Lec 13 : 21 / 04 / 21 (Wed)

2nd order expansion $\quad u = \mathring{\mathrm{I}} - \mathring{\Psi} + v$

$$\begin{array}{cc} \mathring{\mathrm{I}} & \mathring{\Psi} \\ \wr & \wr \\ -\tfrac{1}{2}- & \tfrac{1}{2}- \end{array}$$

$$(\partial_t + 1 - \Delta)\, v = -v^3 - 3\,(v - \mathring{\Psi})\,\mathring{V} + Q(v)$$

$$Q(v) = b_0 + b_1 v + b_2 v^2$$

all have reg $\sim -\frac{1}{2}-$

$$\left\{\begin{aligned} b_0 &= (\mathring{\Psi})^3 - 3\,\underline{\mathring{\mathrm{I}}\,(\mathring{\Psi})^2} \\ b_1 &= 6\,\underline{\mathring{\mathrm{I}}\,\mathring{\Psi}} - 3\,(\mathring{\Psi})^2 \\ b_2 &= -3\,\mathring{\mathrm{I}} + 3\,\mathring{\Psi} \sim -\tfrac{1}{2}- \end{aligned}\right.$$

$\mathring{\mathrm{I}} \odot \mathring{\Psi} \sim -\frac{1}{2}-$

Not well defined (via deterministic analysis)

$\Leftarrow$ use the regularity lemma.

Issue: the product $v \mathring{V}$ does not make sense (2)

$$(1-) + (-1-) < 0.$$

$\Rightarrow$ Impose a structure.

if $s_1 > 0 > s_2$
$\underbrace{s_1 + s_2 > 0}_{\text{NOT true}}$

· Use the paraproduct decomposition

$$v \mathring{V} = \underbrace{v \circledless \mathring{V}}_{\substack{\sim -1- \\ = \text{worst term}}} + \overbrace{\underbrace{v \circleddash \mathring{V}}_{\text{ill-defined}} + v \circledgtr \mathring{V}}^{s_1+s_2}$$

But if we can make sense of it, it will have a better reg than $v \circledless \mathring{V}$

Let $\circledgeq = \circledgtr + \circleddash$

$\Rightarrow$ Write $v = X + Y$.

$$\begin{cases} (\partial_t + 1 - \Delta)\, X = -3\,(X + Y - \Psi^{\cdot})\, ⊜_{<}\, V^{\cdot} \\ (\partial_t + 1 - \Delta)\, Y = -(X+Y)^3 - 3\,(X + Y - \Psi^{\cdot})\, ⊜_{\geq}\, V^{\cdot} \\ \qquad\qquad + Q(X+Y) \end{cases} \tag{3}$$

- X carries the rough regularity of v but Y is smoother.

- $\boxed{X \sim (-1-) + 2 = 1-}$

  $(\cdots) \circledless V^{\cdot}$

- For Y, we ignore the ill-defined $(\cdots) \circleddash V^{\cdot}$.

  $\boxed{Y \sim \left(-\tfrac{1}{2}-\right) + 2 = \tfrac{3}{2}-}$

  $\Psi^{\cdot} \circledgeq V^{\cdot}$

We still need to make sense of the resonant product ④

$$-3\,(X + Y - \Psi) \circledcirc \vee$$

(i) $Y \circledcirc \vee \sim \left(\tfrac{3}{2}-\right) + (-1-) = \tfrac{1}{2} - \ (>0)$

$\Rightarrow$ makes sense deterministically.

