Lec 5 17/03/21 (Wed) $\quad S(t) = e^{-it\Delta}$

$$\Psi = \int_0^t S(t-t')\,\Phi\, dW(t') \qquad \Phi \in HS(L^2; H^s)$$

$$\Phi e_n = \hat{\Phi}_n e_n$$

$$= \sum_{n \in \mathbb{Z}^d} \hat{\Phi}_n e_n \int_0^t e^{i(t-t')|n|^2} d\beta_n(t').$$

Prop: $b < \frac{1}{2}$. Then, $\mathbb{1}_{[0,T]} \Psi \in X^{s,b}$, a.s.

Pf: Recall $\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau - |n|^2 \rangle^b \hat{u}(\tau, n) \|_{\ell_n^2 L_\tau^2}$

$$= \| \langle \partial_x \rangle^s \langle \partial_t \rangle^b S(-t) u(t) \|_{L^2_{x,t}}$$

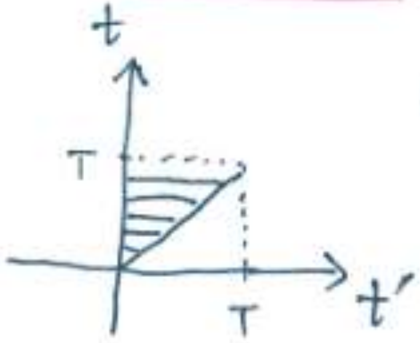
interaction representation

$$\widehat{\mathbb{1}_{[0,T]}(t) \Psi}(t,n) = \mathbb{1}_{[0,T]}(t)\, \hat{\Phi}_n \int_0^t e^{i(\cancel{t}-t')|n|^2} d\beta_n(t')$$

$S(-t)$

$$\cdot \quad F(t) = \underline{S(-t)}\, \mathbb{1}_{[0,T]}(t)\, \Psi(t)$$

$$\boxed{\hat{F}(\tau,n)} = \hat{\Phi}_n \int_{\mathbb{R}} e^{-it\tau}\, \mathbb{1}_{[0,T]}(t) \int_0^t e^{-it'|n|^2} d\beta_n(t')\, dt$$



stoch Fubini ← see Da Prato-Zabczyk: "stoch eqns in ∞ dim"

$$\overset{\downarrow}{=} \hat{\Phi}_n \int_0^T e^{-it'|n|^2} \underbrace{\int_{t'}^T e^{-it\tau}\, dt}_{\lesssim_T \min\left(1, \frac{1}{|\tau|}\right) \sim \frac{1}{\langle\tau\rangle}} d\beta_n(t')$$ ⇐ Wiener integral

$\langle\tau\rangle = \sqrt{1+|\tau|^2}$

$$\cdot \quad \mathbb{E}\left[\| \mathbb{1}_{[0,T]} \Psi \|^2_{X^{s,b}}\right] = \Big\| \langle n\rangle^s \langle\tau\rangle^b \underbrace{\boxed{\| \hat{F}(\tau,n) \|_{L^2(\Omega)}}}_{|\hat{\Phi}_n| \left(\int_0^T |\cdots|^2 dt'\right)^{1/2} \lesssim_T |\hat{\Phi}_n| \frac{1}{\langle\tau\rangle}} \Big\|^2_{\ell^2_n L^2_\tau}$$

$$= \underbrace{\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{\Phi}_n|^2}_{= \|\Phi\|^2_{HS(L^2; H^s)}} \cdot \underbrace{\int_{\mathbb{R}} \langle\tau\rangle^{2(b-1)}\, d\tau}_{< \infty \text{ iff } b < 1/2}$$

We also have, for $p \geq 2$,

$$\| \| \mathbb{1}_{[0,T]} \Psi \|_{X^{s,b}} \|_{L^p(\Omega)} \lesssim_T p^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

chebyshev $\Longrightarrow$

$$P\left( \| \mathbb{1}_{[0,T]} \Psi \|_{X^{s,b}} > \lambda \right) \leq C e^{-c \frac{\lambda^2}{\|\Phi\|_{HS(L^2;H^s)}}}$$

