Lec 15: 05/05/21 (Wed)

· **LWP of parabolic Φ^4_3-model (for smoother initial data):**

$$(\mathrm{SQE})\quad \begin{aligned}
(\partial_t + 1 - \Delta) X &= -3\,(X + Y - \Psi) \mathbin{\textcircled{$<$}} \underset{-1-}{\vee} \\
(\partial_t + 1 - \Delta) Y &= -(X+Y)^3 - 3\,Y \mathbin{\textcircled{$=$}} \vee + 3\, \underset{-\frac{1}{2}-}{\underline{\Psi \mathbin{\textcircled{$=$}} \vee}} \\
&\quad - 3\,(X + Y - \Psi) \mathbin{\textcircled{$>$}} \vee \\
&\quad - 3\,\mathrm{com}_1(X, Y) \mathbin{\textcircled{$=$}} \vee \\
&\quad - 3\,\mathrm{com}_2(X+Y) + \underset{-\frac{1}{2}-}{Q(X+Y)}
\end{aligned}$$

$$X \in C_T C_x^{\frac{1}{2}+2\varepsilon} \cap C_T^{1/8} L_x^\infty =: Z^X(T)$$

$$Y \in C_T C_x^{1+2\varepsilon} \cap C_T^{1/8} L_x^\infty =: Z^Y(T)$$

$$\boxed{Z(T) = Z^X(T) \times Z^Y(T).}$$

Write (SQE) in the Duhamel formulation:

$$X = \Gamma^X(X,Y)(t) := \underline{P(t)X_0} - 3\int_0^t P(t-t')[(X+Y-\Psi)\circledcirc V](t')\,dt'$$

$$Y = \Gamma^Y(X,Y)(t) := \underline{P(t)Y_0} - \int_0^t P(t-t')(X+Y)^3(t')\,dt'$$

$$+ \cdots$$

(1) $\Gamma^X \in C_T C_x^{\frac{1}{2}+2\varepsilon}$?

$$\swarrow \underline{\int_0^t \frac{1}{(t-t')^{(\frac{3}{2}-\varepsilon)/2}}\,dt' < \infty}$$

$$\|\Gamma^X(X,Y)\|_{C_T C_x^{\frac{1}{2}+2\varepsilon}} \lesssim \|X_0\|_{C^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta}\|(X+Y-\Psi)\circledcirc V\|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim \|X_0\|_{C^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta}\|X+Y-\Psi\|_{C_T L_x^\infty}\|V\|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim \|X_0\|_{C^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta}K(K+R)$$

$$\boxed{\text{for } (X,Y) \in B_R \subset Z(T)}$$

② $\Gamma^X(X,Y) \in C_T^{1/8} L_x^\infty$?

$$\Gamma^X(X,Y)(t_2) - \Gamma^X(X,Y)(t_1) \qquad t_2 > t_1$$

$$= \big(P(t_2-t_1) - Id\big) P(t_1) X_0$$

$$+ \big(P(t_2-t_1) - Id\big) \int_0^{t_1} P(t_1-t') \square (t')\, dt'$$

$$+ \int_{t_1}^{t_2} P(t_2-t') \square (t')\, dt' =: I + II + III.$$

use Lemma 2 from Lec 14

Lemma 2

$$\|(1-P(t))f\|_{B_p^\alpha} \lesssim t^{\overbrace{\frac{\beta-\alpha}{2}}^{\geq 0}} \|f\|_{B_p^\beta}$$

$$0 \leq \beta - \alpha \leq 2$$

- $\|I\|_{L_x^\infty} \lesssim \|I\|_{B_\infty^\varepsilon}$

$$\lesssim (t_2-t_1)^{1/8} \|X_0\|_{C_x^{\frac14+\varepsilon}}.$$

↑ Lemma 2.