(ii) $\Psi \circledcirc \vee \rightsquigarrow \overset{\Psi}{\underset{\circledcirc}{\vee}}$

$\left(\tfrac{1}{2}-\right) + (-1-) < 0$

$\sigma_N = \mathbb{E}[\mathsf{I}_N^2]$

Not Wick renormalization

$$\left(\overset{\Psi}{\underset{\circledcirc}{\vee}}\right)_N = \underline{\Psi_N \circledcirc \vee_N - 3\alpha_N \mathsf{I}_N}$$

$$\xrightarrow[N\to\infty]{} \overset{\Psi}{\underset{\circledcirc}{\vee}} \in C_t C_x^{-\frac{1}{2}-}$$

$\mathsf{I}_N = P_{\le N}\mathsf{I}$

$\vee_N = \mathsf{I}_N^2 - \sigma_N$

$\alpha_N \sim \log N$

$\Psi_N = (\partial_t + 1 - \Delta)^{-1}(\mathsf{I}_N^3 - 3\sigma_N \mathsf{I}_N)$

- Construction of stoch objects : See Mourrat-Weber-Xu
  Also, my note (taken by Justin Forlano

Note:

$$\varphi(t) = \int_{-\infty}^{t} P(t-t')\, dW(t')$$

$$\Rightarrow \quad \sigma_N = \mathbb{E}\left[\varphi^2(t,x)\right] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim N, \text{ indep of } t, x.$$

$$\varphi(t) = \int_{-\infty}^{t} P(t-t')\, dW(t')$$

$$= \int_0^t P(t-t')\, dW(t') + \overbrace{P(t) \underbrace{\int_{-\infty}^{0} P(-t')\, dW(t')}_{\sim \mu_1 \,=\, \text{massive G.F.F.}}}^{\text{random lin soln.}}$$

invariant meas for the linear flow.

$$(\partial_t + 1 - \Delta) u = \xi.$$

Back to SQE:

Issue: $X \circledcirc \vee$

$$(1-) + (-1-) < 0$$

$\Leftarrow$ use the structure of X. i.e.

$$X(t) = P(t)X_0 - 3\int_0^t P(t-t')\left[(X + Y - \Psi) \prec \vee\right](t')\,dt'$$

- $P(t)X_0 \circledcirc \vee$, well-defined.
- We "expect"

$$\int_0^t P(t-t')\left[(X+Y-\Psi) \prec \vee\right](t')\,dt'$$

$$\approx (X + Y - \Psi) \prec \underline{\underline{\curlyvee}}$$

where $\curlyvee = (\partial_t + 1 - \Delta)^{-1} \vee \sim (-1-) + 2 = 1-$

$\Rightarrow$ We write

$$X = -3\,(X + Y - \Psi) \olessthan \Upsilon + \mathrm{com}_1(X, Y)$$

$$\mathrm{com}_1(X,Y) := P(t)X_0 - 3\int_0^t P(t-t')\big[(X+Y-\Psi)\olessthan \vee\big](t')\,dt'$$
$$+\, 3\,(X + Y - \Psi) \olessthan \Upsilon$$

$\uparrow$

smoother $\sim 1+\varepsilon$. (NOT true for dispersive PDEs.)

$(1-) + (-1-) < 0.$

$$X \circledequal \vee = -3\big[(X+Y-\Psi)\olessthan \Upsilon\big] \circledequal \vee$$

$$+\, \underbrace{\mathrm{com}_1(X,Y) \circledequal \vee}_{\text{makes sense since } (1+) + (-1-) > 0.}$$

$-1-\varepsilon,\ \forall \varepsilon.$

$\uparrow$

$1+\delta$ for some δ

$(X+Y-\Psi)\prec\Upsilon$ ⇐ high freq behavior is given by Υ

⇒ We 'expect'

$$[(X+Y-\Psi)\prec\Upsilon]\circ\vee \approx (X+Y-\Psi)\prec(\Upsilon\circ\vee)$$

- $\underset{n_1}{[f}\prec\underset{n_2}{g]}\circ\underset{n_3}{h}$ $\quad |n_1|\ll|n_2|$, $\; |n_1+n_2|\sim|n_3| \Rightarrow |n_2|\sim|n_3|$

  $\approx f\prec(g\circ h)$

$$\underset{(1-)\,+\,(-1-)\,<\,0.}{\Upsilon\circ\vee} \rightsquigarrow$$

$\sim 0-$

Let

$$[\olessthan, \circledequal](f, g, h) = (f \olessthan g) \circledequal h - f (g \circledequal h)$$

NO $\olessthan$!