$\boxtimes$

· <u>LWP of the 1-d cubic SNLS in $L^2(\mathbb{T})$</u>, $\Phi \in HS(L^2;L^2)$

Duhamel formulation:

$T \leq 1$.

$$u(t) = \eta(t) S(t) u_0 - i\, \eta\!\left(\tfrac{t}{T}\right) \int_0^t S(t-t')|u|^2 u(t')\, dt'$$

$$+ \mathbb{1}_{[0,T]}(t)\, \Psi(t)$$

$$=: \Gamma(u)$$

$\Uparrow$

$\mathbb{1}_{[0,1]}(t)$

$$\|\eta(t)\int_0^t S(t-t')|u|^2u(t')\,dt'\|_{X^{0,\frac12+0}}$$ ④

Fix $\frac{3}{8} \le b < \frac12$.

Then,
$$\|\Gamma u\|_{X^{0,b}} \lesssim \|u_0\|_{L^2} + T^{\theta}\,\||u|^2u\|_{X^{0,-\frac12+2\theta}}$$
$$+\ C_\omega \leftarrow \mathbb{1}_{[0,1]}\Psi$$

- $\||u|^2u\|_{X^{0,-\frac12+2\theta}}$

$$(X^{s,b})' = X^{-s,-b}$$

$$\overset{\text{duality}}{=} \sup_{\|v\|_{X^{0,\frac12-2\theta}}=1} \underbrace{\left|\int_{\mathbb{R}\times\mathbb{T}} |u|^2u\,\bar{v}\,dx\,dt\right|}$$

$$\le \|u\|^3_{L^4_{x,t}}\|v\|_{L^4_{x,t}}$$

$$\lesssim \|u\|^3_{X^{0,\frac38}}\underbrace{\|v\|_{X^{0,\frac38}}}_{\le 1}$$

Recall on $\mathbb{T}$

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac38}}$$

$$\Rightarrow \quad \|\Gamma u\|_{X^{0,b}} \lesssim \|u_0\|_{L^2} + T^{\theta}\|u\|^3_{X^{0,b}} + C_\omega$$

Also,

$$\|\Gamma u - \Gamma v\|_{X^{0,b}} \lesssim T^{\theta}\left(\|u\|^2_{X^{0,b}} + \|v\|^2_{X^{0,b}}\right)\|u-v\|_{X^{0,b}}$$

$\Rightarrow$ $\Gamma$ is a contraction on $B_R \subset X^{s,b}$

$$R = R_\omega \sim \|u_0\|_{L^2} + \underset{\uparrow\, \|\mathbb{1}_{[0,1]}\Psi\|_{X^{0,b}}}{C_\omega}$$

by choosing $T = T(R) \ll 1$.

Also, LWP in $H^s(\mathbb{T})$; $\Phi \in HS(L^2; H^s)$

$$\Leftarrow \quad \|u_1 \overline{u_2} u_3\|_{X^{s,-\frac{1}{2}+}} \lesssim \prod_{j=1}^{3}\|u_j\|_{X^{s,b}}$$

$$\Leftarrow \quad \langle n\rangle^s \lesssim \langle n_1\rangle^s\langle n_2\rangle^s\langle n_3\rangle^s, \quad s \geq 0 \quad \text{if } n = n_1 - n_2 + n_3.$$

- We constructed $u \in B_R \subset X^{0,\frac{1}{2}-}$. $\not\subset C_t L^2_x$ ⑥

$\Rightarrow$ We need to show $u \in C_t L^2_x$

- lin part : $S(t) u_0 \in C_t L^2_x$
- stoch convolution $\Psi \in C_t L^2_x$ if $\phi \in HS(L^2; L^2)$
- nonlinear part $\in X^{0,\frac{1}{2}+} \subset C_t L^2_x$.

