Lec 16 19/05/21 (Wed)

Paracontrolled distribution:

We say f is paracontrolled by g if we have

$$\boxed{f = f' \olessthan g + R}$$

$f' \sim$ Gubinelli derivative

for some f' and smoother R.

GIP '15: $f \in C^\alpha$, $g \in C^\alpha$, $f' \in C^\alpha$
$R \in C^\beta$, $\beta > \alpha$.

Parabolic Φ^4_3-model: $X = (\cdots) \olessthan \Upsilon + \mathrm{com}_1(X, Y)$

$$\left(\Leftarrow (\partial_t + 1 - \Delta) X = (\cdots) \olessthan \vee \right.$$

$$\Rightarrow v = X + Y = (\cdots) \olessthan \Upsilon + R \quad \swarrow \text{smoother}$$

· <u>Controlled rough path</u> (Gubinelli JFA '04).

We say that <u>Y is controlled by X</u> if

$$Y_{s,t} = Y'_s X_{s,t} + R_{s,t} \qquad Y_{s,t} = Y_t - Y_s$$

for some Y' and smoother R.

$\uparrow$ Gubinelli derivative

<u>ex</u>: $Y, X, Y' \in C^\alpha_t$, $R \in C^{2\alpha}$.

Moral: local fluctuation of $Y \approx$ " of X.

(f paracontrolled by g

$\Rightarrow$ small scale behavior of $f \approx$ " of g.

3.2 quadratic SNLW on $\mathbb{T}^3$

$$
(\text{SNLW}) \quad \begin{cases} (\partial_t^2 + 1 - \Delta) u + u^2 = \xi & \text{on } \mathbb{R}_+ \times \mathbb{T}^3 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}
$$

$d = 3$: ξ has spatial regularity $-\frac{3}{2} - \varepsilon$

$$
\Rightarrow \quad \Psi = \mathcal{I}(\xi) \sim -\tfrac{1}{2} - \varepsilon.
$$

· parabolic case:

$$
(\text{SNLH}) \quad (\partial_t + 1 - \Delta) u + u^2 = \xi.
$$

$$
\Rightarrow \quad \Psi = \mathcal{I}_{heat}(\xi) \sim -\tfrac{1}{2} - \varepsilon.
$$

Da Prato-Debussche: $u = \mathsf{I} + v$.

$$(\text{SNLH}) \Rightarrow (\partial_t + 1 - \Delta) v = -(v + \mathsf{I})^2$$

$$\text{“}=\text{”} - v^2 - \underline{2 v \mathsf{I}} - \underset{-1-}{\mathsf{V}}$$

↑ Wick renormalization

$$\Rightarrow \quad v \sim 1- = (-1-) + 2$$

$$\Rightarrow \quad v\mathsf{I} \quad \leftarrow \text{The product is well defined.}$$

$$(1-) + (-\tfrac{1}{2}-) > 0$$

$\Rightarrow$ can perform a contraction argument in $C_T C_x^{1-}$

Back to the wave case:

$$\text{ᑊ} = \mathcal{I}(\xi) \sim -\tfrac{1}{2}-$$

$$\text{V} = \lim_{N\to\infty} \underbrace{\left(\text{ᑊ}_N^2 - \sigma_N\right)}_{= \text{V}_N} \qquad \text{ᑊ}_N = P_{\le N}\text{ᑊ}$$

$$\sigma_N = \mathbb{E}\left[\text{ᑊ}^2(t,x)\right] \sim tN.$$

$\Rightarrow$ $\text{V}_N \to \text{V}$ in $C_T W_x^{-1-,\infty}$, a.s.

2nd order object: $\underline{\underline{\text{Y} = \mathcal{I}(\text{V})}} = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}\text{V}(t')dt'$

A naive "parabolic thinking" gives

$$0- = \left(-\tfrac{1}{2}-\right) + \left(-\tfrac{1}{2}-\right) + 1 \quad \Leftarrow \text{Not good.}$$

· Multilinear dispersive smoothing gives extra $\frac{1}{2}$-smoothing.

Prop: $\Upsilon_N = \mathcal{I}(\curlyvee_N) \longrightarrow \Upsilon$

in $C_T W_x^{\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-1-\varepsilon,\infty}$, a.s.

