Lec 14 : 28 / 04 / 21 (Wed) ①

$$\mathrm{com}_1(X,Y) = P(t)X_0 - 3\int_0^t P(t-t')[(X+Y-\Psi)\circledcirc \underset{-1-\varepsilon}{\Psi}]\,dt'$$

$$+ 3(X+Y-\Psi) \circledless \underset{1-\varepsilon}{\Psi}$$

Prop (1st commutator)

Let $\varepsilon > 0$, $\beta \in (4\varepsilon, 1+2\varepsilon]$, $p \in [1,\infty]$, $T > 0$

Then,

$$\| \mathrm{com}_1(X,Y)(t) - P(t)X_0 \|_{B_\infty^{1+2\varepsilon}}$$

$$\lesssim K^2 + \int_0^t \frac{K}{(t-t')^{1+2\varepsilon-\beta/2}} \|(X,Y)(t')\|_{B_p^\beta \times B_p^\beta}\, dt'$$

$$+ \int_0^t \frac{K}{(t-t')^{1+2\varepsilon}} \|\delta_{t',t}(X+Y)\|_{L^p}\, dt'$$

$$(\quad \delta_{t_1,t_2} f = f(t_2) - f(t_1)$$

- K = bound on the (given) enhanced data set.

Ξ_j						
reg s_j	$-\frac{1}{2}-\varepsilon$	$-1-\varepsilon$	$\frac{1}{2}-\varepsilon$	$-\varepsilon$	$-\frac{1}{2}-\varepsilon$	$-\varepsilon$

$$\sup_{0\le t\le 1} \| \Xi_j(t) \|_{B^{s_j}_{\infty,\infty}} \le K$$

Furthermore,

$$\sup_{0\le t_1 < t_2 \le 1} \frac{\| \Psi(t_2) - \Psi(t_1) \|_{B^{\frac{1}{4}-\varepsilon}_{\infty,\infty}}}{|t_2 - t_1|^{1/8}} \le K$$

 

<u>Prop'</u>: $\alpha < 1$, $\beta \in \mathbb{R}$, $\gamma \geq \alpha + \beta$, $1 \leq p, p_1, p_2 \leq \infty$ ③

$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

Then,

$$(*) \quad \| [P(t), \circledcirc] (f, g) \|_{B_p^\gamma} \lesssim t^{\underbrace{\frac{\alpha+\beta-\gamma}{2}}_{\leq 0}} \|f\|_{B_{p_1}^\alpha} \|g\|_{B_{p_2}^\beta}.$$

We assume Prop' and prove Prop on com_1.

$$\cdot\ \mathrm{com}_1 (X, Y)(t) - P(t) X_0 = -3 \int_0^t P(t-t') [(X + Y - \Psi) \circledcirc \nabla](t') dt' + 3 (X + Y - \Psi)(t) \circledcirc \Psi(t).$$

$$= -3 \int_0^t [P(t-t'), \circledcirc] (X + Y - \Psi, \nabla)(t') dt'$$

Commutator bet $\int_0^t \cdots dt'$ and $\circledcirc$

$$\left[\begin{aligned} & -3 \int_0^t (X + Y - \Psi)(t') \circledcirc (P(t-t') \nabla(t')) dt' \\ & + 3 (X + Y - \Psi)(t) \circledcirc \boxed{\int_0^t} P(t-t') \nabla(t') dt' \end{aligned} \right.$$

$$= -3\int_0^t \underbrace{[P(t-t'), \varolessthan]}\,(\underbrace{X+Y-\Psi}, v)(t')\,dt' \qquad (4)$$

$$+\,3\int_0^t [\underbrace{\delta_{t',t}(X+Y-\Psi)}_{1/2-\varepsilon}] \varolessthan \Big(P(t-t')v(t')\Big)\,dt'$$

$$=: \mathrm{I} + \mathrm{II}.$$

On I: Use Prop' on $[P(t), \varolessthan]$.