$\Longrightarrow$ $u \in C([0,T]; L^2_x)$. $T = T_\omega \le 1$.

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Aside: LWP of the cubic SNLS in $L^2(\mathbb{T})$ $(\phi \in HS(L^2; L^2))$

- We proved

Zygmund '74: $\| S(t) u_0 \|_{L^4_{x,t}(\mathbb{T}^2)} \lesssim \| u_0 \|_{L^2(\mathbb{T})}$

Moyua-Vega '08 BLMS. : $\| S(t) u_0 \|_{L^4_{I,x}} \lesssim |I|^{\frac{1}{8}} \| u_0 \|_{L^2(\mathbb{T})}$ $\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$

$t^6$

( Bourgain '93: $\| u \|_{L^4_{x,t}} \lesssim \| u \|_{X^{0,\frac{3}{8}}}$

$$\Rightarrow \quad (*) \quad \left\| \int_I S(-t')F(t')\,dt' \right\|_{L^2_x} \lesssim \underline{\underline{|I|^{1/8}}} \, \|F\|_{L^{4/3}_{I,x}}$$

$\|$ duality

$$\sup_{\|f\|_{L^2}=1} \left| \int_I \langle F(t'), S(t')f \rangle_{L^2_x} \, dt' \right|$$

$\wedge|$

$$\|F\|_{L^{4/3}_{I,x}} \underbrace{\|S(t')f\|_{L^4_{I,x}}}_{\text{use } L^4\text{-Strichartz.}}$$

$$\Rightarrow \quad \left\| \int_I S(t-t')F(t')\,dt' \right\|_{\underline{\underline{L^4_{I,x}}}} \lesssim |I|^{\frac{1}{4}} \, \|F\|_{L^{4/3}_{I,x}}$$

$= \underline{\underline{S(t)}} \, S(-t')$

$\Leftarrow$ ① Use $L^4$-Strich & gain $|I|^{1/8}$

② use $(*)$

$\Rightarrow$ Christ-Kiselev lemma (JFA '01)

$$\left\| \int_0^t S(t-t')F(t')\,dt' \right\|_{L^4_{I,x}} \lesssim |I|^{\frac{1}{4}} \|F\|_{L^{4/3}_{I,x}}$$

· Now, run a contraction argument in $L^4_{T,x} \cap C_T L^2_x$

$\Leftarrow$ · $\| |u|^2 u \|_{L^{4/3}_{T,x}} \le \|u\|^3_{L^4_{T,x}}$

· $\Phi \in L^4_{T,x} \cap C_T L^2_x$

· Global well-posedness?

① Ito calculus approach (Martingale approach)

· 1-d cubic SNLS on $\mathbb{T}$ : $i\partial_t u - \partial_x^2 u + |u|^2 u = \phi\xi$

$\phi \in HS(L^2; L^2)$

deterministic case: $\phi = 0$

$$\partial_t \int |u|^2 dx = 2\operatorname{Re}\int \partial_t u \cdot \bar{u}\, dx$$

$$= -2\operatorname{Re}\, i \underbrace{\int \partial_x^2 u \cdot \bar{u}\, dx}_{\overset{\text{IBP}}{=} \int |\partial_x u|^2 dx} + 2\operatorname{Re}\, i \int \underbrace{|u|^2 u \cdot \bar{u}}_{|u|^4} dx$$

$= 0$

$\Rightarrow L^2$-conservation $\Rightarrow$ GWP in $L^2(\mathbb{T})$.