· Second order expansion: $u = \overset{\bullet}{|} - \Upsilon + v$

$$(\text{SNLW}) \Rightarrow (\partial_t^2 + 1 - \Delta) v = -(v + \overset{\bullet}{|} - \Upsilon)^2 + \curlyvee$$

$$\text{"="} - (v - \Upsilon)^2 - 2\underbrace{v\overset{\bullet}{|}}_{-\frac{1}{2}-} + 2\underbrace{\overset{\bullet}{|}\Upsilon}_{-\frac{1}{2}-}$$

$\Rightarrow$ expect $v \sim \frac{1}{2}-$

$\Rightarrow$ $v\overset{\bullet}{|}$: $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0$

↑ NOT well defined.

Paracontrolled ansatz: $\underset{s_1}{v = X} + \underset{s_2}{Y}$ $\qquad 0 < s_1 < s_2$.

$$(\partial_t^2 + 1 - \Delta) X = -2(X + Y - \Upsilon) \textcircled{<} \Psi$$

$$(\partial_t^2 + 1 - \Delta) Y = -(X + Y - \Upsilon)^2 - 2(X + Y - \Upsilon) \underset{\textcircled{>} + \textcircled{=}}{\underset{\uparrow}{\textcircled{\geq}}} \Psi$$

$$\left(\begin{array}{l} \Box v = (\cdots) \textcircled{<} \Psi + R \\ \uparrow \text{ paracontrolled} \end{array} \right.$$

Expect: $X \sim \frac{1}{2}- = (-\frac{1}{2}-) + 1$

For now, ignore $\textcircled{=}$ in the Y-eqn.

$$(X + Y - \Upsilon) \textcircled{>} \Psi \sim (\tfrac{1}{2}-) + (-\tfrac{1}{2}-) \sim 0-$$

$$\Rightarrow Y \sim 1-$$

· Need to make sense of the resonant product

$$\left(X + Y - \stackrel{\bullet\,\bullet}{Y} \right) \mathbin{\textcircled{=}} \stackrel{\bullet}{|}$$

① $Y \mathbin{\textcircled{=}} \stackrel{\bullet}{|}$

$(1-) + (-\tfrac{1}{2}-) = \tfrac{1}{2}- > 0$ good as long as $s_2 > 1/2$

② $\stackrel{\bullet\,\bullet}{Y}{}_{\circ}\!\!\diagup^{\bullet} = \underline{\stackrel{\bullet\,\bullet}{Y} \mathbin{\textcircled{=}} \stackrel{\bullet}{|}}$

$0- = (\tfrac{1}{2}-) + (-\tfrac{1}{2}-)$

· Once again, we need to exploit multilinear dispersive smoothing.

· <u>without</u> renormalization

($\Leftarrow$ there was NO extra renormalization in the parabolic case

Prop: $\Upsilon\!\!\!\!\bullet_{\ominus N} = \Upsilon_N \ominus \mathbf{I}_N \longrightarrow \Upsilon\!\!\!\!\bullet_{\ominus}$ in $C_T W_x^{-\varepsilon,\infty}$, a.s.

3) $X \ominus \mathbf{I}$

$$\left(\tfrac{1}{2}-\right) + \left(-\tfrac{1}{2}-\right) < 0$$

$\Rightarrow$ NOT well defined.

· Use the Duhamel formulation for X:

$$X(t) = \underbrace{\partial_t S(t) X_0 + S(t) X_1}_{=: \underbrace{\vec{S}(t)(X_0, X_1)}_{\text{NOT smoother!!}}} - 2\mathcal{I}\big((X + \Upsilon - \Upsilon) \ominus \mathbf{I}\big)$$

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$$

<u>Lemma:</u> $s_1 > 0$. $(X_0, X_1) \in \mathcal{H}^{s_1}(\mathbb{T}^3)$.

$$Z_N = \left(\vec{S}(t)(X_0, X_1)\right) \circleddash \,\text{?}_N$$

depends on (X_0, X_1)

$$\downarrow$$

$$Z = \left(\vec{S}(t)(X_0, X_1)\right) \circleddash \,\text{?} \quad \text{in } C_T H_x^{s_1 - \frac{1}{2} - \varepsilon}, \quad \underline{\text{a.s.}}$$

need to include this in the enhanced data set.

<u>Note:</u> set of prob 1 depends on the initial data (X_0, X_1).