Write $\mathrm{I} = \mathrm{I}_X + \mathrm{I}_Y + \mathrm{I}_\Psi$.

$$\|\mathrm{I}_\Psi\|_{B_p^{1+2\varepsilon}} \lesssim \int_0^t \big\|[P(t-t'), \varolessthan](\Psi(t'), v(t'))\big\|_{B_p^{1+2\varepsilon}}\,dt'$$

$$\underset{\text{Prop}'}{\lesssim} K^2 \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+2\varepsilon}}\,dt' \qquad \left(\begin{array}{l}\gamma = 1+2\varepsilon\\ \alpha = \frac{1}{2}-\varepsilon\\ \beta = -1-\varepsilon\end{array}\right.$$

$$\lesssim K^2$$

⑤

$$\| I_X + I_Y \|_{B_p^{1+2\varepsilon}}$$

$$\underset{\text{Prop}'}{\lesssim} K \int_0^t \frac{1}{(t-t')^{\frac{2+3\varepsilon-\beta}{2}}} \| (X+Y)(t') \|_{B_p^{\beta}} \, dt'$$

$$\left(\begin{array}{l} \gamma = 1+2\varepsilon \\ \alpha = \beta \\ \beta = -1-\varepsilon \end{array} \right.$$

<u>On II</u>: $\| \mathrm{II} \|_{B_p^{1+2\varepsilon}} \lesssim \int_0^t \| \delta_{t',t} (X+Y-\Psi) \|_{L^p}$ (paraprod est.)

$$\times \| \underbrace{P(t-t') \nabla(t')}_{-1-\varepsilon} \|_{B_\infty^{1+2\varepsilon}} \, dt'$$

$$\underset{\text{Schauder}}{\lesssim} K \int_0^t \frac{1}{\underbrace{(t-t')^{1+\frac{3}{2}\varepsilon}}_{\text{Not integrable near } t=0.}} \| \delta_{t',t} (X+Y-\Psi) \|_{L^p} \, dt'$$

$$\| \delta_{t',t} \Psi \|_{L^p} \lesssim |t-t'|^{1/8}$$

$$\lesssim K \int_0^t \frac{1}{(t-t')^{1+\frac{3}{2}\varepsilon}} \| \delta_{t',t} (X+Y) \|_{L^p} \, dt' + K^2.$$

$\square$

Before proving Prop′ on $[P(t), \circledcirc]$, (6)

we go over some basic estimates on $P(t) = e^{t(\Delta-1)} = e^{-t}e^{t\Delta}$

Lemma 1: $\| \mathcal{F}^{-1}\left(\varphi\left(\frac{\cdot}{2^j}\right)e^{-t|\cdot|^2}\right)\|_{L^1_x} \lesssim e^{-ct2^{2j}}$.

(φ is supported on an annulus $A \sim \{|\xi| \sim 1\}$

Pf: On $\mathbb{R}^d$: Let $g(t,x) = \int e^{ix\cdot\xi}\varphi(\xi)e^{-t|\xi|^2}d\xi$

$$= (1+|x|^2)^{-d} \int \underbrace{(1+|x|^2)^d e^{ix\cdot\xi}}_{=(1-\Delta_\xi)^d e^{ix\cdot\xi}} \cdot \varphi(\xi)e^{-t|\xi|^2}d\xi$$

$$= \underbrace{(1+|x|^2)^{-d}}_{\in L^1_x} \int e^{ix\cdot\xi}(1-\Delta_\xi)^d\left(\varphi(\xi)e^{-t|\xi|^2}\right)d\xi$$

$|\xi| \sim 1$

$$\left| (1-\Delta_\xi)^d \left(\varphi(\xi)\, e^{-t|\xi|^2} \right) \right|$$

$$\lesssim \Big(1 + \underbrace{t|\xi|}_{\overset{?}{t|\xi|^2}}\Big)^{2d} \tilde{\varphi}(\xi)\, e^{-t|\xi|^2} \lesssim e^{-ct}$$

support in $\{|\xi| \sim 1\}$

say $\frac{1}{2} \le |\xi| \le 2$

use $x e^{-x} \le e^{-cx} \quad \forall x \ge 0$

- $$g_\lambda(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \varphi\left(\frac{\xi}{\lambda}\right) e^{-t|\xi|^2} d\xi$$

$$= \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x\cdot\xi} \varphi(\xi)\, e^{-t\lambda^2|\xi|^2} d\xi$$

$$= \lambda^d g(\lambda^2 t, \lambda x)$$

$$\Rightarrow \|g_\lambda(t,x)\|_{L^1_x} = \|g(t\lambda^2, x)\|_{L^1_x} \lesssim e^{-ct\lambda^2}.$$