$$\cdot\ \|\mathrm{II}\|_{L^\infty_x} \lesssim (t_2-t_1)^{\frac{1}{8}} \left\| \int_0^{t_1} P(t_1-t')\, \square(t')\, dt' \right\|_{C^{\frac{1}{4}+\varepsilon}}$$

$$\swarrow$$

$$(X+Y-\Psi) \olessthan \vee \in C^{-1-\varepsilon}$$

$$\lesssim \underline{(t_2-t_1)^{\frac{1}{8}}}\ T^\theta\ K(K+R).$$

$$\cdot\ \|\mathrm{III}\|_{L^\infty_x} \lesssim \|\mathrm{III}\|_{C^\varepsilon_x} \lesssim \int_{t_1}^{t_2} \frac{1}{(t_2-t')^{\frac{1}{2}+\varepsilon}} \|\square\|_{C_T C_x^{-1-\varepsilon}}\, dt'$$

$$\lesssim (t_2-t_1)^{1/8}\, T^\theta K(K+R).$$

$$(0 < t_1 < t_2 \le T.)$$

$$\Rightarrow \boxed{\|\Gamma^X(X,Y)\|_{C_T^{1/8} L^\infty_x} \lesssim \|X_0\|_{C^{\frac{1}{2}+2\varepsilon}} + T^\theta K(K+R)}.$$

3 $\Gamma^{Y}(X,Y) \in C_T C_x^{1+2\varepsilon}$?

- $$\| \mathrm{I}(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{1}{4}}} \| (X+Y)^3 \|_{C_T C_x^{\frac{1}{2}+2\varepsilon}} \, dt'$$

$$\lesssim T^{\theta} R^3 \quad \text{by algebra property of } C_x^s,\ s \geq 0$$

- Similarly,

$$\| \underline{\mathrm{III}} + \underline{\mathrm{VII}}(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+\frac{3}{2}\varepsilon}} \| \quad + Q(X+Y) \|_{C_T C_x^{-\frac{1}{2}-\varepsilon}} \, dt'$$

$$\lesssim T^{\theta} \quad K\left(1 + (K+R)^2\right).$$

⑥

$$\cdot\ \|\mathrm{II}(t)\|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{1+\varepsilon}{2}}} \underbrace{\|Y \odot V\|_{C_T C_x^{\varepsilon}}}_{\lesssim \|Y\|_{C_T C_x^{1+2\varepsilon}} \|V\|_{C_T C_x^{-1-\varepsilon}}} dt'$$

$$\lesssim T^{\theta} K R.$$

$$\cdot\ \|\underline{\mathrm{IV}}(t)\|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+2\varepsilon}} \underbrace{\|(X+Y-\Psi) \odot V\|_{C_T C_x^{-\frac{1}{2}-2\varepsilon}}}_{\lesssim \|X+Y-\Psi\|_{C_T C_x^{\frac{1}{2}-\varepsilon}} \|V\|_{C_T C_x^{-1-\varepsilon}}} dt'$$

$$\lesssim T^{\theta} K (K+R).$$

$$\bullet \quad \| \mathrm{V}(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{1+\varepsilon}{2}}} \| \overset{1+2\varepsilon}{\mathrm{com}_1(X,Y)} \circleddash \overset{-1-\varepsilon}{V} \|_{C_T C_x^{\varepsilon}} \, dt'$$

$$\lesssim T^{\theta} (K^2 + K R).$$

$$\left(\Leftarrow \quad \int_0^t \frac{K}{\underbrace{(t-t')^{1+2\varepsilon}}_{\frac{1}{(t-t')^{\frac{7}{8}+2\varepsilon}} \frac{1}{(t-t')^{1/8}}}} \boxed{\| \delta_{t',t} (X+Y) \|_{L_x^{\infty}}} \, dt' \lesssim T^{\theta} K R \right.$$

$$\underbrace{\frac{1}{(t-t')^{\frac{7}{8}+2\varepsilon}}}_{\text{locally integrable}}$$

$\Leftarrow$ In this step, we needed the Hölder-in-time regularity of Ψ

$$\bullet \quad \| \mathrm{VI}(t) \|_{C^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+\frac{5\varepsilon}{2}}} \| \mathrm{com}_2 (X+Y) \|_{C_T C_x^{\frac{1}{2}-3\varepsilon}} \, dt'$$

$$\lesssim T^{\theta} \| X + Y - \Psi \|_{C_T C_x^{\frac{1}{2}-\varepsilon}} \| Y \|_{C_T C_x^{1-\varepsilon}} \| V \|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim T^{\theta} K^2 (K + R).$$

Hence, we obtain

(8)

$$\|\Gamma^Y(X,Y)\|_{C_T C_x^{1+2\varepsilon}} \lesssim \|Y_0\|_{C_x^{1+2\varepsilon}} + T^\theta (K+R)^3$$

④ $\Gamma^Y(X,Y) \in C_T^{1/8} L_x^\infty$?

