Lec 2: 24/02/21 (Wed)

· Stochastic convolution (for SNLS)

$$\Psi(t) = \int_0^t S(t-t')\,\underline{\underline{\phi}}\, dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} e_n \underbrace{\int_0^t e^{i(t-t')|n|^2} \phi_n\, d\beta_n(t')}_{\text{Wiener integral}}$$

$S(t) = e^{-it\Delta}$

$\widehat{S(t)f}(n) = e^{it|n|^2}\hat{f}(n)$

$\underline{\phi(e_n) = \hat{\phi}_n e_n}$

<u>Prop</u>: $s\in\mathbb{R}$, $\phi \in HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$

Then, $\Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}$ a.s. $\forall\, r \le \infty$, $\alpha \gneq 0$

$$\|\phi\|_{HS(L^2;H^s)} = \Big(\sum_{n\in\mathbb{Z}^d} \|\phi(e_n)\|_{H^s}^2\Big)^{\frac12} = \Big(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{\phi}_n|^2\Big)^{\frac12}$$

<u>$r=2$</u>: $\underline{\Psi \in C_t H_x^s}$. a.s.

(<u>On $\mathbb{R}^d$</u>: $\Psi \in L^q_{\substack{t\\0\le t\le T}} W_x^{s,r}$. $q \lneq \infty$, $\underline{r \le \frac{2d}{d-2}}$. de Bouard · Debussche '03, Oh - Pocovnicu - Wang Kyoto '20.

Pf: $t \le \tau$

$$\mathbb{E}\left[\Psi(x,t)\overline{\Psi(y,\tau)}\right] = \sum_{n\in\mathbb{Z}^d} |\widehat{\Phi}_n|^2 e_n(x-y) \underbrace{\int_0^t e^{i(t-t')|n|^2} e^{-i(\tau-t')|n|^2} dt'}$$

$$= \int_0^t e^{i(t-\tau)|n|^2} dt'$$

$$= t\, e^{i(t-\tau)|n|^2}$$

$\Leftarrow$ Apply $\langle \nabla_x \rangle^s$ and $\langle \nabla_y \rangle^s$

$$\langle \nabla \rangle = \sqrt{1-\Delta}$$

$$\widehat{\langle \nabla \rangle^s f}(n) = \langle n \rangle^s \widehat{f}(n)$$

$$\Rightarrow \mathbb{E}\left[\langle \nabla_x \rangle^s \Psi(x,t) \cdot \overline{\langle \nabla_y \rangle^s \Psi(y,\tau)}\right] = \underbrace{\sum_{n\in\mathbb{Z}^d} |\widehat{\Phi}_n|^2 \langle n \rangle^{2s}}_{\text{HS norm}} e_n(x-y) \cdot t\, e^{i(t-\tau)|n|^2}.$$

Set $x=y$, $t=\tau$

$= 1$ when $x=y$, $t=\tau$.

- $$\mathbb{E}\left[|\langle \nabla \rangle^s \Psi(x,t)|^p\right] \le p^{p/2}\left(\mathbb{E}\left[|\langle \nabla \rangle^s \Psi(x,t)|^2\right]\right)^{p/2}$$

$$= p^{p/2} t^{p/2} \|\Phi\|^p_{HS(L^2; H^s)}$$

- $p \ge r$, $r < \infty$

$$\Big\| \|\Psi(t)\|_{W^{s,r}_x} \Big\|_{L^p(\Omega)} \underset{\text{Mink ineq}}{\le} \Big\| \|\langle \nabla \rangle^s \Psi(x,t)\|_{L^p(\Omega)} \Big\|_{L^r_x}$$

$$\|\Psi(t)\|_{W^{s,r}_x} = \|\langle \nabla \rangle^s \Psi(x,t)\|_{L^r_x}$$

$$\lesssim p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2; H^s)} \cdot \quad \forall p \ge r$$

$\Leftarrow$ · $\Psi(t) \in W_x^{s,r}(\mathbb{T}^d)$, a.s. $(r<\infty)$

· Here, we used the fact $|\mathbb{T}^d| \sim 1$.