and set

$$\mathrm{corr}_2(X+Y) = [\olessthan, \circledequal]\big(-3(X+Y-\Psi), \curlyvee, \vee\big)$$

$$\Rightarrow \boxed{X \circledequal \vee = -3\left[(X+Y-\Psi) \underset{\circledequal}{\curlyvee}\right] + \mathrm{corr}_2(X+Y) + \mathrm{corr}_1(X,Y) \circledequal \vee}$$

0 –

Back to $Q(v)$: and

(10)

$$= \circledless + \circeq + \circledgtr$$

use stoch analysis, i.e. the reg. lemma.

$$\mathbb{E}\left[\left| \circeq (n) \right|^2 \right] \quad\quad \mathbb{E}\left[\left| (n) \right|^2 \right] \lesssim \langle n \rangle^{-3-1}$$

$$\tfrac{1}{2}- \qquad\qquad \langle n \rangle^{-d-2s_0}$$

CHECK

$$\lesssim \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_2 \rangle^4}$$

$$|n| \sim |n_2|$$

$$\sim \frac{1}{\langle n \rangle^3} \sim \langle n \rangle^{-3-2\cdot 0}$$

Summation

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha} \frac{1}{\langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

① $\alpha+\beta > d$, $\alpha, \beta < d$

② If $|n_1| \sim |n_2|$, then we only need $\alpha+\beta > d$.

$$\Rightarrow = \circledneq + \circeq$$

No renormalization needed

$\mathsf{I}(\Psi)^2$: Need to consider $\mathsf{I} \,\textcircled{$=$}\, (\Psi)^2$

$$\mathsf{I}\,\textcircled{$=$}\,(\Psi)^2 = 2\,\underbrace{\mathsf{I}\,\textcircled{$=$}\,[\Psi \,\textcircled{$<$}\, \Psi]} + \underbrace{\mathsf{I}\,\textcircled{$=$}\,[\Psi \,\textcircled{$=$}\, \Psi]}_{\left(-\frac{1}{2}-\right)+(1-)\,=\,\frac{1}{2}- \,>\,0}$$

$\left(\frac{1}{2}-\right)+\left(\frac{1}{2}-\right) = 1-$

$\Rightarrow$ makes sense

$$\Psi\,\Psi_{\textcircled{$=$}} + [\textcircled{$<$}, \textcircled{$=$}](\Psi, \Psi, \mathsf{I})$$

$$\Psi \,\textcircled{$=$}\, \mathsf{I}$$

$$\left(\frac{1}{2}-\right)+(0-) = \frac{1}{2}- > 0$$

$$\Rightarrow \mathsf{I}(\Psi)^2 = \mathsf{I}\,\textcircled{$\neq$}\,(\Psi)^2 + \mathsf{I}\,\textcircled{$=$}\,[\Psi \,\textcircled{$=$}\, \Psi] + 2\,\Psi\,\Psi_{\textcircled{$=$}}$$

$$+ 2[\textcircled{$<$}, \textcircled{$=$}](\Psi, \Psi, \mathsf{I})$$

Finally, we arrive at

$$
(*)\quad
\begin{cases}
(\partial_t + 1 - \Delta) X = -3\,(X + Y - \Psi) \circlearrowleft\!\!\!\!<\; \vee \\[6pt]
(\partial_t + 1 - \Delta) Y = -(X+Y)^3 - 3\, Y \circleddash \vee + 3\, \underline{\Psi \circleddash \vee} \\[4pt]
\qquad + 9\left[(X+Y-\Psi)\, \vee \circleddash \vee \right] - 3\,\mathrm{com}_2 (X+Y) \\[4pt]
\qquad - 3\,\mathrm{com}_1 (X,Y) \circleddash \vee + \underline{Q(X+Y)} \\[4pt]
\qquad - 3\,(X + Y - \Psi) \circlearrowleft\!\!\!\!>\; \vee
\end{cases}
$$

$-\frac{1}{2}-$

· <u>Enhanced data set</u>

$$
(u_0, \xi) \longmapsto (X_0, Y_0, \overset{-\frac12-}{\dot{\mathsf{I}}}, \overset{-1-}{\vee}, \overset{\frac12-}{\Psi}, \overset{0-}{\Psi \circleddash \dot{\mathsf{I}}}, \underbrace{\overset{-\frac12-}{\Psi \circleddash \vee}, \overset{0-}{\vee \circleddash \vee}})
$$