For SNLS, we use Ito's lemma on

$$M(u) = \int |u|^2 dx = \sum |\hat{u}(n)|^2 = \sum (p_n^2 + q_n^2)$$

$$p_n = \mathrm{Re}\, \hat{u}(n) \,, \quad q_n = \mathrm{Im}\, \hat{u}(n).$$

SNLS:
$$d\hat{u}_n = \left( i n^2 \hat{u}_n + i \mathcal{F}_x \left( |u|^2 u \right)(n) \right) dt$$
$$- i \hat{\Phi}_n \, d\beta_n$$

$$\Rightarrow \quad dp_n = \left( -n^2 q_n - \mathrm{Im}\, \mathcal{F}_x \left( |u|^2 u \right)(n) \right) dt$$
$$+ \underbrace{\mathrm{Im} \left( \hat{\Phi}_n \, d\beta_n \right)}_{= \mathrm{Im}\, \hat{\Phi}_n \, d(\mathrm{Re}\, \beta_n) + \mathrm{Re}\, \hat{\Phi}_n \, d(\mathrm{Im}\, \beta_n)}$$

$$dq_n = \left( n^2 p_n + \mathrm{Re}\, \mathcal{F}_x \left( |u|^2 u \right)(n) \right) dt$$
$$\underbrace{- \mathrm{Re} \left( \hat{\Phi}_n \, d\beta_n \right)}_{= - \mathrm{Re}\, \hat{\Phi}_n \, d(\mathrm{Re}\, \beta_n) + \mathrm{Im}\, \hat{\Phi}_n \, d(\mathrm{Im}\, \beta_n)}$$

· Itô's lemma : $dX = f\,dt + g\,dB$

Consider $F(X)$

Then, $dF = \partial_x F\, dX + \frac{1}{2}\partial_x^2 F (dX)^2$

2nd order Taylor exp

$$= \partial_x F\,(f\,dt + g\,dB) + \frac{1}{2}\partial_x^2 F \cdot g^2 dt$$

$(dt)^2 = 0$

$dt\,dB = dB\,dt = 0$

$(dB)^2 = dt$

To be understood under

$F(b) - F(a) = \int_a^b \cdots$

$$dM \overset{\text{Ito}}{=} \underbrace{2\sum\left(p_n\,dp_n + q_n\,dq_n\right)}_{2\sum\left(p_n \operatorname{Im}(\hat{\Phi}_n d\beta_n) - q_n \operatorname{Re}(\hat{\Phi}_n d\beta_n)\right)} + \underbrace{\sum\left((dp_n)^2 + (dq_n)^2\right)}_{2\|\Phi\|^2_{HS(L^2;L^2)}\,dt}$$

$d(\operatorname{Re}\beta_n)\,d(\operatorname{Im}\beta_n) = 0$

· Burkholder - Davis - Gundy inequality : $X$, (local) martingale ⑫

$1 \le p < \infty$.

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] \sim \mathbb{E}\left[\langle X_t\rangle_{[0,T]}^{p/2}\right]$$

quadratic variation

For Itô process $dX = f\,dt + g\,dB$

$$\langle X\rangle_{[0,t]} = \int_0^t g^2\,dt'.$$

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ex: $2\sum_n p_n \operatorname{Im}\hat{\Phi}_m \, d(\operatorname{Re}\beta_n)$

Take $\mathbb{E}\left[\sup_{t\in[0,T]} \int_0^t \cdots \right]$

and apply B-D-G ineq.

$$\lesssim \mathbb{E}\left[\left(\int_0^T \underbrace{\sum_n p_n^2 |\hat{\Phi}_n|^2}_{\le \|u\|_{L^2}^2 \|\Phi\|_{HS(L^2;L^2)}} dt\right)^{1/2}\right]$$

$$\sum a_n b_n \le \sum a_n \cdot \sum b_n$$

Hide in LHS

$$\le \mathbb{E}\left[\left(\sup_{t\in[0,T]} M(u)(t)\right)^{1/2} \cdot T^{1/2} \|\Phi\|_{HS(L^2;L^2)}\right]$$

$$\le \varepsilon\, \mathbb{E}\left[\sup_t M(u)(t)\right] + \frac{1}{\varepsilon} T \|\Phi\|_{HS}^2$$

Start with

$$\mathbb{E}\Big[\sup_{t\in[0,T]} M(u)(t)\Big]$$

$$\Rightarrow \mathbb{E}\Big[\sup_{t\in[0,T]} M(u)(t)\Big] \lesssim \|u_0\|_{L^2}^2$$

$$+ C\big(T, \|\Phi\|_{HS(L^2;L^2)}\big)$$

$\Rightarrow$ a.s. existence up to time $T$.