- Duhamel $\circleddash$? : In the parabolic case, we introduced two commutators.

$$\text{com}_1 = [\text{Duhamel integral}, \circledless], \quad \text{com}_2 = [\circledless, \circleddash]$$

Bad news: In the wave case,

$$\left[\int_0^t S(t-t') \cdots dt', \circledcirc\right]$$ does NOT provide any smoothing

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Main idea: directly study the following paracontrolled operators

Given $w \in C(\mathbb{R}_+; H^{s_1}(\mathbb{T}^3))$, $0 < s_1 < 1/2$,

define

$$\boxed{\mathcal{J}_{\circledcirc}(w)(t) = \mathcal{I}(w \circledcirc \,\mathsf{I}\,)(t)}$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \sum_{\substack{n = n_1 + n_2 \\ \boxed{|n_1| \ll |n_2|}}} \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle} \widehat{w}(t', n_1)\, \widehat{\mathsf{I}}(t', n_2)\, dt'$$

signifies the para product (formal)

<u>Goal</u>: Make sense of $\mathcal{J}_{\circledcirc}(w) \circleddash \dot{\uparrow}$

$$\left(\text{with } w = X + Y - \dot{Y}. \right)$$

· Divide $\mathcal{J}_{\circledcirc}$ into <u>good and bad parts</u>

Fix $\theta > 0$ small.

$$\mathcal{J}_{\circledcirc} = \mathcal{J}^{(1)}_{\circledcirc} + \mathcal{J}^{(2)}_{\circledcirc}$$

restriction of $\mathcal{J}_{\circledcirc}$ onto $\{|n_1| \gtrsim |n_2|^{\theta}\}$

"$|n_1|$ is NOT too small"

Want $\sim \frac{1}{2} + 2\varepsilon$.

$$\langle m \rangle^{\frac{1}{2}+2\varepsilon} \frac{1}{\langle m \rangle} = \langle m \rangle^{-\frac{1}{2}+2\varepsilon} \lesssim \langle n_1 \rangle^{\frac{4\varepsilon}{\theta}} \underline{\langle m_2 \rangle^{-\frac{1}{2}-2\varepsilon}}$$

$\searrow$ put on $\widehat{\dot{\uparrow}}(t', n_2)$

$$\lesssim \langle n_1 \rangle^{s_1-\varepsilon} \langle n_2 \rangle^{-\frac{1}{2}-2\varepsilon}$$

by choosing $\varepsilon = \varepsilon(s_1, \theta) > 0$ suff. small.

· <u>Lemma</u> : $0 < s_1 < 1/2$. Given small $\theta > 0$,

$\exists$ small $\varepsilon = \varepsilon(s_1, \theta) > 0$ s.t.

given any $\Xi \in C(\mathbb{R}_+, W^{-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$,

the paracontrolled operator

$$\mathcal{J}^{(1), \Xi}_{\circledcirc} = \mathcal{J}^{(1)}_{\circledcirc} \text{ with } 1 \text{ replaced by } \Xi$$

belongs to $\mathcal{L}_2 = \mathcal{L}(C_T H^{s_1}_x \,;\, C_T H^{\frac{1}{2}+2\varepsilon}_x)$.

(i.e. NO stochastic analysis is needed.

· As for $\mathcal{J}^{(2)}_{\circledcirc}$, i.e. $|m_1| \ll |m_2|^\theta$, the positive regularity of w does not help.

$\Rightarrow$ We use stochastic analysis to directly study $\mathcal{J}_{\circledless, \circleddash}$

$$\boxed{\mathcal{J}_{\circledless, \circleddash}(w)(t) = \mathcal{J}^{(2)}_{\circledless}(w)(t) \circleddash \varphi(t)}$$

$$= \mathcal{I}(w \circledless \varphi)(t) \circleddash \varphi(t).$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \int_0^t \sum_{n_1} \widehat{w}(t', n_1) \underline{A_{n,n_1}(t, t')} dt'.$$

$$A_{n,n_1}(t,t') = \mathbb{1}_{[0,t]}(t') \sum_{\substack{n - n_1 = n_2 + n_3 \\ |n_1| \ll |n_2|^\theta \\ |n_1 + n_2| \sim |n_3|}} \frac{\sin((t-t')\langle n_1 + n_2 \rangle)}{\langle n_1 + n_2 \rangle} \boxed{\widehat{\varphi}(t', n_2)\, \widehat{\varphi}(t, n_3)}$$

