Lec 3 : 03 / 03 / 21 (Wed)

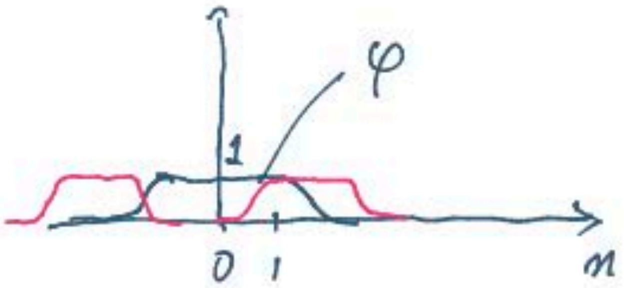
Last time : LWP of 1-d SNLS in $H^s(\mathbb{T}^d)$, $s > \frac{d}{2}$
$\phi \in HS(L^2; H^s)$

· Littlewood-Paley decomposition.

$$f = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} P_N f \qquad N \in 2^j , \ j \geq 0.$$

$$= \sum_{j=0}^{\infty} P_j f$$

P_N = LP projector = "projection" onto freq $\{ |n| \sim N = 2^j \}$.



$\varphi(\xi)$ radial, supported on $\{ |\xi| \leq 2 \}$
$\equiv 1$ on $\{ |\xi| \leq 6/5 \}$

$$\varphi_j(\xi) = \varphi\left(\frac{\xi}{2^j}\right) - \varphi\left(\frac{\xi}{2^{j-1}}\right) , \ j \geq 1$$

$$\widehat{P_j f}(\xi) = \varphi_j(\xi) \hat{f}(\xi).$$

$\Leftarrow$ Need to normalize φ_j

$$\psi_j(\vec{\xi}) = \frac{\varphi_j(\xi)}{\sum_{k=0}^{\infty} \varphi_k(\xi)} \leftarrow \text{finite sum}$$

$\Rightarrow \quad P_j f = \mathcal{F}^{-1}(\psi_j f)$

- LP theorem: $\quad 1 < p < \infty$

$$\|f\|_{L^p} \sim \Big\| \underbrace{\Big(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} |P_N f|^2 \Big)^{1/2}}_{\text{square function.}} \Big\|_{L^p}$$

$$\left(\begin{array}{l} \text{if } p \geq 2 \\ \leq \big\| \underbrace{\|P_N f\|_{L^p}}_{\text{simpler object}} \big\|_{\ell^2_{N,\text{dyadic}}} \end{array} \right.$$

Sobolev spaces

$$\|f\|_{H^s} \sim \Big(\sum_{\substack{N\ge 1\\ \text{dyadic}}} N^{2s} \|P_N f\|_{L^2_x}^2 \Big)^{1/2}$$

Besov spaces $B^s_{p,q}$ or $B^{s,p}_q$

$$\|f\|_{B^s_{p,q}} = \Big\| \underset{\overset{\shortparallel}{2^{js}}}{N^s} \|P_N f\|_{L^p_x} \Big\|_{\ell^q_{\substack{N\ge 1\\ \text{dyadic}}}} \quad = \ell^s_j(\mathbb{Z}_{\ge 0})$$

- $H^s = B^s_{2,2}$
- $$\begin{array}{ccccc} B^s_{p,1} & \subset & W^{s,p} & \subset & B^s_{p,\infty} \\ \cup & & \shortparallel & & \\ B^{s+\varepsilon}_{p,\infty} & & L^p_s & & \end{array}$$

- <u>Hölder-Besov space</u> $\mathcal{C}^s = B^s_{\infty,\infty}$, $s \in \mathbb{R}$

  - natural extension of the classical Hölder space. $\underset{0<s<1}{C^s}$.

$$\|f\|_{C^s} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^s}$$

$$\overset{\|}{\dot{\Lambda}^s} = \text{(homog) Lipschitz space}$$

$$\Lambda^s = \dot{\Lambda}^s \cap L^\infty$$

<u>FACT:</u> $\dot{\Lambda}^s = \dot{B}^s_{\infty,\infty}$, $0 < s < 1$

$$\|f\|_{\dot{B}^s_{p,q}} = \Big\| 2^j \|P_j f\|_{L^p_x} \Big\|_{\ell^q_j(\mathbb{Z})}$$