$u_0 = X_0 + Y_0$ — need 2^{nd} renormalization

$$
\longmapsto (X, Y) \longmapsto u = \dot{\mathsf{I}} - \Psi + X + Y
$$

$$\sigma_N \sim N, \qquad \alpha_N \sim \log N$$

$$\mathrm{V}_N = \Psi_N^2 - \sigma_N \,, \qquad \Psi\!\!\!\mid_N = \Psi_N^3 - 3\sigma_N \Psi_N$$

$$\mathrm{V}\!\!\!\!\mathrm{Y}_N = \mathrm{Y}_N \circleddash \mathrm{V}_N - \alpha_N \qquad\qquad \mathrm{Y}_N = (\partial_t + 1 - \Delta)^{-1}\, \mathrm{V}_N$$

$$\left(\mathbb{E}\Big(\underset{n_1 + n_4 = 0,\ n_2 + n_3 = 0}{\mathrm{Y}_N \circleddash \mathrm{V}_N} \Big) \cdot\sim \right.$$  $\sim \log N$

$$n_1 + n_4 = 0$$
$$n_2 + n_3 = 0$$

$$\Psi\!\!\!\!\mathrm{V}_N = \Psi\!\!\!\mid_N \circleddash \mathrm{V}_N - 3\alpha_N \Psi_N$$

$$\Psi\!\!\!\!\Psi = \Psi\!\!\!\mid \circleddash \Psi \qquad (\text{no renormalization})$$

Fix $N \in \mathbb{N}$. Start with $\circledast_N$

and get an eqn for $u_N = \mathsf{I}_N - \mathsf{Y}_N + X_N + Y_N$.

$$\begin{cases} (\partial_t + 1 - \Delta) u_N + u_N^3 - C_N u_N = \xi_N \\ u_N|_{t=0} = (u_0)_N \end{cases} \qquad \xi_N = P_{\leq N}\xi$$

where $\boxed{C_N = 3\,\underset{\substack{\sim \\ N}}{\sigma_N} - 9\,\underset{\substack{\sim \\ \log N}}{\alpha_N}}$

$$\Rightarrow (\partial_t + 1 - \Delta) u + u^3 - \infty \cdot u = \xi.$$

$L = \partial_t + 1 - \Delta$. We drop the subscript N.

- $L \varphi = \xi$, $\quad L \Psi = \Psi = \varphi^3 - 3\sigma_N \varphi$
- $Q(v) = -(v - \Psi)^3 + v^3 - 3(v - \Psi)\varphi$

$$
\begin{aligned}
Lu &= L\varphi - L\Psi + LX + LY \\
&= \xi - \varphi^3 + 3\sigma_N \varphi - 3(v - \Psi) \prec \vee \\
&\quad - v^3 - 3(v - \Psi) \succ \vee + 9\left[(v - \Psi) \circleddash \mathcal{Y}\right] \\
&\quad - 3\,\mathrm{com}_2(X+Y) - 3\,\mathrm{com}_1(X,Y) \circleddash \vee \\
&\quad - 3\,Y \circleddash \vee + 3\,\mathcal{T} + Q(v)
\end{aligned}
$$

$$= \Psi \circleddash \vee - 3\alpha_N \varphi$$

$$-3\left[(X+Y-\Psi) \prec Y\right] \circleddash \vee$$

$$\cancel{+\,3\,(X+Y-\Psi)\left[Y \circleddash \vee\right]}$$

$\cancel{Y \circleddash \vee - \alpha_N}$

$-9\alpha_N v$

$+9\alpha_N \Psi$

$$= \xi - \varphi^3 - v^3 + (3\sigma_N - 9\alpha_N)\varphi - 9\alpha_N(v - \Psi) \qquad (16)$$