$|n_1 + n_2| \sim |n_3| \leftarrow$ signifies $\circleddash$

$$= \mathbb{1}_{[0,t]}(t') \sum \frac{\sin((t-t')\langle n_1+n_2\rangle)}{\langle n_1+n_2\rangle} \Big(\hat{\rho}(t', n_2)\, \hat{\rho}(t, n_3) - \mathbb{1}_{n_2+n_3=0}\, \sigma_{n_2}(t,t') \Big)$$

$$+ \mathbb{1}_{[0,t)}(t') \cdot \mathbb{1}_{n=n_1} \sum_{\substack{n_2 \\ |n| \ll |n_2|^\theta}} \frac{\sin((t-t')\langle n+n_2\rangle)}{\langle n+n_2\rangle} \sigma_{n_2}(t,t')$$

NOT abs conv. $\qquad \frac{1}{\langle n+n_2\rangle} \frac{1}{\langle n_2\rangle^2}$

$$=: \mathcal{A}^{(1)}_{n,n_1}(t,t') + \underbrace{\mathcal{A}^{(2)}_{n}(t,t')}$$

deterministic counter term

$0 \le t_2 \le t_1$

$$\sigma_n(t_1,t_2) = \mathbb{E}\big[\hat{\rho}(t_1, n)\, \hat{\rho}(t_2, n) \big]$$

$$= \frac{\cos((t_1-t_2)\langle n\rangle)}{2\langle n\rangle^2} t_2 + O\Big(\frac{1}{\langle n\rangle^3}\Big)$$

- exploit dispersion, stationary phase, symmetrization ($n_2 \leftrightarrow -n_2$).

i.e. the order of summations matter.
$\Rightarrow$ only conditionally convergent.

<u>Prop:</u> $s_2 < 1$. $\exists$ small $\theta(s_2) > 0$ and $\varepsilon > 0$ s.t.

$$\mathcal{J}_{\ominus,\ominus} \in \mathcal{L}_1 = \mathcal{L}\left(C_T L^2_x \cap \underline{C^1_T H^{-1-\varepsilon}_x}\,;\, C_T H^{s_2-1}_x\right) \text{ a.s.}$$

$\mathcal{J}^N_{\ominus,\ominus}$ (with Υ replaced by Υ_N) $\to \mathcal{J}_{\ominus,\ominus}$ in $\mathcal{L}_1$, a.s.

GKO, JEMS

can drop this assumption

Oh-Okamoto-Tolomeo '20.

We arrive at the following system:

$$
(\mathrm{SNLW}')\qquad
\begin{aligned}
(\partial_t^2+1-\Delta)X &= -2(X+Y-\Upsilon)\mathbin{\textcircled{$<$}}\Psi\\
(\partial_t^2+1-\Delta)Y &= -(X+Y-\Upsilon)^2-2(X+Y-\Upsilon)\mathbin{\textcircled{$>$}}\Psi\\
&\quad -2Y\mathbin{\textcircled{$=$}}\Psi+2\,\underline{\mathcal{Y}}-2\,\underline{\underline{Z}}\\
&\quad +4\,\mathcal{J}^{(1)}_{\textcircled{$<$}}(X+Y-\Upsilon)\mathbin{\textcircled{$=$}}\Psi\\
&\quad +\underline{4\,\mathcal{J}_{\textcircled{$<$},\textcircled{$=$}}}(X+Y-\Upsilon)
\end{aligned}
$$

$$(X,\partial_tX,Y,\partial_tY)\big|_{t=0}=(X_0,X_1,Y_0,Y_1).$$

<u>Thm:</u> $\frac{1}{4} < s_1 < \frac{1}{2} < s_2 \leq s_1 + \frac{1}{4}$.

Then, (SNLW′) is locally well-posed in $\mathcal{H}^{s_1} \times \mathcal{H}^{s_2}$.