· When $r=\infty$, $\underline{\varepsilon r > d}$

$$\Big\| \|\Psi(t)\|_{W_x^{s,\infty}} \Big\|_{L^p(\Omega)} \overset{\text{Sobolev}}{\lesssim} \Big\| \|\Psi(t)\|_{W_x^{s+\varepsilon,r}} \Big\|_{L^p(\Omega)}$$

$$\overset{\text{for } p\geq r}{\lesssim} p^{1/2} t^{1/2} \|\phi\|_{HS(L^2;H^{s+\varepsilon})}.$$

In other words, $\Psi(t) \in W_x^{s-\varepsilon,\infty}$, if $\phi \in HS(L^2;H^s)$

· $P\left(\|\Psi(t)\|_{W_x^{s,r}} > \lambda\right)$, $r<\infty$

$$\leq C e^{-c\lambda^2 / t\|\phi\|_{HS(L^2;H^s)}}$$

· Sobolev embedding thm

$$\|f\|_{L^\infty} \lesssim \|f\|_{W^{s,r}}$$

$sr > d$

· Sobolev inequality

Grafakos, Benyi-Oh

$$\|f\|_{L^q} \lesssim \|f\|_{\dot{W}^{s,p}}$$

f, mean 0 on $\mathbb{T}^d$. $\quad \frac{s}{d} = \frac{1}{p} - \frac{1}{q} \quad 1<p<q<\infty$

- Given $h \in \mathbb{R}$, let

$$\delta_h \Psi(x,t) = \Psi(x, t+h) - \Psi(x,t)$$

$$\Rightarrow \mathbb{E}\left[\delta_h \Psi(x,t) \overline{\delta_h \Psi(y,t)} \right]$$

$$= \mathbb{E}\left[\Psi(x,t+h) \overline{\Psi(y,t+h)} \right]$$

$$- \mathbb{E}\left[\Psi(x,t) \overline{\Psi(y,t+h)} \right]$$

$$- \mathbb{E}\left[\Psi(x,t+h) \overline{\Psi(y,t)} \right]$$

$$+ \mathbb{E}\left[\Psi(x,t) \overline{\Psi(y,t)} \right]$$

$t > 0$
$h > 0$

$$= \sum_{n \in \mathbb{Z}^d} |\widehat{\Phi}_n|^2 e_n(x-y) \underbrace{\left\{ (t+h) - t e^{-ih|n|^2} - t e^{ih|n|^2} + t \right\}}_{=: F_n(t,h) \to 0 \text{ as } h \to 0}$$

and $|F_n(t,h)| \lesssim |t| + |h|$.

$$|F_n(t,h)| \le |h| + t\,\underbrace{|1-e^{-ih|n|^2}|} + t\,\underbrace{|1-e^{+ih|n|^2}|}$$ ⑤

$$\lesssim |h| + t\min(1, |h||n|^2)$$

$$\lesssim |h| + t\,|h|^{\alpha}|n|^{2\alpha}, \quad 0\le\alpha\le 1$$

$$\Rightarrow \|\langle\nabla\rangle^s \delta_h \Psi(x,t)\|_{L^p(\Omega)}$$

$$\le p^{1/2}\|\langle\nabla\rangle^s \delta_h \Psi(x,t)\|_{L^2(\Omega)}$$

$$\lesssim p^{1/2}(1+T)^{1/2}|h|^{\alpha/2}\Big(\sum_n \langle n\rangle^{2(s+\alpha)}|\hat{\Phi}_n|^2\Big)^{1/2}, \quad \forall t\in[0,T]$$

• $p\ge r$, $r<\infty$:

$$\big\|\,\|\delta_h\Psi(t)\|_{W_x^{s,r}}\big\|_{L^p(\Omega)} \qquad \text{Mink integ ineq}$$

$$\lesssim C_T\, p^{1/2}\, \|\Phi\|_{HS(L^2;H^{s+\alpha})}\, |h|^{\alpha/2}, \quad \forall t\in[0,T]$$