- <u>$s>0$</u>: $\mathcal{C}^s$ is an alegebra

$$\|fg\|_{\mathcal{C}^s} \lesssim \|f\|_{\mathcal{C}^s} \|g\|_{\mathcal{C}^s}$$

- $\boxed{\text{1-d SNLW}}$: $(\partial_t^2 + 1 - \Delta)u + u^k = \xi$ on $\mathbb{T}$

Duhamel

- $$u(t) = \underbrace{\partial_t S(t) u_0 + S(t) u_1}_{\cos(t\langle\nabla\rangle) u_0 + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} u_1} - \int_0^t S(t-t') u^k(t') dt' + \underbrace{\int_0^t S(t-t') dW(t')}_{\Psi = \text{stoch convolution}}$$

$$S(t) = \frac{\sin t\langle\nabla\rangle}{\langle\nabla\rangle} \qquad = \int_0^t S(t-t') \xi(dt')$$

Denote RHS by $\Gamma(u) = \Gamma_{(u_0,u_1),\xi}(u)$

- Recall $\Psi = \Psi_{\text{wave}} \in C_t W_x^{\frac{1}{2}-,\infty}(\mathbb{T})$

- $\| \partial_t S(t) u_0 + S(t) u_1 \|_{H^s} \lesssim \| (u_0, u_1) \|_{\mathcal{H}^s}$

$$\mathcal{H}^s = H^s \times H^{s-1}.$$

$$\Rightarrow \|\Gamma(u)\|_{C_T H^s} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \int_0^t \underbrace{\|u^k(t')\|_{H^{s-1}}}\, dt' + \underbrace{\|\Psi\|_{C_T H^s}}_{\substack{\le C_\omega < \infty \\ \text{a.s.}}}$$

$s < 1/2$

$$\begin{aligned} &\le \|u^k(t')\|_{L^2} \\ &= \|u(t')\|^k_{L^{2k}} \\ &\overset{\text{Sob}}{\lesssim} \|u(t')\|^k_{H^s} \qquad s \ge \underbrace{\frac{1}{2} - \frac{1}{2k}}_{< 1/2} \end{aligned}$$

$\Rightarrow$ By choosing $\frac{1}{2} - \frac{1}{2k} \le s < 1/2$, we have

$$\|\Gamma(u)\|_{C_T H^s} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + T\|u\|^k_{C_T H^s} + \|\Psi\|_{C_T H^s}$$

Similarly,

$$\|\Gamma(u) - \Gamma(v)\|_{C_T H^s} \lesssim T\left(\|u\|^{k-1}_{C_T H^s} + \|v\|^{k-1}_{C_T H^s}\right)\|u - v\|_{C_T H^s}.$$

$$R = R_\omega \sim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|\Psi\|_{C([0,1];H^s)}$$

$\Rightarrow$ Γ is a contraction on $B_R \subset C_T H^s$, $T = T_\omega = T(R_\omega) \ll 1$.

· <u>Rmk:</u> By a similar argument, we can show LWP for

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \end{pmatrix}$$

$\Leftarrow$ important for global-in-time study.

· $\boxed{\text{1-d SNLH}}$: $(\partial_t + 1 - \Delta) u + u^k = \xi$.

· <u>Schauder estimate</u>: $1 \le p \le q \le \infty$

$$\| D^\alpha P(t) f \|_{L^q_x} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{\alpha}{2}} \|f\|_{L^p_x}$$

$P(t) = e^{t(\Delta - 1)}$ on $\mathbb{R}^d$ or $\mathbb{T}^d$, $t > 0$, $\alpha \ge 0$.

Pf: On $\mathbb{R}^d$.