$$+ Q(v) - 3(v - \Psi)\oplus v - 3(Y - \Psi)\ominus v$$

$$-3\Big[\underbrace{-3[(v-\Psi)\odot \Upsilon]\ominus v + \mathrm{com}_1(X,Y)\ominus v}_{= X\ominus v}\Big] \to -3(v-\Psi)\ominus v$$

$$= \xi - \varphi^3 - v^3 + (3\sigma_N - 9\alpha_N)\varphi - 3(v - \Psi)v$$

$$-9\alpha_N(v - \Psi) + Q(v) \qquad v = \varphi^2 - \sigma_N$$

$$= \xi - \varphi^3 - \cancel{v^3} + (3\sigma_N - 9\alpha_N)\underbrace{(\varphi + v - \Psi)}_{=u}$$

$$-3(v - \Psi)\varphi^2 + Q(v)$$

$$Q(v) = -(v-\Psi)^3 + \cancel{v^3} - 3(v-\Psi)^2\varphi$$

$$= \xi - (\varphi + v - \Psi)^3 + 3(\sigma_N - 9\alpha_N)u$$

$$= \xi - u^3 + c_N u.$$

Prop (2nd commutator) Mourrat-Weber CMP '17 GIP.

$\alpha < 1$, $\beta, \gamma \in \mathbb{R}$, $1 \le p, p_1, p_2, p_3 \le \infty$

s.t. $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.

Then, $[\varolessthan, \varodot] : (f, g, h) \mapsto (f \varolessthan g) \varodot h - f(g \varodot h)$

extends to a continuous trilinear map:

$$B^{\alpha}_{p_1} \times B^{\beta}_{p_2} \times B^{\gamma}_{p_3} \longrightarrow B^{\alpha+\beta+\gamma}_{p}$$

where $B^s_p = B^s_{p,\infty}$.

$\beta + \gamma < 0$

$\alpha = \frac{1}{2}-$ $\quad \beta = 1-$ $\quad \gamma = -1-$

Our case: $\underbrace{\mathrm{com}_2(X+Y)}_{\frac{1}{2}-} = [\varolessthan, \varodot]\left(-3(X + Y - \Psi), Y, V\right)$.

$\alpha + \beta + \gamma = \frac{1}{2} - > 0$.

· Idea of the proof:

(**) $\| [\circledcirc, \circleddash](f,g,h) \|_{B_p^{\alpha+\beta+\gamma}} \lesssim \|f\|_{B_{p_1}^{\alpha}} \|g\|_{B_{p_2}^{\beta}} \|h\|_{B_{p_3}^{\gamma}}$

For proving (**), we instead consider

$$\| [\circledcirc, \circleddash](\langle\nabla\rangle^{-\alpha} f, \langle\nabla\rangle^{-\beta} g, \langle\nabla\rangle^{-\gamma} h) \|_{B_p^{\alpha+\beta+\gamma}}.$$

$$\lesssim \|f\|_{B_{p_1}^{0}} \|g\|_{B_{p_2}^{0}} \|h\|_{B_{p_3}^{0}}$$

$$\mathcal{F}(\quad)(n) = \sum_{\underline{|j-k|\le 2}} \sum_{n=n_1+n_2+n_3} \underbrace{\boxed{\frac{\langle n\rangle^{\alpha+\beta+\gamma}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta}\langle n_3\rangle^{\gamma}}}}_{=m(\bar{n})} \underline{\varphi_k(n_3)}$$

$(f \circledcirc g)\circleddash h$ $\quad g \circleddash h$

$$\times \Big(\sum_{\underline{\ell \le q-2}} \underbrace{\varphi_\ell(n_1)\varphi_q(n_2)}_{f\circledcirc g} \underline{\varphi_j(n_1+n_2)} - \underline{\varphi_j(n_2)} \Big)$$

$$\times \hat{f}(n_1)\, \hat{g}(n_2)\, \hat{h}(n_3)$$

Roughly speaking, we want to bound the weight

$$\sum_{|j-k|\le 2} m(\bar{n})\,\varphi_k(n_3)\Big(\sum_{l<q-2}\varphi_l(n_1)\varphi_q(n_2)\varphi_j(n_1+n_2)-\varphi_j(n_2)\Big)$$

write it as (drop the summations in j,k).