Proceed as in case ② for $\Gamma^X(X,Y)$.

Write

$$\Gamma^Y(X,Y)(t_2) - \Gamma^Y(X,Y)(t_1) =: \mathrm{I} + \mathrm{II} + \mathrm{III}$$

as on page ③

$$\cdot\ \|\mathrm{I}\|_{L_x^\infty} \lesssim \|\mathrm{I}\|_{B_x^\varepsilon} \overset{\text{Lemma 2}}{\lesssim} (t_2 - t_1)^{1/8} \|Y_0\|_{C_x^{\frac14+\varepsilon}}$$

$$\cdot\ \|\mathrm{II}\|_{L_x^\infty} \lesssim (t_2-t_1)^{1/8} \Big\| \int_0^{t_1} P(t_1-t') \,\boxed{}\, dt' \Big\|_{C_x^{\frac14+\varepsilon}}$$

$$\lesssim (t_2-t_1)^{1/8} T^\theta (K+R)^3$$

Apply the result from case ③.

$$\cdot \ \|\mathrm{III}\|_{L^\infty_x} \lesssim \|\mathrm{III}\|_{C^\varepsilon_x} \lesssim \int_{t_1}^{t_2} \frac{1}{(t_2-t')^{\frac{1}{4}+\frac{3\varepsilon}{2}}} \|\square\|_{C_T C_x^{-\frac{1}{2}-2\varepsilon}} \, dt'. \qquad (9)$$

$$\lesssim (t_2-t_1)^{1/8} T^\theta (K+R)^3.$$

By similar computations, we can obtain difference estimates

on $\Gamma(X_1, Y_1) - \Gamma(X_2, Y_2)$

where $\Gamma = (\Gamma^X, \Gamma^Y)$

$\Rightarrow$ Γ is a contraction on $B_R \subset Z(T)$

$R \sim \|X_0\|_{C^{\frac{1}{2}+2\varepsilon}} + \|Y_0\|_{C^{1+2\varepsilon}}$

by choosing $T = T(R, K) > 0$ suff. small.

· As in the 1-d, 2-d cases, we can also handle rougher initial data in $C^{-3/5} \times C^{-3/5}$

(Point : $-\frac{3}{5} < -\frac{1}{2}$ ~ regularity of the Gibbs meas on $\mathbb{T}^3$

· Catellier - Chouk, Mourrat - Weber CMP '17.

$\Leftarrow$ Need to use the $Y(T)$-type norm. ($\times$ beta function)

$$
\begin{aligned}
\|(X,Y)\|_{Z(T)} = \max\Big\{ & \|X\|_{C_T C_x^{-3/5}},\ \sup_{0<t\le T} t^{3/5}\|X(t)\|_{C_x^{\frac{1}{2}+2\varepsilon}}, \\
& \sup_{0<t_1<t_2\le T} \frac{t_1^{1/2}\|X(t_2)-X(t_1)\|_{L_x^\infty}}{|t_2-t_1|^{1/8}}, \\
& \|Y\|_{C_T C_x^{-3/5}},\ \sup_{0<t\le T} t^{17/20}\|Y(t)\|_{C^{1+2\varepsilon}},\ \sup_{0<t_1<t_2\le T} \frac{t_1^{1/2}\|Y(t_2)-Y(t_1)\|_{L_x^\infty}}{|t_2-t_1|^{1/8}} \Big\}.
\end{aligned}
$$

· On the construction of the stoch. objects

· Mourrat - Weber - Xu
· Note by Justin.