- $(X, Y) \in Z_T^{s_1, s_2} = X_T^{s_1} \times Y_T^{s_2}$

$$X_T^{s_1} = C_T H_x^{s_1} \cap C_T^1 H_x^{s_1 - 1} \cap L_T^8 W_x^{s_1 - \frac{1}{4}, \frac{8}{3}} \qquad (8, \tfrac{8}{3}),\ \tfrac{1}{4}\text{-admis}$$

$$Y_T^{s_2} = C_T H_x^{s_2} \cap C_T^1 H_x^{s_2 - 1} \cap L_T^4 W_x^{s_2 - \frac{1}{2}, 4} \qquad (4, 4),\ \tfrac{1}{2}\text{-admis.}$$

- enhanced data set

$$\Xi = (X_0, X_1, Y_0, Y_1, \Psi, \underline{\mathbb{Y}}, \mathbb{Y}^{\ominus}, \underset{\vec{S}(X_0,X_1) \ominus \Psi}{\overset{\shortparallel}{\mathbb{Z}}}, \mathfrak{J}^{\ominus,\ominus})$$

$$\in \mathcal{H}^{s_1} \times \mathcal{H}^{s_2} \times C_T W_x^{-\frac{1}{2}-\varepsilon, \infty} \times \underline{\left(C_T W_x^{\frac{1}{2}-\varepsilon} \cap C_T^1 W_x^{-1-\varepsilon, \infty} \right)}$$

$$\times\, C_T W_x^{-\varepsilon, \infty} \times C_T H_x^{s_1 - \frac{1}{2} - \varepsilon} \times \mathcal{L}_1 =: \mathcal{X}_T^{s_1, s_2, \varepsilon}$$

- soln map is continuous from $\mathcal{X}_T^{s_1, s_2, \varepsilon} \longrightarrow Z_T^{s_1, s_2}$

$$\Xi \longrightarrow (X, Y).$$

- Proof is done by the Strichartz esti & a contraction argument.

- $$u_N = \overset{\bullet}{|}_N - \overset{\bullet\,\bullet}{\mathrm{Y}}_N + \underbrace{X_N + Y_N}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$u = \overset{\bullet}{|} - \overset{\bullet\,\bullet}{\mathrm{Y}} + X + Y \quad \text{in } C_T H_x^{-\frac{1}{2}-\varepsilon}$$

· **Remarks**: · quadratic nonlin $\Leftarrow$ neither defocusing or focusing.

but the Gibbs measure

$$d\rho = Z^{-1} e^{-\sigma \int :u^3: dx \; - A\left(\int :u^2: dx\right)^{\gamma}} d\mu$$

can be constructed in the weakly nonlinear regime ($0<\sigma\ll 1$)

(When $\sigma \gg 1$, ρ is non-normalizable.)

$\Rightarrow$ a.s. GWP. (O-Ok-Tol '21)

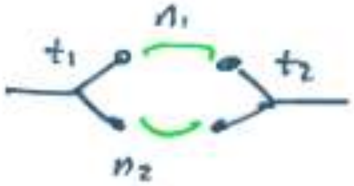
· The real question is the hyperbolic Φ^4_3-model.

Bringmann '20: Hartree nonlinearity $(V * u^2)u$

$\widehat{V}(n) \sim \langle n \rangle^{-\varepsilon}$, $\forall \varepsilon > 0$.

- <u>Key point on $\overset{\bullet}{Y}$:</u> By the regularity lemma, we study

$$\mathbb{E}\left[|\widehat{\overset{\bullet}{Y}}(t,n)|^2 \right]$$



$$= 4 \sum_{\substack{n = n_1 + n_2 \\ n_1 \neq \pm n_2}} \int_0^t \frac{\sin (t - t_1)\langle n \rangle}{\langle n \rangle} \int_0^{t_1} \frac{\sin (t - t_2)\langle n \rangle}{\langle n \rangle}$$

$$\times \; \sigma_{n_1}(t_1, t_2)\, \sigma_{n_2}(t_1, t_2)\, dt_2\, dt_1$$

$$+ \quad \sim_{n_1 = n_2}$$

$$\frac{\cos (t_1 - t_2)\langle n_1 \rangle}{2\langle n_1 \rangle^2}\, t_2 + O\left(\frac{1}{\langle n \rangle^3}\right)$$

$$0 \leq t_2 \leq t_1$$

$\Leftarrow$ expand sines and cosines in complex exponentials

$$\Rightarrow \sum_{\substack{n=n_1+n_2\\ n_1\neq \pm n_2}} \frac{e^{i(\varepsilon_1+\varepsilon_2)t\langle n\rangle}}{\langle n\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2} \int_0^t e^{-it_1K_1(\bar n)}\int_0^{t_1} t_2^2\, e^{-it_2K_2(\bar n)}\,dt_2\,dt_1$$

$$K_1(\bar n) = \varepsilon_1\langle n\rangle - \varepsilon_3\langle n_1\rangle - \varepsilon_4\langle n_2\rangle$$

$$K_2(\bar n) = \varepsilon_2\langle n\rangle + \varepsilon_3\langle n_1\rangle + \varepsilon_4\langle n_2\rangle$$

$\varepsilon_j \in \{\pm 1\}$.