(but for ANY finite $T$).

$\Rightarrow$ GWP. in $L^2(\mathbb{T})$

· ② pathwise approach:

ex: 3-d cubic SNLW (defocusing) on $\mathbb{T}^3$

$$\partial_t^2 u + (1-\Delta) u + u^3 = \Phi \xi, \qquad \Phi \in HS(L^2; H^{s-1}), \quad s>0$$

$$\Rightarrow \Psi \in C_t W_x^{s-\varepsilon, \infty}$$

Duhamel: $u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^3(t') dt'$

$S(t) = \frac{\sin t\langle \nabla \rangle}{\langle \nabla \rangle}$ $\quad + \Psi.$

"Da Prato–Debussche trick" '03:

McKean '95, Bourgain '96

$\boxed{\text{1st order expansion} \quad \text{Write: } u = \Psi + v}$

postulate "smoother".

" $v(t) \in H^1$ ".

$$V(t) = \text{lin soln} + \int_0^t \underline{S(t-t')}\,(v+\Psi)^3(t')\,dt' \quad \cancel{+\Psi}$$

$$\overset{\shortparallel}{u-\Psi}$$

one gain of deriv

$$\|(\Gamma v, \partial_t \Gamma v)\|_{C_T \mathcal{H}^1} \leq \underline{\|(u_0, u_1)\|_{\mathcal{H}^1}} + T\, \|(v+\Psi)^3\|_{L^\infty_T L^2_x}$$

$$\mathcal{H}^1 = H^1 \times L^2$$

$$\underline{v}\,,\ \partial_t v$$

$$\|(v+\Psi)^3\|_{L^\infty_T L^2_x} = \|v+\Psi\|^3_{L^\infty_T L^6_x} \lesssim \|v\|^3_{L^\infty_T L^6_x} + \underbrace{\|\Psi\|^3_{L^\infty_T L^6_x}}_{\leq C_\omega}$$

$$\overset{\text{Sob}}{\lesssim} \|v\|^3_{C_T H^1_x}$$

$\Rightarrow$ LWP. (for $(v, \partial_t v) \in C_T \mathcal{H}^1$).

as long as $\Phi \in HS(L^2; H^{-1})$

(s.t. $\Psi \in C_T L^6_x$ )

- Moral: Fix a target time $T_0 \gg 1$. global target time

Suppose $Z \in C_{T_0} L^6_x$.

Note: The $C_{T_0} L^6_x$-norm of $Z$ may be very large BUT it is a fixed number (for fixed $T_0 \gg 1$)

The argument above yields LWP in $\mathcal{H}^1(\mathbb{T}^3)$:

$$\begin{cases} \partial_t^2 v + (1-\Delta) v + (v+Z)^3 = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1), \end{cases}$$

where the local existence time $T = T\left(\|(u_0,u_1)\|_{\mathcal{H}^1}, K\right) > 0$ with $K = \|Z\|_{C_{T_0} L^6_x}$

small

Then, if we can control $\|(v(t), \partial_t v(t))\|_{\mathcal{H}^1}$ on $[0, T_0]$, then we can iterate the LWP argument on $[jT, (j+1)T]$ to conclude existence on the entire interval $[0, T_0]$.

Since the choice of $T_0$ was arbitrary, this yields GWP.

$\Rightarrow$ Main goal: Control $\| (v(t), \partial_t v(t)) \|_{\mathcal{H}^1}$
and show that it is finite on each finite time interval.