Lec 12 : 16/04/21 (Fri)

(2.3) Global-in-time aspects:

(2.3.i) Parabolic $P(\Phi)_2$-model:

- $$(\partial_t + 1 - \Delta) u + \underset{\substack{\uparrow}}{u^k} = \sqrt{2}\,\xi. \quad \text{on } \mathbb{T}^2, \quad k \in 2\mathbb{N}+1$$

We renormalize the nonlinearity $\Rightarrow$ $:u^{k+1}:$

Pathwise approach: Control the L^p-norm of $v = u - \Psi$.

$\boxed{p = p(k)}$

- Why is this enough?

$$\begin{cases} (\partial_t + 1 - \Delta) v + \sum_{j=0}^{k} \binom{k}{j} :\Psi^j: v^{k-j} = 0 \\ v\big|_{t=0} = u_0. \end{cases}$$

As in Lec 11, work with the $Y(T)$-norm:

$$\|u\|_{Y(T)} = \sup_{0<t<T} t^\theta \|u(t)\|_{C^\sigma}$$

$\sigma = 2\varepsilon > 0$

$\theta = \varepsilon - \frac{s}{2}$

$\left(-\frac{2}{k} < s < 0.\right)$

$$t^\theta \|\Gamma(v)(t)\|_{C^\sigma = B^\sigma_{\infty,\infty}}$$

$$\lesssim \underbrace{t^\theta \, t^{-\frac{\sigma}{2} - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{\infty}\right)}}_{\substack{\lesssim 1 \\ \text{for } 0<t\ll 1.}} \; \underbrace{\sup_m \| \overset{\text{LP projection}}{\overset{\downarrow}{P_m}} u_0 \|_{L^p}}_{\lesssim \|u_0\|_{L^p}}$$

$+$ Duhamel term.

$$\Rightarrow \text{Need } \theta - \frac{\sigma}{2} - \frac{1}{p} \geq 0 \Rightarrow \frac{1}{p} \leq -\frac{s}{2} \Rightarrow p \geq -\frac{2}{s} \gg 1$$

when $s \to 0-$

i.e. $k \to \infty$.

$\Rightarrow$ LWP of (SNLH_ν) in $L^p(\mathbb{T}^2)$.

and the local existence time $\sim \left(\|u_0\|_{L^p(\mathbb{T}^2)} + \begin{matrix}\text{Wick powers}\\ \text{of } \Psi\end{matrix} \right)^{-\gamma}$

For fixed $T \gg 1$,
this part may be large
BUT finite a.s.

$\cdot \displaystyle\sum_{j=1}^{k} \|:\Psi^j:\|_{C([0,T];C^{-\varepsilon})} < C_\omega < \infty$

$\Rightarrow$ As long as we control $\sup_{0<t<T} \|v(t)\|_{L^p}$

(for each $T \gg 1$), we obtain GWP.

→ Compute $\partial_t \|v\|_{L^p}^p$ and use the equation and the control on the stochastic term to get a bound on

$$\sup_{0<t<T} \|v(t)\|_{L^p} .$$

Ginzburg - Landau.

See Trenbeth '19(?) on SCGL
(= Schrödinger - heat)

$\mathbb{T}^2$.

$$\partial_t u = (a_1 + i a_2)(\Delta - 1) u - (c_1 + i c_2)|u|^{k-1} u + \sqrt{2}\,\xi$$

a_1 = heat
a_2 = Schrödinger.

With $r = \left|\frac{a_1}{a_2}\right|$, then GWP for $r \geq C(k)$.

i.e. heat part is suff. strong.

$\mathbb{R}^2$: Mourrat - Weber A. Prob '17.

GWP of the parabolic $P(\Phi)_2$ - model on $\mathbb{R}^2$.

⇐ weighted Besov spaces

· Invariant measure argument:

Gibbs measure on $\mathbb{T}^2$:

$$d\rho = Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1}dx} \underset{\uparrow\, e^{-\|u\|^2_{H^1}}du.}{\underline{\underline{d\mu_1}}} \qquad k \in 2\mathbb{N}+1$$

A typical function u under μ_1 is NOT a function.