On $\mathbb{T}^d$: use the Poisson summation formula. ⑧

$$g_\lambda^{per}(t,x) = \mathcal{F}^{-1}_{\mathbb{T}^d}\left(\varphi\left(\frac{\cdot}{\lambda}\right) e^{-t|\cdot|^2}\right).$$

$$\|g_\lambda^{per}(t,x)\|_{L^1_x(\mathbb{T}^d)} = \Big\|\sum_{n\in\mathbb{Z}^d} \underbrace{\varphi\left(\frac{n}{\lambda}\right) e^{-t|n|^2}}_{\widehat{g_\lambda}(n)} e^{in\cdot x}\Big\|_{L^1_x(\mathbb{T}^d)}$$

g_λ on $\mathbb{R}^d$

$$\overset{\text{Poisson}}{=} \Big\|\sum_{n\in\mathbb{Z}^d} g_\lambda(x+n)\Big\|_{L^1(\mathbb{T}^d)}$$

$$= \|g_\lambda\|_{L^1_x(\mathbb{R}^d)} \lesssim e^{-ct\lambda^2}$$

$\boxtimes$

<u>Lemma 2:</u> $\|(1-P(t))f\|_{B^{\alpha}_{p,q}} \lesssim t^{\underbrace{\frac{\beta-\alpha}{2}}_{\geq 0}} \|f\|_{B^{\beta}_{p,q}}$

for $0 \leq \beta - \alpha \leq 2$. $(\Rightarrow \beta \geq \alpha)$

<u>Pf:</u> <u>Claim:</u> For $\hat{f}$ supported on $\{|m| \sim 2^j\}$,

$$\|(1-P(t))f\|_{L^p} \lesssim (t2^{2j} \underset{\substack{\uparrow \\ \text{min.}}}{\wedge} 1)\|f\|_{L^p}.$$

Assume Claim:

$$2^{\alpha j}\|(1-P(t))P_j f\|_{L^p} \overset{\text{Claim}}{\lesssim} 2^{\alpha j}(t2^{2j} \wedge 1)\|P_j f\|_{L^p}$$

$$\lesssim t^{\frac{\beta-\alpha}{2}} \underbrace{\left((t2^{2j})^{\frac{\alpha-\beta}{2}}(t2^{2j} \wedge 1)\right)}_{\lesssim 1 \quad \forall j, t.} 2^{\beta j}\|P_j f\|_{L^p}$$

$\square$

Pf of Claim:

By Lemma 1, we have

$$\| P(t) P_j f \|_{L^p} \lesssim e^{-c t 2^{2j}} \| P_j f \|_{L^p}$$

Young & Lemma 1.

$\Rightarrow$ claim for $t 2^{2j} \geq 1$.

· Suffices to prove $\| (1 - P(t)) P_j f \|_{L^p} \lesssim t 2^{2j} \| f \|_{L^p}$

(and $t 2^{2j} \ll 1$.)

On $\mathbb{R}^d$, let

$$g_\lambda(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \varphi\left(\frac{\xi}{\lambda}\right)\left(1 - e^{-t\langle\xi\rangle^2}\right) d\xi$$

$$= \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x\cdot\xi} \varphi(\xi) \left(1 - e^{-t\lambda^2|\xi|^2 - t}\right) d\xi.$$

(11)

$$\| g_\lambda(t,x) \|_{L^1_x}$$

$$= \underbrace{\Big\| \int_{|\xi|\sim 1} e^{ix\cdot\xi} \varphi(\xi) \left(1 - e^{-t\lambda^2|\xi|^2 - t}\right) d\xi \Big\|_{L^1_x(\mathbb{R}^d)}}$$

① On $Q = \{|x| \lesssim 1\}$.

