Lec 4 10/03/21 (Wed)

- 1-d cubic SNLS

$$i\partial_t u - \partial_x^2 u + |u|^2 u = \phi \xi$$

(i) on $\mathbb{R}$. $\phi \in HS(L^2; L^2)$

de Bouard - Debussche '03
Oh-Pac-Wang '20

$$\Psi = \int_0^t S(t-t')\phi\, dW(t') \in C_T L^2_x \cap L^q_T W^{s,r}_x \quad \text{if } \phi \in HS(L^2; H^s)$$

on $[0,T]$

$$\forall\, q < \infty, \quad r \le \frac{2d}{d-2}$$

$(d = 1, 2, \ r < \infty.$

$$S(t) = e^{-it\Delta} \qquad \widehat{S(t)f}(\xi) = e^{it|\xi|^2}\hat{f}(\xi)$$

- $$\|S(t)f\|_{L^2(\mathbb{R}^d)} \underset{\text{Plancherel}}{=} \|e^{it|\xi|^2}\hat{f}\|_{L^2_\xi(\mathbb{R}^d)} = \|f\|_{L^2_x}$$

- unitary on $H^s(\mathbb{R}^d)$.

- (q,r) is called <u>Schrödinger admissible</u> if

$$\underline{\frac{2}{q}+\frac{d}{r}=\frac{d}{2}} \qquad 2\le q,r\le\infty$$

(scaling condition) $\qquad (q,r,d)\neq(2,\infty,2)$

- <u>Strichartz estimates</u> on $\mathbb{R}^d$ (<u>NOT</u> on $\mathbb{T}^d$)

① (homog estu) : $r \ge 2$

$$\|S(t)f\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{L^2_x}$$

② (dual homog est): func of t and x.

$$\Big\|\int_{\mathbb{R}} S(-t')\underline{F(t')}\,dt'\Big\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}$$

③ (nonhomog est / retarded est)

$$\Big\|\int_0^t S(t-t')F(t')\,dt'\Big\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}$$

(q,r), $(\tilde q,\tilde r)$, admissible

- On time averaged sense, there is a smoothing (in terms of integrability, NOT in terms of differentiability)

Idea of proof: · dispersive estimate $\|S(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}}\|f\|_{L^1_x}$

$\Leftarrow$ · soln formula $S(t)f = \frac{1}{(4\pi i t)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4ti}} f(y)\,dy$

or

· Method of stationary phase (see my lecture note)

$$S(t)f(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \hat{f}(\xi)\,d\xi$$

· unitary in $L^2(\mathbb{R}^d)$

$$\overset{\Longrightarrow}{\text{interpolate}} \quad \|S(t)f\|_{L^p_x} \lesssim |t|^{-(\frac{d}{2}-\frac{d}{p})}\|f\|_{L^{p'}}, \quad p \geq 2.$$

$T = S(t)$, bdd : $L^2 \to B = L^q_t L^r_x$

$\Leftrightarrow$ Dual op, bdd : $B' = L^{q'}_t L^{r'}_x \to L^2_x$

$$\underline{T^* F = \int_{\mathbb{R}} S(-t') F(t') dt'}.$$

$$\left(\begin{aligned} \langle S(t) f, F \rangle_{L^2_{t,x}} &= \iint S(t) f \, \overline{F(t,x)} \, dt \, dx \\ &= \int f(x) \, \overline{\int S(-t) F(t,x) \, dt} \, dx \\ &= \langle f, T^* F \rangle_{L^2_x} \end{aligned} \right.$$

$\Leftrightarrow$ TT^* bdd : $B' \to B$.

$$\underbrace{TT^* F}_{\text{put in } L^q_t L^r_x,\ r \geq 2} = \int_{\mathbb{R}} \underline{\underline{S(t-t')}} F(t') dt'.$$

put in $L^q_t L^r_x$, $r \geq 2$

$$\| S(t-t') F(t') \|_{L^r_x} \lesssim \underline{\underline{|t-t'|^{-(\frac{d}{2} - \frac{d}{r})}}} \, \| F(t') \|_{L^{r'}_x}$$

$\Rightarrow$ integrate in t'

convolution

Use Hardy-Littlewood-Sobolev ineq

$1 < p, q, r < \infty$, $\quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$$\| |x|^{-d/p} * g \|_{L^r(\mathbb{R}^d)} \lesssim \|g\|_{L^q(\mathbb{R}^d)}$$

$|x|^{-d/p} \notin L^p(\mathbb{R}^d)$
but "almost".
H-L-S ineq is an endpt ver of Young's ineq

$\Rightarrow$ TT^* bdd : $B \to B'$ (nonendpt)

$\Rightarrow$ ① & ②.

As for ③, $\displaystyle\int_0^t = \int_{\mathbb{R}} \mathbb{1}_{[0,t]}(t')$ and prove by hand.

or use Christ-Kiselev lemma (see Tao's book) endpt case : $q=2$, $r=\frac{2d}{d-2}$ by Keel - Tao AJM '99.