$-\frac{1}{2}-$ $\quad -1- \quad$ $\frac{1}{2}-$ $\quad 0- \quad$ $-\frac{1}{2}-$ $\quad 0-$

same proof as 2-d case

· $\underline{\underline{\Psi}}$: $\mathbb{E}[|\hat{\Psi}(t,n)|^2] = 3! \times (t,n)$  $(t,-n)$

$$= 6 \underbrace{\sum_{n=n_1+n_2+n_3}}_{2\,\text{sums}} \prod_{j=1}^{3} \frac{1}{\langle n_j \rangle^2}$$

divergent.

$$\lesssim \sum_{n_1} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n-n_1 \rangle^1} = \infty$$

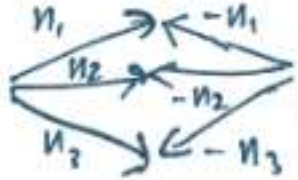
$$\left(\quad \Phi = \int_{-\infty}^{t} P(t-t')\,dW(t') \right.$$

$$\mathbb{E}[|\hat{\Phi}(t,n)|^2] = \frac{1}{\langle n \rangle^2}$$

$$\mathbb{E}\left[\hat{\varphi}(t_1,n)\overline{\hat{\varphi}(t_2,n)}\right] \qquad t_2 < t_1$$ (12)

$$= \int_{-\infty}^{t_2} e^{-(t_1+t_2-2t')\langle n\rangle^2}\,dt'$$

$$= \frac{e^{-|t_1-t_2|\langle n\rangle^2}}{\langle n\rangle^2} \lesssim \frac{1}{\langle n\rangle^2}\left(\frac{1}{|t_1-t_2|\langle n\rangle^2}\right)^{\gamma},\ \forall \gamma\geq 0$$

$$\Rightarrow \mathbb{E}\left[\hat{\Psi}(t_1,n)\,\hat{\Psi}(t_2,-n)\right] = 6\times(t_1,n)$$  $$(t_2,-n)$$

$$\lesssim \frac{1}{|t_1-t_2|^{\gamma}} \sum_{n=n_1+n_2+n_2} \frac{1}{\langle n_1\rangle^{2+2\gamma}}\,\frac{1}{\langle n_2\rangle^2}\,\frac{1}{\langle n_3\rangle^2}$$

$$\lesssim \frac{1}{|t_1-t_2|^{\gamma}} \sum_{n_1} \frac{1}{\langle n_1\rangle^{2+2\gamma}}\,\frac{1}{\langle n-n_1\rangle}$$

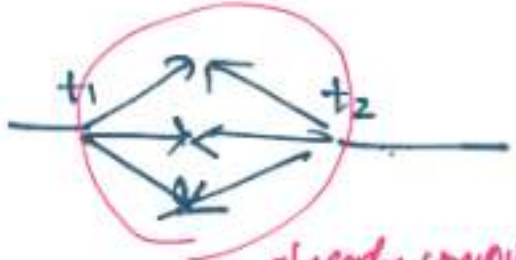
$$\lesssim \frac{1}{|t_1-t_2|^{\gamma}}\,\frac{1}{\langle n\rangle^{2\gamma}}$$

$$\left(\ \sum \frac{1}{\langle n_1\rangle^{\alpha}}\frac{1}{\langle n-n_1\rangle^{\beta}} \lesssim \frac{1}{\langle n\rangle^{\alpha+\beta-d}}\right.$$

for $\alpha+\beta>d$

$\alpha,\beta<d$.

13

$$\mathbb{E}\left[\left| \hat{\Psi}(t,m) \right|^2 \right] = 6 \times (t,m)$$  $$(t,-n)$$

already computed

$$\lesssim \int \hat{P}_{t-t_1}(m) \int \hat{P}_{t-t_2}(m) \frac{1}{|t_1-t_2|^{\gamma}} \frac{1}{\langle m \rangle^{2\gamma}} \, dt_2 \, dt_1$$

$$\left(\quad \hat{P}_t(m) = e^{-t\langle m \rangle^2} \mathbb{1}_{t \geq 0} \right.$$

$$\overset{C-S}{\leq} \frac{1}{\langle m \rangle^{2\gamma}} \| \hat{P}_{t-t_1}(m) \|_{L^2_{t_1}} \underbrace{\left\| \hat{P}_{t-\cdot}(m) * \frac{1}{|\cdot|^{\gamma}}(t_1) \right\|_{L^2_{t_1}}}$$