$$m(\bar{n})\,\varphi_k(n_3)\sum_{l<q-2}\varphi_l(n_1)\varphi_q(n_2)\big(\varphi_j(n_1+n_2)-\varphi_j(n_2)\big)$$

$$+\ m(\bar{n})\,\varphi_k(n_3)\Big(\sum_{l<q-2}\varphi_l(n_1)\varphi_q(n_2)-1\Big)\varphi_j(n_2)$$

$$=:\ \mathrm{I}+\mathrm{II}.$$

$\underline{\mathrm{I}:}$ $\quad |n_1| \ll |n_2| \sim |n_3|$

$$|\varphi_j(n_1+n_2) - \varphi_j(n_2)| = \underbrace{\left|\varphi\left(\frac{n_1+n_2}{2^j}\right) - \varphi\left(\frac{n_2}{2^j}\right)\right|}_{= \int_0^1 \varphi'\left(\frac{n_2+n_1 t}{2^j}\right)\frac{n_1}{2^j}\,dt}$$

$$\leq |\varphi'(n^*)| \frac{|n_1|}{2^j}$$

or MVT

$|n_2| \sim 2^j$ $\qquad \lesssim \dfrac{|n_1|}{|n_2|}$

$$\Rightarrow \quad \mathrm{I} \lesssim \underbrace{\frac{m(\bar{n})\,|n_1|}{|n_2|}}\ \mathbb{1}_{|n_1| \ll |n_2| \sim |n_3|}$$

$\Rightarrow |n| \lesssim \max(|n_2|, |n_3|)$

$\sim |n_2| \sim |n_3|$

$$\sim \frac{\langle n\rangle^{\alpha+\beta+\gamma} \langle n_1\rangle^1}{\langle n_1\rangle^{\alpha} \langle n_2\rangle^{\beta+1} \langle n_3\rangle^{\gamma}}$$

$\alpha+\beta+\gamma > 0$

$\alpha < 1$

$$\lesssim \frac{\langle n_2\rangle^{\alpha+\beta+\gamma} \langle n_1\rangle^{1-\alpha}}{\langle n_2\rangle^{\beta+\gamma+1}} \sim \left(\frac{\langle n_1\rangle}{\langle n_2\rangle}\right)^{1-\alpha} \lesssim 2^{-\varepsilon j} = N_2^{-\varepsilon}$$

$\underline{\alpha < 1}$

$\underline{\mathrm{II}:}$ $\qquad \mathrm{II} = m(\bar{m})\, \varphi_h(n_3)\, \varphi_j(n_2) \Big(\underbrace{\sum_{l<q-2} \varphi_l(n_1)\, \varphi_q(n_2) - 1}_{} \Big)$

$$\sum_{l \geq q-2}$$

$$\Rightarrow |n_1| \gtrsim |n_2|$$

$$\sum_{l,q} \varphi_l(n_1)\,\varphi_q(n_2)$$

$$\sum_l \varphi_l(\xi) \equiv 1$$

$\Rightarrow$ We have $|n_1| \gtrsim |n_2| \sim |n_3|$.

$$\Rightarrow |n| \lesssim |n_1|$$

$\alpha+\beta+\gamma > 0$

$$\Rightarrow \mathrm{II} \lesssim \frac{\langle n \rangle^{\alpha+\beta+\gamma}}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\beta+\gamma}} \lesssim \frac{\langle n_1 \rangle^{\cancel{\alpha}+\beta+\gamma}}{\cancel{\langle n_1 \rangle^{\alpha}} \langle n_2 \rangle^{\beta+\gamma}} \sim \frac{\langle n_1 \rangle^{\beta+\gamma}}{\langle n_2 \rangle^{\beta+\gamma}} \lesssim 1$$

$\beta+\gamma < 0$

$|n_1| \gtrsim |n_2|$.

$\square$

Next class, we prove a bound on $\mathrm{com}_1(X,Y)$ 22

which comes down to $[P(t), \langle\rangle](f,g)$