$$(\text{RHS}) \le \| \cdots \|_{L^\infty_x(Q)} \overset{H-Y}{\le} \| \varphi(\xi) \underbrace{(1 - e^{-t\lambda^2|\xi|^2 - t})}\|_{L^1_\xi}$$

$$\overset{MVT}{\le} t(\lambda^2|\xi|^2 + 1)$$

$$\sim t(\lambda^2 + 1)$$

$$\sim t\lambda^2. \qquad \lambda = 2^j \text{ for } j \ge 0.$$

② On Q^c

Multiply by $\underset{\overset{\Uparrow}{L^1_x}}{|x|^{-2d}} \cdot \underline{\underline{|x|^{2d}}}$

$$\underset{\substack{\Uparrow \\ L^1_x}}{|x|^{-2d}}\ \underbrace{|x|^{2d}\int e^{ix\cdot\xi}\,\varphi(\xi)\left(1-e^{-t\lambda^2|\xi|^2-t}\right)d\xi}$$

$$-\Delta_\xi^d\, e^{ix\cdot\xi}$$

$$\overset{IBP}{=}\int_{|\xi|\sim 1} e^{ix\cdot\xi}\,(-\Delta_\xi)^d\left(\varphi(\xi)\cdot\left(1-e^{-t\lambda^2|\xi|^2-t}\right)\right)d\xi$$

- all the derivatives hit $\varphi(\xi)$

$$1-e^{-t\lambda^2|\xi|^2-t}\lesssim t\lambda^2 \text{ as before}$$

- If at least one derivative hits the 2^{nd} factor,

we get $t\lambda^2|\xi|\lesssim t\lambda^2$.

(and use the boldness of $e^{-t\lambda^2|\xi|^2-t}$.

By the Poisson summation formula as on page (8),
we obtain the same bd on $\mathbb{T}^d$.

$\boxtimes$

Proof of Prop' on page (3):

$$[P(t), \textcircled{<}](f,g) = \sum_{k=0}^{\infty} \Big[P(t)\big(P_{\le k-2}f \cdot P_k g\big) - P_{\le k-2}f \cdot P_k P(t) g \Big]$$

$$=: \sum_{k=0}^{\infty} h_k$$

$$P_{\le k-2} = \sum_{j=0}^{k-2} P_j$$

Note: $\widehat{h_k}$ is supported on $C 2^k A$

annulus $\{|\xi| \sim 1\}$.

Define $G_{k,t}$ by

$$G_{k,t} = \mathcal{F}^{-1}\left(\tilde{\varphi}\left(\tfrac{\cdot}{2^k}\right) e^{-t\langle\cdot\rangle^2}\right)$$

$\uparrow$ wider support. $\tilde{\varphi}(\cdot) = \varphi\left(\tfrac{\cdot}{2}\right) + \varphi(\cdot) + \varphi(2\cdot)$

$$\Rightarrow \quad P(t) h_k = G_{k,t} * h_k$$

$$\Rightarrow \quad h_k = G_{k,t} * \left(P_{\le k-2} f \cdot P_k g\right) - P_{\le k-2} f \left(G_{k,t} * P_k g\right)$$

$$= -\int G_{k,t}(y)\, P_k g(x-y) \underbrace{\left[P_{\le k-2} f(x) - P_{\le k-2} f(x-y)\right]}_{\overset{FTC}{=} -\int_0^1 \nabla P_{\le k-2} f(x-sy) \cdot y \, ds} dy$$

$$= \int_0^1 \int P_k g(x-y) \underbrace{\tilde{G}_{k,t}(y)}_{= y G_{k,t}(y)} \cdot \nabla P_{\le k-2} f(x-sy) \, dy \, ds$$

Let $h_{k,s} = \int P_k g(x-y)\, \widetilde{G}_{k,t}(y) \cdot \nabla P_{\leq k-2} f(x-sy)\, dy.$