$$\lesssim \| \hat{P}_{t-\cdot}(m) \|_{L^q_{t_1}} \qquad \text{H-L-S ineq} \quad \frac{1}{2} + 1 = \frac{1}{1/\gamma} + \frac{1}{q}$$

$$\lesssim \frac{1}{\langle m \rangle^{2/q}} \qquad \Rightarrow \frac{1}{q} = \frac{3}{2} - \gamma.$$

$$\lesssim \frac{1}{\langle m \rangle^{1+2\gamma}} \frac{1}{\langle m \rangle^{3-2\gamma}}$$

$$= \langle m \rangle^{-3-2\left(\frac{1}{2}\right)}$$

$$\Rightarrow \quad \Psi(t) \in C_x^{\frac{1}{2}-\varepsilon}(\mathbb{T}^3).$$

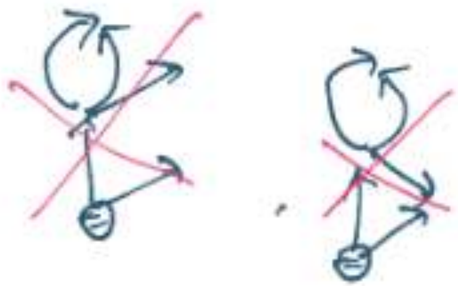
For time difference, we have terms like

$$\hat{P}_{t+h-t_1}(n) - \hat{P}_{t-t_1}(n) = \underbrace{\left(e^{-h\langle n\rangle^2} - 1\right)}_{\underset{MVT}{\lesssim}\ h^\sigma \langle n\rangle^{2\sigma}} \hat{P}_{t-t_1}(n)$$

(P.125)
See the last part of Justin's note.

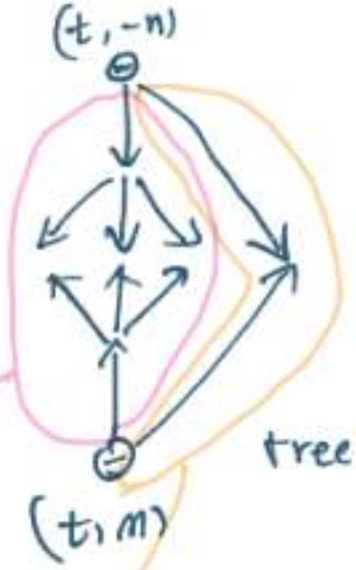
·  (t,n) =  + $3\times$  $=: \mathrm{I} + \mathrm{II}$

(t,n) $\Uparrow$ $\mathcal{H}_4$ (t,n) $\Uparrow$ $\mathcal{H}_2$

 $\Leftarrow$ removed by the Wick renormalization

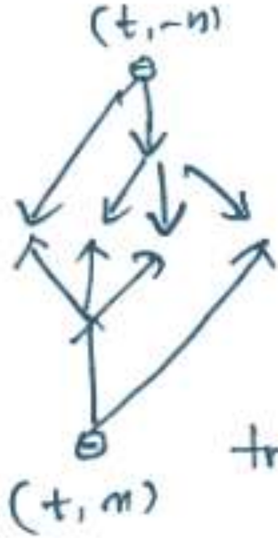
$\phi^3 \rightsquigarrow$ 

$\mathbb{E}\left[|\mathcal{I}(t,n)|^2\right] = 6 \times$



tree 1

$+\ 3 \cdot 3! \times$



tree 2.

tree 2 $\lesssim$ tree 1 by Jensen's inequality

(in this case, by C-S)

See Lec9 in Justin's note.

$$\sum_{\substack{n=n_1+n_2 \\ |n_1|\sim|n_2|}} \frac{1}{\langle n_1\rangle^4}\,\frac{1}{\langle n_2\rangle^2} \lesssim \langle n\rangle^{-3} = \langle n\rangle^{-3-2\cdot 0}.$$