$\Rightarrow$ $\underline{\int_{\mathbb{T}^2} u^{k+1}\,dx} = \infty$, a.s.

$\Rightarrow$ Need to renormalize the potential energy.

We use $$\int_{\mathbb{T}^2} :u^{k+1}:\,dx = \lim_{N\to\infty} \int_{\mathbb{T}^2} :(P_{\le N}u)^{k+1}:\,dx$$

Recall: $:(P_{\le N}u)^{k+1}: \longrightarrow :u^{k+1}:$ in $W^{-\varepsilon,\infty}(\mathbb{T}^2)$ or $\mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$

a.s. / $L^p(\Omega)$.

- 70's : Euclidean quantum field theory.

$$e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} :u^{k+1}: dx} \in L^p(d\mu_1), \ \forall p<\infty$$

- hypercontractivity of OU process / Wiener chaos estimate due to Nelson '65.
- Nelson's estimate

- See my course from 2017 (Chap 3)

  Also. Oh-Thomann ←

  - Glimm-Jaffe, Simon, Da Prato-Tubaro '06.

⇒ Use Bourgain's invariant measure argument (Bourgain '96,
Oh-Robert-Tzvetkov
for SNLW on 2-d mfd

(2.3. ii) 2-d SNLW / SdNLW

pathwise approach: known only $k=3$. (GKOT)

$k\geq 5$: OPEN.

With $v = u - \Psi$, we have

$$\left(\partial_t^2 + 1 - \Delta\right) v + v^3 + \underbrace{3 v^2 \Psi + 3 v :\Psi^2: + :\Psi^3:}_{\text{rough perturbation}} = 0$$

. Two difficulties:

① $v(t) \in H^{1-\varepsilon}(\mathbb{T}^2) \setminus H^1(\mathbb{T}^2)$.

$\Rightarrow$ can not use the energy

$$E(\vec{v}) = \frac{1}{2}\int |\langle\nabla\rangle v|^2 dx + \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{4}\int v^4 dx$$

We need to smooth out v

$\Rightarrow$ I- method

② Even if v <u>were</u> in H^1, v does not satisfy (deterministic) NLW.

$\Rightarrow$ $E(\vec{v})$ is not conserved.

· If the noise is a bit smoother, $\Psi_\varepsilon = (\partial_t^2 + 1 - \Delta)^{-1} \underline{\underline{\langle \nabla_x \rangle^{-\varepsilon}}} \xi$

$\in C_t L^\infty_x$.

$\Rightarrow$ GWP by the Gronwall argument (due to Burq – Tzvetkov '14)

$$\partial_t E(\vec{v})(t) \le C(\Psi) E(t)$$

· <u>I-method</u> (= method of almost conservation law)

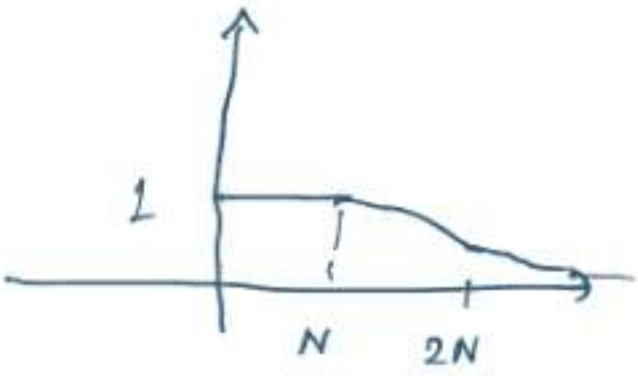
Colliander – Keel – Staffilani – Takaoka – Tao '02.

(after Bourgain's high-low method '98)

· $N \in \mathbb{N}$. Let

$0<s<1$

$$m_N(n) = \begin{cases} 1, & |n| \le N \\ \dfrac{N^{1-s}}{|n|^{1-s}}, & |n| \ge 2N \end{cases}$$



low freq: identity, high freq: integration

- $\|If\|_{W^{a+\sigma,p}} \lesssim N^{\sigma} \|f\|_{W^{a,p}}$ $\forall\ 0 \le \sigma \le 1-s$, $\forall\ 1 < p < \infty$ ⇐ LP theory.