Integrate in t_1 first.

$$\left|\int_0^t t_2^2\, e^{-it_2K_2(\bar n)}\, \frac{e^{-itK_1(\bar n)} - e^{-it_2K_1(\bar n)}}{-iK_1(\bar n)}\,dt\right|$$

$$\lesssim \frac{C(T)}{1+|K_1(\bar n)|}$$

Need to bound

$$I = \sum_{n=n_1+n_2} \frac{1}{\langle n\rangle^2 \langle n_1\rangle^2 \langle n_2\rangle^2} \underline{\frac{1}{1+|K_1(\bar n)|}} \qquad \left(\overset{\text{WTS}}{\lesssim} \langle n\rangle^{-4+}\right)$$

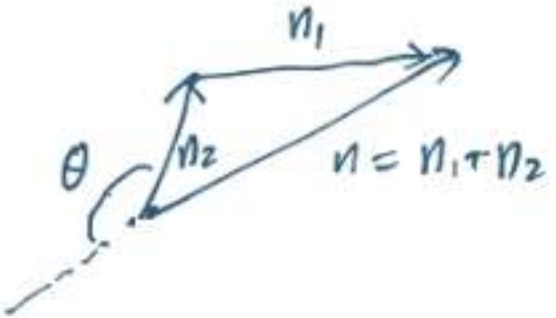
$$\Rightarrow \quad s < 1/2$$

· <u>Bad case</u> $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \pm 1, \mp 1)$

$$|K_1(\bar n)| = \langle n\rangle + \langle n_2\rangle - \langle n_1\rangle$$

$\left(\begin{array}{l}\text{Assume} \\ |n| \gg 1 \\ |n_1| \geq |n_2|\end{array}\right.$

$$\Rightarrow \langle n_1\rangle \sim \langle n\rangle + \langle n_2\rangle$$



<u>law of cosines</u>

$$\underline{|n|^2 + |n_2|^2 - |n_1|^2} = 2|n||n_2| \cos(\angle(n, n_2))$$

$$\Rightarrow |K_1(\bar n)| = \frac{(\langle n\rangle + \langle n_2\rangle)^2 - \langle n_1\rangle^2}{\langle n\rangle + \langle n_1\rangle + \langle n_2\rangle} = \frac{2\langle n\rangle\langle n_2\rangle + \boxed{|n|^2 + |n_2|^2 - |n_1|^2} + 1}{\langle n\rangle + \langle n_1\rangle + \langle n_2\rangle}$$

$$\Rightarrow \quad |K_1(\bar{n})| \gtrsim \frac{|n|\,|n_2|\,(1-\cos\theta)}{\langle n_1\rangle} \qquad \theta = \angle(n_2, -n)$$

<u>case 1:</u> $1-\cos\theta \gtrsim 1.$ (large angle)

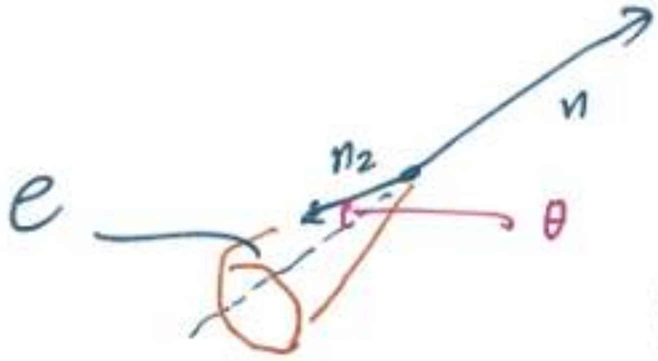
$$I \lesssim \sum \frac{1}{\langle n\rangle^3 \langle n_1\rangle \langle n_2\rangle^3} \qquad \langle n_1\rangle \sim \max(\langle n\rangle, \langle n_2\rangle)$$

$$\lesssim \langle n\rangle^{-4+}$$

<u>case 2:</u> $1-\cos\theta \ll 1$ (nearly) resonant.