drop the condition $\alpha, \beta < d$

$\Longrightarrow$  $(t) \in C_x^{0-}(\mathbb{T}^3)$

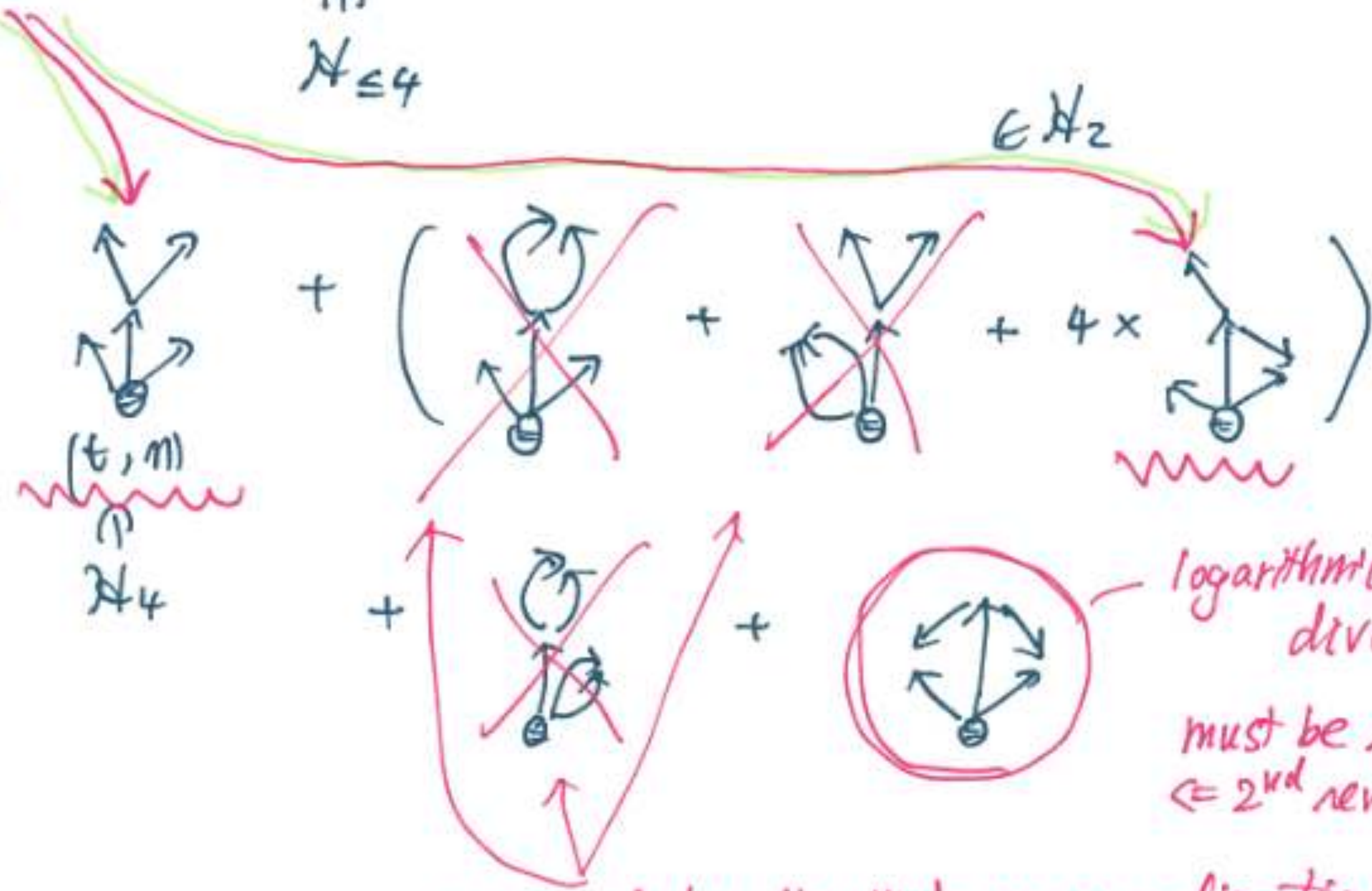
·  : Let's consider

$\mathcal{I}(\Phi_N^2)\,\Phi_N^2$ (without the Wick renormalization)