$\Rightarrow$ Kolmogorov conti criterion

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2}-\frac{1}{p}-} W_x^{s-\alpha,r}, \quad \text{a.s. if } \Phi\in HS(L^2;H^s)$$

(here, we replaced $s+\alpha$ by s)

$$\overset{p\to\infty}{\Longrightarrow} \quad \Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}, \text{ a.s. if } \Phi \in HS(L^2; H^s)$$

$$r \le \infty, \quad \alpha > 0$$

- When $r=2$, it suffices to study the continuity property of

$$\widetilde{\Psi}(t) = S(-t)\Psi(t) = \int_0^t S(-t')\Phi\, dW(t')$$

$$\Rightarrow E\left[\widetilde{\Psi}(x,t)\overline{\widetilde{\Psi}(y,\tau)}\right] = \sum_n |\widehat{\Phi}_n|^2 e_n(x-y)\cdot \underline{\underline{t}}. \qquad t \le \tau$$

By repeating analogous computation,

$$\Big\| \, \| \delta_h \widetilde{\Psi}(t) \|_{H_x^s} \Big\|_{L^p(\Omega)} \le C_T\, p^{1/2}\, \|\Phi\|_{HS(L^2;H^s)}\, |h|^{1/2}$$

$\Rightarrow$ Kolmogorov conti criterion,

$$\widetilde{\Psi} \in C_t^{\frac12 -} H_x^s, \text{ a.s.}$$

$$\rightarrow \Psi \in C_t H_x^s, \text{ a.s.} \qquad \left\{ \begin{aligned} &\Psi(t+h) - \Psi(t) \\ &= S(t+h)\left[\widetilde{\Psi}(t+h) - \widetilde{\Psi}(t)\right] \\ &\quad + \left[S(t+h) - S(t)\right]\widetilde{\Psi}(t) \end{aligned} \right.$$

$\boxtimes$

ex: $\phi = Id \Rightarrow \phi \in HS(L^2; H^s)$, $s < -\frac{d}{2}$

$\Rightarrow \Psi_{Sch} \in C_t H_x^s$, a.s., $s < -\frac{d}{2}$

spatial regularity of (space-time) white noise

· Wave & heat? $\phi = Id$.

$$\Psi_{wave}(t) = \int_0^t \frac{\sin(t-t')\langle\nabla\rangle}{\boxed{\langle\nabla\rangle}} dW(t'), \quad (\partial_t^2 + 1 - \Delta)\Psi = \xi$$

one gain of derivative

$$\Psi_{heat}(t) = \int_0^t P(t-t')\, dW(t'), \quad (\partial_t + 1 - \Delta)\Psi = \xi$$

$$P(t) = e^{t(\Delta-1)}$$

In the first step, we get $\int_0^t e^{-2t'\langle n\rangle^2} dt'$ $\downarrow$ $\frac{1}{\langle n\rangle^2}$

$$\widehat{P(t)f}(n) = e^{-t(1+|n|^2)} \hat{f}(n)$$

$\mathbb{E}[\Psi(x,t)\Psi(y,\tau)]$ $\quad e^{-t\langle n\rangle^2} e^{-\tau\langle n\rangle^2}$

Taking a square root, it gives one gain of deriv.

$\Rightarrow \quad \Psi_{\text{wave}}, \Psi_{\text{heat}} \in C_t W_x^{s,\infty}(\mathbb{T}^d)$, a.s. $\quad s < -\frac{d}{2}+1$

$(\phi = Id)$

Chap 1: One-dimensional case

$$\text{SNLS}: \begin{cases} i\partial_t u - \Delta u + |u|^{k-1} u = \phi \xi \\ u|_{t=0} = u_0 \end{cases}, \quad x \in \mathbb{T}^d.$$

- $s > \frac{d}{2}$: $\quad \phi \in HS(L^2; H^s)$
  $u_0 \in H^s(\mathbb{T}^d)$

Duhamel: $\quad u(t) = S(t) u_0 + i \int_0^t S(t-t') |u|^{k-1} u(t') \, dt' + \Psi(t)$

$$=: \Gamma_{u_0,\phi}(u)$$

- When $s > \frac{d}{2}$, H^s is an algebra.