· <u>Back to 1-d cubic SNLS on $\mathbb{R}$</u>

Duhamel formulation: $u(t) = S(t)u_0 - \int_0^t S(t-t')|u|^2 u(t')dt'$

$+ \Psi.$

$(q,r) = (\infty, 2), (8,4)$. $\frac{2}{8}+\frac{1}{4}=\frac{1}{2}.$

admissible

· $\Gamma u :=$ RHS of Duhamel

$X(T) = C_T L^2_x \cap L^8_T L^4_x$ $\qquad \frac{7}{8} = \frac{1}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$

$\Rightarrow \|\Gamma u\|_{X(T)} \lesssim \|u_0\|_{L^2_x} + \underbrace{\||u|^2 u\|_{L^{8/7}_T L^{4/3}_x}} + \underset{\substack{\vee\! \mid \\ C_\omega < \infty \text{ a.s.}}}{\|\Psi\|_{X(T)}}$

$\lesssim T^{1/2} \|u\|^3_{L^8_T L^4_x} \leq T^{1/2} \|u\|^3_{X(T)}$

$T \le 1$ $\cdot\ \|\Gamma u\|_{X(T)} \le C_1 \left(\|u_0\|_{L^2_x} + \|\Psi\|_{X(1)} \right) + T^{1/2} \|u\|^3_{X(T)}$

Also, difference estimate.

$\Rightarrow$ LWP in $L^2(\mathbb{R})$ if $\phi \in HS(L^2; L^2)$

· <u>On $\mathbb{T}$?</u>

· Strichartz estimates on $\mathbb{T}^d$

· only local in time.

· NOT as good as those on $\mathbb{R}^d$

· proof, much harder. analytic number theory (HL circle method)

· Zygmund '74, Bourgain '93, '13

Bourgain-Demeter '15, Killip-Vişan '16.

· L^4 - Strichartz on $\mathbb{T}$ (Zygmund '74)

$$\underbrace{\Big\| \sum_{n\in\mathbb{Z}} e^{inx} e^{itn^2} \hat{u}_0(n) \Big\|}_{= F(t,x)}{}_{L^4_{x,t}(\underset{\mathbb{T}_t\times\mathbb{T}_x}{\overset{\shortparallel}{\mathbb{T}^2}})} \lesssim \|u_0\|_{L^2(\mathbb{T})}$$

Pf:

$$\|F\bar{F}\|_{L^2(\mathbb{T}^2)} = \Big\| \sum_{n_1}\sum_{n_2} \hat{u}_0(n_1)\overline{\hat{u}_0(n_2)}\, \underline{e^{it(n_1^2-n_2^2)}}\, \underline{e^{i(n_1-n_2)x}} \Big\|_{L^2(\mathbb{T}^2)}$$

· write this sum as a Fourier series in t and x

$$\sum_{(\tau,n)\in\mathbb{Z}^2} a(\tau,n)\, e^{i\underline{(nx+\tau t)}}.$$

where
$$a(\tau,n) = \sum_{(n_1,n_2)\in \underline{P(\tau,n)}} \hat{u}_0(n_1)\overline{\hat{u}_0(n_2)}$$

$$P(\tau,n) = \{(n_1,n_2) : n_1^2 - n_2^2 = \tau,\ n_1 - n_2 = n\}$$

• Given $(\tau, n) \neq (0,0)$ and $(\tau, n) = (n_1^2 - n_2^2,\ n_1 - n_2)$,

there exists at most one soln (n_1, n_2)

$$\left(\begin{array}{l} \tau - n^2 = n_1^2 - n_2^2 - n^2 = -2n_2(n_2 - n_1) = 2 n n_2 \\ \Rightarrow \text{ determines } n_2 \quad (\text{and } n_1 = n + n_2) \end{array} \right.$$

• Also, $a(0,0) = \sum_n |\hat{u}_0(n)|^2$

Putting together,

$$\|F\|_{L^4(\mathbb{T}^2)} \overset{\text{Plancherel}}{=} \Big(\sum_{\tau, n \in \mathbb{Z}} |a(\tau, n)|^2 \Big)^{1/4}$$

$$\sim \Big(\sum_{(\tau, n) \neq (0,0)} \cdots \Big)^{\frac{1}{4}} + \underbrace{|a(0,0)|^{1/2}}_{= \|u_0\|_{L^2}}$$

$$\lesssim \Big(\sum_{n_1, n_2} |\hat{u}_0(n_1) \overline{\hat{u}_0(n_2)}|^2 \Big)^{\frac{1}{4}}$$

↖ disjoint sum in n_1 and n_2

$$\sim \|u_0\|_{L^2} \quad \square$$

- Zygmund's L^4-Strichartz is NOT enough to prove LWP of cubic NLS in $L^2(\mathbb{T})$ (any $H^s(\mathbb{T})$, $s \le \frac{1}{2}$)

'80~: Kenig - Ponce - Vega, harmonic analytic approach on $\mathbb{R}^d$.