$\cap$

$\mathcal{H}_{\leq 4}$

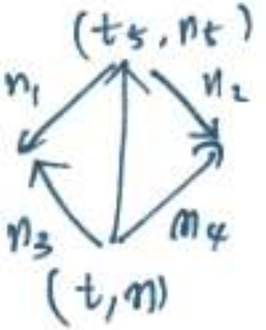
F.T in x $\Longrightarrow$



logarithmically divergent

must be removed
⇐ 2nd renormalization

removed by the Wick renormalization.
(diverges ~ N



$$= \sum_{\substack{n_1,\dots,n_5}} \left.\begin{array}{l} n_1+n_3=0\\ n_2+n_4=0\\ n_5=n_1+n_2\\ n=n_3+n_4+n_5\end{array}\right\} \Rightarrow n=0 \quad \int \hat{P}_{t-t_5}(n_5)\left(\int \hat{P}_{t_5-t_1}(n_1)\,\hat{P}_{t-t_1}(-n_1)\,dt_1\right) \times\left(\int \hat{P}_{t_5-t_2}(n_2)\,\hat{P}_{t-t_2}(-n_2)\,dt_2\right) dt_5$$

$$= \mathbb{1}_{\{n=0\}} \sum_{\substack{n_1,n_2,n_5\\ n_5=n_1+n_2}} \boxed{\int \hat{P}_{t-t_5}(n_5)\, \frac{e^{-|t-t_5|\langle n_1\rangle^2}}{2\langle n_1\rangle^2}\cdot\frac{e^{-|t-t_5|\langle n_2\rangle^2}}{2\langle n_2\rangle^2}\, dt_5}.$$

$$\lesssim \frac{1}{\langle n_1\rangle^2}\,\frac{1}{\langle n_2\rangle^2}\,\frac{1}{\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_5\rangle^2}$$

2 summations

deg 6.

$\Rightarrow$ log divergent!!

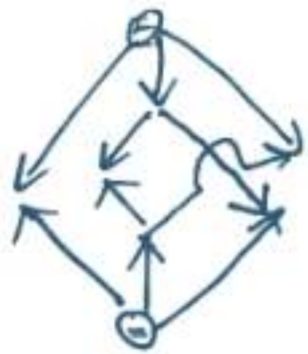
· We only consider the $\mathcal{H}_4$- contribution.

$I(t,n) =$  $\in \mathcal{H}_4$

(t,n)

$$\mathbb{E}\left[\,|I(t,n)|^2\,\right]$$

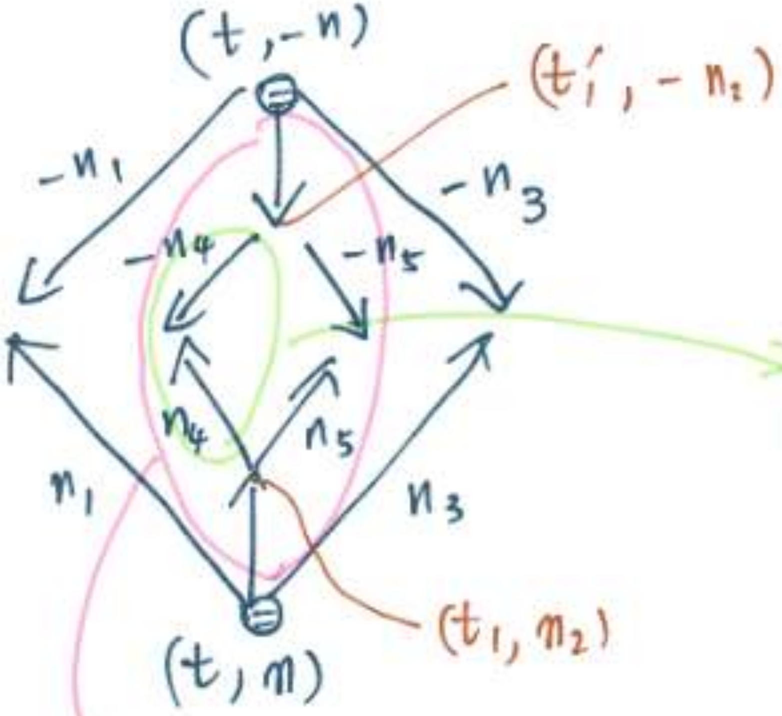
$(t,-n)$

$= 2 \times 2$  + terms like 

(t,n)

tree 1

dominated by tree 1
via Jensen's ineq.