· Bahouri – Chemin – Danchin
· Grafakos

$$e^{t\Delta} f(x) = \int K_t(x-y) f(y)\, dy$$

$$\hat{K_t}(\xi) = e^{-t|\xi|^2}$$

$$= \hat{K}(t^{1/2}\xi), \quad \text{where } K = K_1.$$

inv F.T. $\Rightarrow$ $K_t(x) = \frac{1}{t^{d/2}} K\left(\frac{x}{t^{1/2}}\right)$

Apply Young

$\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$

$$\Rightarrow \|K_t\|_{L^r_x} = t^{-d/2} \left\|K\left(\frac{x}{t^{1/2}}\right)\right\|_{L^r_x} = t^{-d/2}\, t^{\frac{d}{2r}}\, C_K.$$

ch of var

$$\sim t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}$$

$\Rightarrow$ By Young's ineq, $\|e^{t\Delta} f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}$

· $\alpha > 0$: $D^\alpha(e^{t\Delta} f) = D^\alpha(K_t * f) = (D^\alpha K_t) * f$

$$\widehat{D^\alpha K_t}(\xi) = |\xi|^\alpha e^{-t|\xi|^2} \qquad D = \sqrt{-\Delta} = |\nabla|$$

$$= t^{-\frac{\alpha}{2}} \underbrace{(t^{1/2}|\xi|)^\alpha e^{-t|\xi|^2}}$$

$$=: \widehat{G_t}(\xi) = \widehat{G}(t^{1/2}\xi), \qquad G = G_1 \in \mathcal{S}(\mathbb{R}^d)$$

repeating the same comp
$\Rightarrow$

$$\| D^\alpha K_t \|_{L^r} = t^{-\frac{\alpha}{2}} \| G_t(x) \|_{L^r} \sim t^{-\frac{\alpha}{2}} t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$$

$\Rightarrow$ Apply Young's ineq. $\qquad \frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$

- On $\mathbb{T}^d$: $\qquad e^{t\Delta} f = R_t * f, \quad \widehat{R_t}(n) = e^{-t|n|^2}$.

  - can not use the scaling argument.

  $\Rightarrow$ use the Poisson summation formula: $|f(x)| + |\hat{f}(x)| \lesssim \langle x \rangle^{-d-\varepsilon}$
  f on $\mathbb{R}^d$

$$\sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in x} = \sum_{n \in \mathbb{Z}^d} f(x+n)$$

Pf: $F(x) = \sum_{n\in\mathbb{Z}^d} f(x+n)$ periodic func on $\mathbb{T}^d$.

$\|$

$\sum_{n\in\mathbb{Z}^d} \hat{F}(n) e^{inx}$

$$\hat{F}(n) = \int_{\mathbb{T}^d} F(x) e^{-in\cdot x} dx = \int_{[-1/2,1/2)^d} \sum_m f(x+m) e^{-in\cdot x} dx.$$

$$= \underbrace{\sum_{m\in\mathbb{Z}^d} \int_{m+\mathbb{T}^d}}_{=\int_{\mathbb{R}^d}} f(y) e^{-in\cdot y} dy = \hat{f}(n)$$

$\boxtimes$

· Back to the proof of the Schauder esti. on $\mathbb{T}^d$.

$$\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$$

$$\hat{R}_t(n) = \hat{K}_t(n) = e^{-t|n|^2}$$

$$\| R_t \|_{L^r(\mathbb{T}^d)}$$

$$= \| \sum \hat{K}_t(n) e^{in\cdot x} \|_{L^r(\mathbb{T}^d)}$$

$$\overset{\text{Poisson}}{=} \| \sum_{n \in \mathbb{Z}^d} K_t(x+n) \|_{L^r(\mathbb{T}^d)}$$

$$\langle n \rangle^{-\beta} \langle n \rangle^{\beta} \qquad \beta r' > d$$

$$\lesssim \| \underbrace{\left(\sum \langle n \rangle^{-\beta r'}\right)^{1/r'}}_{<\infty} \| \langle n \rangle^{\beta} K_t(x+n) \|_{\ell^r_n} \|_{L^r_x(\mathbb{T}^d)}$$

$$\lesssim \| \langle x \rangle^{\beta} K_t(x) \|_{L^r_x(\mathbb{R}^d)}$$

$$\sim \| K_t(x) \|_{L^r_x(\mathbb{R}^d)} + \| |x|^{\beta} \underbrace{K_t(x)}_{= \frac{1}{t^{d/2}} K\left(\frac{x}{t^{1/2}}\right)} \|_{L^r_x(\mathbb{R}^d)}$$

$$|x|^\beta \le \left|\frac{x}{t^{1/2}}\right|^{\beta}, \quad \underline{0 < t \le 1}$$

$\Rightarrow$ Repeat the prev argument.

