_N = \operatorname{span}\{e_n, |n|\le N\}$.

Under $\mu_{s,N}$, we have

$$u_N(x) := \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x}$$

Take $N \to \infty$.

$\{g_n\}$, indep, standard $\mathbb{C}$-valued Gaussian r.v.'s.

· my course from 2017
· Also mini-courses

· Kuo: Gaussian meas in Banach sp.
· Janson
· Nualart

$N \geq M \geq 1$ ⑨

$$\mathbb{E}\left[\|u_N - u_M\|_{H^\sigma}^2\right] = \sum_{M < |n| \leq N} \langle n \rangle^{2\sigma} \frac{\mathbb{E}[|g_n|^2]}{\langle n \rangle^{2s}}$$

$$= \sum_{M < |n| \leq N} \langle n \rangle^{2\sigma - 2s} \to 0$$

iff $2\sigma - 2s < -d$

$$\Longleftrightarrow \boxed{\sigma < s - \frac{d}{2}}$$

- $\{u_N\}_{N \in \mathbb{N}}$ forms a Cauchy seq in $L^2(\Omega; H^\sigma(\mathbb{T}^d))$
  (also in $L^p(\Omega; H^\sigma(\mathbb{T}^d))$, $p < \infty$)

$\Rightarrow$ limit:

$$\omega \in \Omega \longmapsto u(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^s} e^{in \cdot x}$$

$$\mu_s = \lim_{N \to \infty} \mu_{s,N}$$

= induced probability measure under <u>this</u> map $= P \circ u^{-1}$.

<u>Moral</u> : μ_s is NOT a probability measure on $H^s(\mathbb{T}^d)$
and we needed to enlarge the space to H^σ, $\sigma < s - \frac{d}{2}$

<u>ex</u>: $W^{\sigma,p}$, $\boxed{\sigma < s - \frac{d}{2}}$

$B =$ $\mathcal{C}^\sigma = B^\sigma_{\infty,\infty}$

Also, Fourier-Lebesgue spaces $\mathcal{F}L^{\sigma,p}$, $\sigma < s - \frac{d}{p}$

and we say $(\mu_s, H^s, \underline{\underline{B}})$ is an abstract Wiener space.

$\boxed{d=1}$ Gibbs measure for NLW

$$d\rho = Z^{-1} \underbrace{e^{-\frac{1}{k+1}\int u^{k+1}\,dx}}_{\text{depends only on } u} d\mu_1 \otimes d\mu_0(u, \partial_t u)$$

$u \in H^{\frac{1}{2}-}(\mathbb{T})$, a.s. $\underset{\text{Sob}}{\Rightarrow}$ $u \in L^r_x$, a.s. $r < \infty$.

$\Rightarrow \underset{\text{a.s.}}{0 <} e^{-\frac{1}{k+1}\int u^{k+1}\,dx} \leq 1$

$$\Rightarrow \quad e^{-\frac{1}{k+1}\int u^{k+1}\,dx} \in L^p(\mu_1) \qquad p \le \infty$$

and a.s. positive

$$\Rightarrow \quad \rho \sim \mu_1 \otimes \mu_0$$

equivalent (i.e. $\ll$ and $\gg$)

ρ is a prob mease on $H^{\frac{1}{2}-}(\mathbb{T}) \times H^{-\frac{1}{2}-}(\mathbb{T})$.

Aside: If $\mathcal{L}(v) = \mu_0$, then

$$v = \sum_{n \in \mathbb{Z}^d} g_n e^{in\cdot x} \Leftarrow \text{spatial white noise}$$

same intensity of randomness at each freq.