$n = n_1 + n_2 + n_3$

$n_2 = n_4 + n_5$

$|n_1 + n_3| \sim |n_2|.$

$$\mathbb{E}\left[\hat{\imath}(t_1, n_4)\,\hat{\imath}(t_1', -n_4)\right] = \frac{e^{-|t_1 - t_1'|\langle n_4\rangle^2}}{2\langle n_4\rangle^2}$$

$$\int \hat{P}_{t-t_1}(n_2)\,\hat{P}_{t-t_1'}(-n_2) \underbrace{\sum_{n_2 = n_4 + n_5} \frac{1}{|t_1 - t_1'|^\gamma}\,\frac{1}{\langle n_4\rangle^{2+2\gamma}} \times \frac{1}{\langle n_5\rangle^2}}_{\lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}}}\, dt_1'\, dt_1$$

$$\lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}} \|\hat{P}_{t-t_1}(n_2)\|_{L^2_{t_1}} \times \left\| \left(\frac{1}{|\cdot|^\gamma} \times \hat{P}_{t-\cdot}(n_2)\right)(t_1) \right\|_{L^2_{t_1}}$$

Apply H-L-S.

$\frac{1}{2} + 1 = \frac{1}{1/\gamma} + \frac{1}{q}$

$\Rightarrow \frac{2}{q} = 3 - 2\gamma$

$$\lesssim \frac{1}{\langle n_2\rangle^{1+2r}} \frac{1}{\langle n_2\rangle} \frac{1}{\langle n_2\rangle^{3/8}} \quad = 3-2r$$

$$= \frac{1}{\langle n_2\rangle^{5}}$$

$$\Rightarrow \mathbb{E}\left[|I(t,n)|^2\right] \lesssim \sum_{\substack{n=n_1+n_2+n_3\\ |n_1+n_3|\sim|n_2|}} \frac{1}{\langle n_2\rangle^5}\frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_3\rangle^2}$$

$$\lesssim \sum_{\substack{n_2\\ |n-n_2|\sim|n_2|}} \frac{1}{\langle n_2\rangle^5}\frac{1}{\langle n-n_2\rangle}$$

$$\sim \langle n\rangle^{-3} = \langle n\rangle^{-3-2\cdot 0} \quad \Rightarrow \quad \underline{s<0}$$

•  $(t,n) =$  $\in \mathcal{H}_5$ $+ \; 6 \times$  $\in \mathcal{H}_3$

$+ \; 6 \times \Big($  $-$  $\times \uparrow \Big)$

$\underbrace{\qquad (t,n) \qquad\qquad (t,0) \qquad\qquad (t,n) \qquad}_{\in \mathcal{H}_1}$

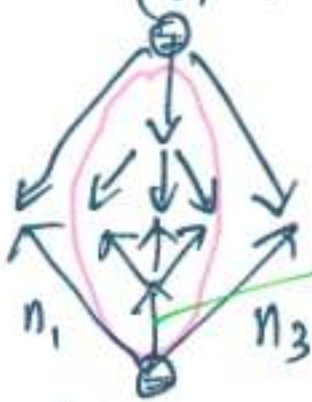
$\sim \log N$

diverging logarithmically

2^{nd} renormalization

Only consider the $\mathcal{H}_5$-contribution

$$\mathbb{E}\left[\left| \dots (t,n) \right|^2\right] \overset{\text{Jensen}}{\lesssim}$$



$$\lesssim \frac{1}{\langle n_2\rangle^4}.$$ as in 

$$\Longrightarrow \mathbb{E}\left[|\cdots(t,n)|^2\right] \lesssim \sum_{\substack{n=n_1+n_2+n_3\\ |n_1+n_3|\sim|n_2|}} \frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_3\rangle^2}\frac{1}{\langle n_2\rangle^4}$$

$$\lesssim \sum_{\substack{n_2\\ |n-n_2|\sim|n_2|}} \frac{1}{\langle n-n_2\rangle}\frac{1}{\langle n_2\rangle^4}$$

$$\lesssim \langle n\rangle^{-2} = \langle n\rangle^{-3-2(-\frac{1}{2})}$$

$$\Longrightarrow \boxed{s < -1/2}$$