$$\Rightarrow \quad 0 \le \theta \ll 1$$

$$\Rightarrow \quad 1-\cos\theta \sim \theta^2 \ll 1$$



Dyadically decompose $|n_2| \sim N_2$

$N_2 \geq 1$, dyadic

For fixed $n \in \mathbb{Z}^3$,

$n_2 \in$ cone $\mathcal{C}$ height $\sim N_2 \cos\theta \sim N_2$

$|n_2| \sim N_2$ radius of the base

$\sim N_2 \sin\theta \sim N_2 \theta$

$\Rightarrow \mathrm{vol}(\mathcal{C}) \sim N_2^3 \theta^2$

$$\Rightarrow I \lesssim \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^3 \max(\langle n \rangle, N_2)\, \cancel{N_2^3 \theta^2}} \times \underbrace{\cancel{N_2^3 \theta^2}}_{\sum_{n_2}}$$

$$\lesssim \langle n \rangle^{-4+}$$

Actually. $\displaystyle\sum_{n_2} 1 \lesssim \underline{\underline{1}} + \mathrm{vol}(\mathcal{C})$

So, to be correct, we go back to

$$I = \sum_{n = n_1 + n_2} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2} \frac{1}{\underline{\underline{1}} + |K_1(\bar{n})|} \mathbb{1}_{0 \le \theta \ll 1}$$

$$\uparrow$$

$$\sum_{\substack{N_2 \ge 1 \\ \text{dyadic}}} \sum_{|n_2| \sim N_2} \lesssim \underline{\underline{1}} + \mathrm{vol}(\mathcal{C})$$

$$\sum_{\substack{N_2 \ge 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^2 \max(\langle n \rangle^2, N_2^2)\, N_2^2} \lesssim \langle n \rangle^{-4}.$$

Key point on $\Upsilon^{\ominus}$ = $\Upsilon \odot \mathfrak{I}$

New difficult term

$0<\gamma\ll 1$

$$\sum_{\substack{n_2\in\mathbb{Z}^3\\ |n+n_2|\sim|n_2|\\ |n|\ll|n_2|^\gamma}} \frac{\sin(t-t')\left(\boxed{\langle n+n_2\rangle-\langle n_2\rangle}\right)}{\langle n\rangle^2\langle n+n_2\rangle\langle n_2\rangle^2} \qquad \left(\overset{\text{WTS}}{\lesssim} \langle n\rangle^{-3}\right)$$

$\Theta^{+} + \dfrac{\langle n, n_2\rangle}{\langle n_2\rangle}$

can drop $|n+n_2|\sim|n_2|$

$$\sum_{\substack{n_2\in\mathbb{Z}^3\\ |n|\ll|n_2|^\gamma}} \quad \Leftarrow \quad \text{symmetrize } n_2 \longleftrightarrow -n_2.$$

Let $\Theta^{\pm}(n,n_2) = \underline{\langle n\pm n_2\rangle - \langle n_2\rangle} \mp \dfrac{\langle n, n_2\rangle}{\langle n_2\rangle} = \underline{O\left(\dfrac{\langle n\rangle^2}{\langle n_2\rangle}\right)}$

$$\text{Sum} = \sum_{\substack{n_2 \in "\mathbb{Z}^3/2" \\ |n| \ll |n_2|^\gamma}} \frac{1}{\langle n\rangle^2 \langle n+n_2\rangle\langle n_2\rangle^2}$$

$$\times \Big[\sin(t-t_1)\Big(\frac{\langle n, n_2\rangle}{\langle n_2\rangle} + \underline{\Theta^+(n,n_2)}\Big)$$

$$- \sin(t-t_1)\Big(\frac{\langle n, n_2\rangle}{\langle n_2\rangle} - \underline{\Theta^-(n,n_2)}\Big)\Big]$$

$$\overset{MVT}{\lesssim} \sum_{|n| \ll |n_2|^\gamma} \frac{1}{\langle n\rangle^2 \langle n_2\rangle^3} \underline{\Big(\frac{\langle n\rangle^2}{\langle n_2\rangle}\Big)^\delta} \quad \text{for any } \delta \in [0,1]$$

$$\lesssim \langle n\rangle^{-3+}$$

- On $\mathcal{J}$ ⓒ, ⓔ:
  - symmerization: $n_2 \leftrightarrow -n_2$
  - integration by parts in time to handle $\sin(t-t')\underline{(\langle n+n_2\rangle + \langle n_2\rangle)}$