Also, $\mathbb{E}\left[v(x)\overline{v(y)}\right] = \sum_{n\in\mathbb{Z}^d} e^{in\cdot(x-y)} = \delta_0(x-y)$

Bourgain '94:

Bourgain's invariant measure argument.

$$
\begin{tikzcd}
\text{NLW} \arrow[rr, leftrightarrow, "\text{invariant}"] & & \rho \\
\text{"finite dim'l" approx} \arrow[u, "N\to\infty"', "\text{convergence}"] \arrow[rr, leftrightarrow, "\text{invariant}"] & & \rho_N = \text{truncated Gibbs meas} \arrow[u, "\text{convergence (in } L^p \text{ for density)}"']
\end{tikzcd}
$$

finite $p \geq 2$

$$\partial_t^2 u_N + (1-\Delta) u_N + P_{\leq N}\big((P_{\leq N} u)^k\big) = 0$$

$$\boxed{d\rho_N = Z_N^{-1} \, e^{-\frac{1}{k+1} \int (P_{\leq N} u)^{k+1}} \, d\mu_1 \otimes d\mu_0(u, \partial_t u)}$$

low freq $(|n| \leq N)$: truncated Gibbs $\Leftarrow$ finite dim'l Hamiltonian system

high freq : Gaussian measure. (invariant under lin dynamics)

Main idea: Use the invariance of ρ_N

for the truncated dynamics. (for $(u_N, \partial_t u_N)$)

N fixed Prop: Given $T>0$, $\varepsilon>0$, $\exists\, \Omega_{N,T,\varepsilon} \subset \Omega$

s.t. ① $\rho_N(\Omega^c_{N,T,\varepsilon}) < \varepsilon$

② For each $\omega \in \Omega_{N,T,\varepsilon}$, the soln $(u_N^\omega, \partial_t u_N^\omega)$ exists on $[-T, T]$ AND

$$\sup_{-T\le t\le T} \|(u_N(t), \partial_t u_N(t))\|_{\mathcal{H}^{1/2-}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$$

indep of N.

Pf: Fix $K \gg 1$.

$\Phi_N(t) : (u_0, u_1) \mapsto (u(t), \partial_t u(t))$, $\mathcal{H}^{1/2-} \to \mathcal{H}^{1/2-}$

Set
$$\Omega_{N,T,\varepsilon} = \bigcap_{j=-[T/\delta]}^{[T/\delta]} \Phi_N(-j\delta)\, B_K,$$

where B_K = ball of radius K in $\mathcal{H}^{1/2-}(\mathbb{T})$

$\delta \sim K^{-\theta}$ local existence time for initial data of size K.

① $$\rho_N\left(\Omega^c_{N,T,\varepsilon}\right) \le \sum_{j=-[T/\delta]}^{[T/\delta]} \rho_N\left(\cancel{\Phi_N(-j\delta)}\, B_K^c\right)$$

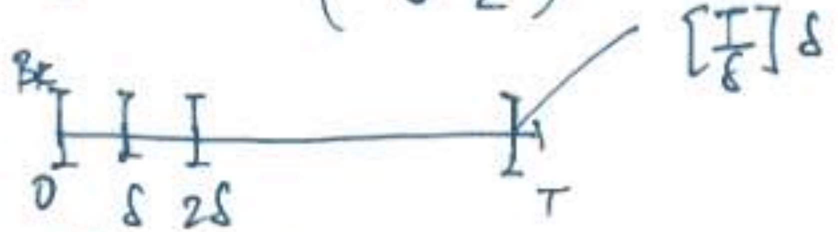
ρ_N invariant under Φ_N

$$\sim \frac{T}{\delta}\, \rho_N(B_K^c) \quad \overset{\text{C.S.}}{\lesssim} \left(\mu_1 \otimes \mu_0 (B_K^c)\right)^{1/2}$$

$$\lesssim T\, \underbrace{K^{\theta}\, e^{-cK^2}}_{\lesssim e^{-c'K^2}} < \varepsilon$$

by choosing $K \sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}$

②



By LWP argument, we have $\sup_{-T \le t \le T} \|(u_N(t), \partial_t u_N(t))\|_{\mathcal{H}^{\frac{1}{2}-}} \le CK$

$$\sim \left(\log \frac{T}{\varepsilon}\right)^{1/2}.$$

indep of N.

$\square$

- The set $\Omega_{N,T,\varepsilon}$ depends on N
  but the constants in the bd are indep of N.

$\Rightarrow$ can deduce the same log bound for true soln $(u, \partial_t u)$
(by a PDE approx argument.) $\Rightarrow$ a.s. GWP.

i.e. used an invariant measure in place of a conservation law.