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

$$\left(\begin{array}{l} \cdot \ \langle n_1 + n_2 \rangle^s \lesssim \langle n_1 \rangle^s + \langle n_2 \rangle^s, \quad s \geq 0. \\ \cdot \ \|\hat{f} * \hat{g}\|_{\ell^2_n} \leq \|\hat{f}\|_{\ell^2_n} \underbrace{\|\hat{g}\|_{\ell^1_n}}_{\underset{C\text{-}S.}{\lesssim} \|g\|_{H^s}} \end{array} \right.$$

$$\|\Gamma_{u_0,\phi}(u)\|_{C_T H^s_x}$$

$C_T H^s_x = C([0,T]; H^s)$

$S(t)$, unitary on H^s

$$\leq \underline{\underline{\|u_0\|_{H^s}}} + \underbrace{\Big\| \overset{\leq T}{\int_0^t} \underbrace{\|\cancel{S(t-t')}\, |u|^{k-1} u(t')\|_{H^s_x}}_{\lesssim \|u(t')\|^k_{H^s_x}} dt' \Big\|_{L^\infty_T}}_{\leq C_1 T \|u\|^k_{C_T H^s_x}},$$

$k \in 2\mathbb{N}+1$.

$$+ \underline{\underline{\|\Psi\|_{C_T H^s_x}}}$$

$$R = \underline{\underline{R_\omega}} = 2\left(\|u_0\|_{H^s} + \|\Psi\|_{C([0,1];H^s)} \right)$$

random

NOT T.

$$\|\Gamma_{u_0,\phi}(u)\|_{C_T H^s} \le \frac{1}{2} R + C_1 T R^k \qquad T \le 1.$$

$$\le R. \qquad \forall u \in B_R \subset C_T H^s_x$$

by choosing $T = T\omega = T(R\omega) \ll 1$

$$T \sim R^{-1/\underline{\underline{k-1}}}$$

$\Rightarrow \Gamma_{u_0,\phi} : B_R \circlearrowleft.$

$$\|\Gamma_{u_0,\phi}(u) - \Gamma_{u_0,\phi}(v)\|_{C_T H^s_x} \le C_2 T \left(\|u\|^{k-1}_{C_T H^s_x} + \|v\|^{k-1}_{C_T H^s_x} \right) \times \|u - v\|_{C_T H^s_x}$$

$$|u|^{k-1} u - |v|^{k-1} v \qquad \le \underbrace{C_2 T R^{k-1}}_{\le 1/2} \|u - v\|_{C_T H^s_x}$$

$\Leftarrow$ write in a telescoping sum.

& Young's ineq $\quad ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$

$\Rightarrow$ $\Gamma_{u_0,\Phi}$ is a contraction on $B_R \subset C_T H^s_x$.

By Banach fixed pt thm, $\exists!$ $u \in B_{R_\omega}$ s.t. $u = \Gamma_{u_0,\Phi}(u)$

$[0, T_\omega]$

- time of local existence is random
- conti dependence on $u_0 \in H^s$ and $\Phi \in HS(L^2; H^s)$

Suppose $u = \Gamma_{u_0,\Phi_1}(u)$, $v = \Gamma_{v_0,\Phi_2}(v)$

$$\Rightarrow \|u - v\|_{C_T H^s_x} \lesssim \|u_0 - v_0\|_{H^s} + \underbrace{\|\Psi_1 - \Psi_2\|_{C_T H^s_x}}_{\int_0^t S(t-t')(\Phi_1 - \Phi_2)\, dW(t')}$$

$$\leq C_\omega \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}$$

· Here, we find the nonlinear part on LHS.

·

$$\|\Psi_1 - \Psi_2\|_{C_T H^s_x} \leq \underset{\uparrow \text{ random const}}{C_\omega} \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}$$

$$\Leftarrow \quad \mathbb{E}\left[\|\Psi_1 - \Psi_2\|^2_{C_T H^s_x}\right] \leq C_T \|\Phi_1 - \Phi_2\|^2_{HS(L^2; H^s)}$$

and Chebyshev's ineq.