- Fourier restriction norm method: Bourgain '93

$X^{s,b}$ - spaces

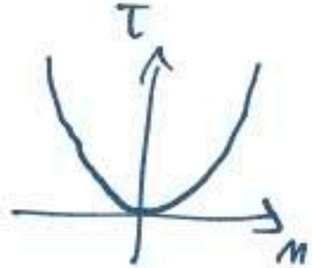
(wave: hyperbolic Sobolev space by Klainerman-Machedon '93

$$\|u\|_{X^{s,b}} = \| \underbrace{\langle n \rangle^s}_{\text{Sob weight in } x} \underbrace{\langle \tau - |n|^2 \rangle^b}_{\text{modulation}} \hat{u}(\tau, n) \|_{\ell^2_n L^2_\tau (\mathbb{Z}^d \times \mathbb{R})}$$

- useful for "perturbative" study

"close to being linear soln"

linear Schrödinger: $i\partial_t u - \Delta u = 0$



space-time F.T. $\Longrightarrow$ $\quad -(\tau - |n|^2)\,\hat{u}(\tau, n) = 0$

$\hat{u}(\tau, n)$ is a measure supported on $\{\tau = |n|^2\}$

- $\langle \tau - |n|^2 \rangle^b$, $b > 0$, "penalizes" functions whose Fourier support is away from $\{\tau = |n|^2\}$

- Basic properties:

① $b > 1/2$, $\quad X^{s,b} \subset C_t H^s_x$

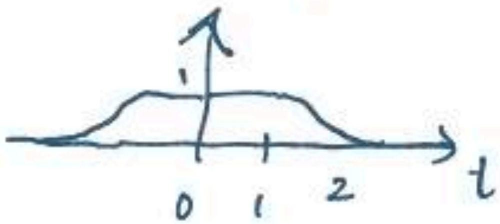
($b = 1/2$, need some other spaces

$b = 1/2$, may use $B^{1/2}_{2,1}$

may V^2, U^2 = predual of V^2

② $\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{H^s_x H^b_t} := \|\langle\partial_x\rangle^s \underset{?\ \langle\tau\rangle^b}{\langle\partial_t\rangle^b} (S(-t)u(t))\|_{L^2_{t,x}}$

③ $\eta(t)$ = smooth cutoff func on $[0,1]$



$$\| \eta(t) S(t) f \|_{X^{s,b}} \lesssim \| f \|_{H^s}.$$

Pf: $\widehat{\eta(t) S(t) f}(\tau, n) = \hat{\eta}(\tau - |n|^2) \hat{f}(n)$

$$(\mathrm{LHS}) = \| \langle n \rangle^s \langle \underbrace{\tau - |n|^2}_{=:\tilde{\tau}} \rangle^b \hat{\eta}(\underbrace{\tau - |n|^2}_{\tilde{\tau}}) \hat{f}(n) \|_{l_n^2 L_\tau^2}$$

$$= \| \eta \|_{H^b} \| f \|_{H^s}$$

(4) Duhamel term: $\underline{b > 1/2}$

$$\| \eta(t) \int_0^t S(t-t') F(t')\, dt' \|_{X^{s,b}} \lesssim \| F \|_{X^{s,b-1}}$$

· Bourgain '93, Tao's book

· Andreia's 1st project (2018) $\Leftarrow$ by taking space-time Fourier transform

$$\cdot \quad \| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') F(t')\, dt' \|_{X^{s,b}} \lesssim T^{\theta} \| F \|_{X^{s,b-1+\theta}}$$

cutoff on $[-T,T]$

$b > 1/2$, $\quad \theta > 0$, small, $\quad T \leq 1$

($\underline{\text{but}}$ $\quad \| \eta\left(\tfrac{t}{T}\right) S(t) f \|_{X^{s,b}} \lesssim \underline{T^{\frac{1}{2}-b}} \| f \|_{H^s}$, $\quad b > 0$, $b \neq 1/2$

bad for $b > 1/2$, $T \ll 1$.

We consider

$$u(t) = \eta(t)\, S(t)\, u_0 - \eta\left(\frac{t}{T}\right) \int_0^t S(t-t')\, |u|^2 u(t')\, dt'$$

$$+ \eta(t) \int_0^t S(t-t')\, \Phi\, dW(t').$$

⑤ L^4 - Strichartz estimate by Bourgain '93

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{3}{8}}} \longleftarrow \text{better than } \tfrac{1}{2}+$$

Zygmund '74: $\|S(t) f\|_{L^4_{t,x}} \lesssim \|f\|_{L^2_x}$ (local int)

$\Rightarrow \|\eta(t)\, u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,b}}, \quad b > \frac{1}{2}$

transference principle

· Tao's book: by Tzvetkov.

<u>Prop:</u> $b < 1/2$. $\quad \Phi \in HS(L^2; H^s)$

$$\| \mathbb{1}_{[0,1]} \Psi \|_{X^{s,b}} \le C_\omega < \infty, \quad a.s.$$

$\Psi \sim$ B.M. in time

<u>Next time:</u>
- Pf of Prop on Ψ
- LWP of 1-d cubic SNLS in $L^2(\mathbb{T})$