$\Rightarrow$ Schauder est for $e^{t\Delta}$ on $\mathbb{T}^d$, $0 < t \le 1$.

· As for $P(t) = e^{t(\Delta - 1)} = e^{-t} e^{t\Delta}$.

$\uparrow$

can absorb $t^{\beta/2}$, $\forall\, t$.

$\Rightarrow$ Schauder est for $P(t)$ on $\mathbb{T}^d$, $t > 0$

$\boxtimes$

Cor:
$$\| P(t) f \|_{C^{s_1}} \lesssim t^{-\frac{s_1 - s_2}{2}} \| f \|_{C^{s_2}}, \quad s_1 \ge s_2$$

· Back to SNLH on $\mathbb{T}$:

$$u(t) = P(t) u_0 - \int_0^t P(t - t') u^k(t')\, dt' + \underbrace{\int_0^t P(t - t')\, dW(t')}_{=: \Psi}$$

· $\Psi \in C_t C_x^{1/2 -}$

$$\underset{\substack{\| \\ W^{s,\infty}}}{\frac{1}{2} - 2\varepsilon} \supset C^{s+\varepsilon}$$

$\Gamma(u) =$ RHS of the Duhamel formulation.

$$\Rightarrow \|\Gamma(u)(t)\|_{C^s_x} \lesssim \|u_0\|_{C^s_x} + \int_0^t \|u(t')\|^k_{C^s_x}\, dt' + \|\Psi(t)\|_{C^s_x}.$$

$s > 0$

$s < \frac{1}{2}$

$$\Rightarrow \|\Gamma(u)\|_{C_T C^s_x} \lesssim \|u_0\|_{C^s_x} + T\|u\|^k_{C_T C^s_x} + \|\Psi\|_{C_T C^s_x}.$$

$T \leq 1$

& difference estimate

$\Rightarrow$ LWP in $\mathcal{C}^s(\mathbb{T}^d)$, $\quad 0 < s < \frac{1}{2}$.

Rougher data? $\quad u_0 \in \mathcal{C}^s, \ s<0$

$$\| P(t) u_0 \|_{C^\sigma} \lesssim \underbrace{t^{-\frac{\sigma-s}{2}}}_{\text{blows up as } t \to 0+} \|u_0\|_{C^s}, \qquad \sigma \geq s.$$

· $\sigma > s$:

$$\|u\|_{Y^\sigma(T)} = \sup_{0 \leq t \leq T} t^\theta \|u(t)\|_{C^\sigma}$$

$$\Rightarrow \quad t^\theta \|\Gamma(u)(t)\|_{C^\sigma} \lesssim t^\theta t^{-\frac{\sigma-s}{2}} \|u_0\|_{C^s}$$

$$+ \, t^\theta \int_0^t (t')^{-\theta k} \Big(\underbrace{(t')^\theta \|u(t')\|_{C^\sigma}}_{\text{put in } \sup_t} \Big)^k dt'$$

$$+ \, t^\theta \|\Psi(t)\|_{C^\sigma}$$

need $\theta k < 1$

$\Rightarrow$ Take $\sup\limits_{0 \leq t \leq T}$ and get estimate for the $Y^\sigma(T)$-norm

$$\theta - \frac{\sigma - s}{2} \geq 0$$

$s<0$

$$\|\Gamma(u)\|_{C_T C^s_x} \lesssim \|u_0\|_{C^s} + \underbrace{\left\| \int_0^t (t')^{-\theta k} \left((t')^{\theta} \|u(t')\|_{C^\sigma} \right)^k dt' \right\|_{L^\infty_T}}_{T^{1-\theta k} \|u\|^k_{Y^\sigma(T)}} + \|\Psi\|_{C_T C^s_x},$$

C^s is not an algebra.

- $\theta - \frac{\sigma - s}{2} \geq 0$

$$\Rightarrow \underline{\underline{s}} \geq \sigma - 2\theta \qquad \sigma > 0$$
$$> -2\theta \qquad \theta < \frac{1}{k}$$
$$> -\frac{2}{k}$$

(1) Run a contraction argument on a ball in $Y^\sigma(T)$ and show $u \in C_T C^s_x$

$$\Rightarrow u \in C([0,T_\omega]; C^s) \cap C((0,T_\omega]; C^\sigma).$$