Claim: $\| h_{k,s} \|_{L^p_x} \lesssim \| \widetilde{G}_{k,t} \|_{L^1} \| \nabla P_{\leq k-2} f \|_{L^{p_1}} \| P_k g \|_{L^{p_2}}$

unif in $s \in [0,1]$

Pf of Claim: Write $|\widetilde{G}_{k,t}| = |\widetilde{G}_{k,t}|^{\frac{1}{p'}} |\widetilde{G}_{k,t}|^{\frac{1}{p}}$

$$\Rightarrow |h_{k,s}(x)| \lesssim \| \widetilde{G}_{k,t} \|_{L^1}^{1-\frac{1}{p}} \Big(\int |\widetilde{G}_{k,t}(y)|\, |P_k g(x-y)|^p \times |\nabla P_{\leq k-2} f(x-sy)|^p dy \Big)^{1/p}$$

$$\Rightarrow \| h_{k,s} \|_{L^p_x}^p \lesssim \| \widetilde{G}_{k,t} \|_{L^1}^{p-1} \int |\widetilde{G}_{k,t}(y)| \underbrace{\int |P_k g(x-y)|^p |\nabla P_{\leq k-2} f(x-sy)|^p dx}_{\text{Apply Hölder.}}\, dy$$

$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

$\square$

Now, we estimate $\|\widetilde{G}_{k,t}\|_{L^1}$

$$\| \widetilde{G}_{k,t} \|_{L^1} = \| y\, G_{k,t} \|_{L^1} \qquad y = (y_1, \dots, y_d)$$

· Suffices to study $y_1 G_{k,t}(y)$

On the Fourier side,

$$\widehat{y_1 G_{k,t}}(\xi) \sim \partial_{\xi_1}\left(\varphi\left(\frac{\xi}{2^k}\right) e^{-t\langle\xi\rangle^2}\right)$$

$$= \left(\frac{1}{2^k}\, \partial_{\xi_1} \varphi\left(\frac{\xi}{2^k}\right) - 2\xi_1 t\, \varphi\left(\frac{\xi}{2^k}\right)\right) e^{-t\langle\xi\rangle^2}.$$

Lemma 1 on page 6 $\Longrightarrow$

(**) $$\|\widetilde{G}_{k,t}\|_{L^1} \lesssim 2^{-k}\left(1 + t\, 2^{2k}\right) e^{-ct 2^{2k}}$$

$$2^{-k} 2^{2k} t \underbrace{\frac{\xi_1}{2^k}\, \varphi\left(\frac{\xi}{2^k}\right)}_{\tilde{\varphi}}$$

(17)

$$\Rightarrow 2^{k\gamma} \| h_k \|_{L^p} \leq \int_0^1 2^{k\gamma} \| h_{k,s} \|_{L^p} \, ds$$

claim & (**)

$$\lesssim 2^{k(\gamma-\alpha-\beta)} (1+t2^{2k}) e^{-ct2^{2k}}$$

$$\times 2^{k(\alpha-1)} \| \nabla P_{\leq k-2} f \|_{L^{p_1}} \times 2^{k\beta} \| P_k g \|_{L^{p_2}}$$

$$\cdot \| \nabla P_{k\leq 2} f \|_{L^{p_1}} \leq \sum_{j \leq k-2} 2^{j(1-\alpha)} \cdot \boxed{\| P_j \nabla f \|_{L^{p_1}} \, 2^{j(\alpha-1)}}$$

$\alpha < 1$

$$\sup_j \cdots = \| \nabla f \|_{B^{\alpha-1}_{p_1}}$$

$$\lesssim 2^{k(1-\alpha)} \| \nabla f \|_{B^{\alpha-1}_{p_1}}$$

$$\underbrace{2^{k(\gamma-\alpha-\beta)}}\,(1+\underline{t2^{2k}})\,e^{-c\underline{t2^{2k}}}$$

$$= \underbrace{t^{\frac{\alpha+\beta-\gamma}{2}}}\,(t2^{2k})^{\frac{\gamma-\alpha-\beta}{2}}$$

$$x^{\frac{\gamma-\alpha-\beta}{2}}(1+x)\,e^{-cx} \lesssim 1, \quad \forall x \geq 0$$

if $\gamma \geq \alpha+\beta$

$\square$

<u>Rmk</u>: The commutator estimate such as $[P(t), \Theta]$ in the wave case does not provide any smoothing.