- $\|f\|_{H^s} \lesssim \|If\|_{H^1} \lesssim \underline{\underline{N^{1-s}}} \|f\|_{H^s}$

⇒ Now, study the I-SNLW$_v$.

$$(\partial_t^2 + 1 - \Delta) Iv + I(v^3) + \underline{3I(v^2\Psi) + 3I(v:\Psi^2:) + I(:\Psi^3:)} = 0$$

⇒ $E(\overrightarrow{Iv})$ is NOT conserved for two reasons:

① $I(v^3)$ $\left(\neq (Iv)^3 \right)$ ⇒ Need a commutator estimate $I(v^3) - (Iv)^3$. (standard).

② <u>perturbation terms</u> for rough $:\Psi^j:$

Lemma: $p < \infty$

$$\Big\| \| I\Psi \|_{L^p_{T,x}} \Big\|_{L^p(\Omega)} \lesssim p^{1/2}\, T^{1/2 + 1/p}\, \underline{\underline{(\log N)^{1/2}}}$$

$$I(v^2 \Psi) \rightarrow (Iv)^2 \cdot \underline{\underline{I\Psi}} + \text{error.}$$

⇒ At the end, we obtain

$$E(I\vec{v}) \lesssim C + \int_0^t E(I\vec{v})\, \underline{\log E(I\vec{v})}(t')\, dt'.$$

→ double exponential bound.

- Invariant measure argument for SdNLW

$$(\partial_t^2 + \partial_t + 1 - \Delta)\, u + u^k = \xi, \qquad k \in 2\mathbb{N}+1.$$

- Gibbs meas : $\vec{\rho}\,(du, d(\partial_t u)) = \underset{\substack{\uparrow \\ \Phi_2^{k+1}\text{- measure}}}{\underline{\rho(du)}} \otimes \underset{\substack{\uparrow \\ \text{white noise}}}{\mu_0}(d\partial_t u))$

is formally invariant.

- Duhamel formulation for $v = u - \Psi$:

$$v(t) = \partial_t \mathcal{D}(t)\, u_0 + \mathcal{D}(t)(u_0 + u_1)$$
$$- \sum_{j=0}^{k} \binom{k}{j} \int_0^t \mathcal{D}(t-t')(:\Psi^j:\, v^{k-j})(t')\,dt'.$$

where $\mathcal{D}(t) = e^{-t/2}\, \dfrac{\sin\left(t\sqrt{\frac{3}{4}-\Delta}\right)}{\sqrt{\frac{3}{4}-\Delta}}.$ ← one deg of smoothing

- The same LWP argument (by Sobolev) as in SNLW works.

$\Rightarrow$ LWP $\Rightarrow$ a.s. GWP & invariance of $\vec{\rho}$.

<u>Note:</u> For damped NLW, the same Strichartz estimates hold <u>locally in time</u>.

↑ on $\mathbb{T}^2$.

SPDE s.t. a given meas is invariant.

↓

<u>Rmk:</u> · Parabolic Φ_2^{k+1}- model = (parabolic) <u>stochastic quantization equation</u> (for Φ_2^{k+1}- measure)

↖ Parisi - Wu '81

- SdNLW = hyperbolic Φ_2^{k+1}- model
  = canonical SQE. ← Ryang et al.

With $w = \partial_t u$,

$$\partial_t \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E}{\partial u} \\ \frac{\partial E}{\partial w} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{\partial E}{\partial w} + \xi \end{pmatrix}$$

Langvin eqn.

③ 3-d case:

(3.1) parabolic Φ^4_3-model:

$$(\partial_t + 1 - \Delta) u + u^3 = \xi.$$

· Recall on $\mathbb{T}^3$:

$$\Psi = (\partial_t + 1 - \Delta)^{-1} \xi \sim \underset{\displaystyle -\frac{d}{2}+1-}{\underset{\shortparallel}{-\frac{1}{2}-}}$$

$\Rightarrow$ Ψ^3 (and hence u^3) does NOT make sense.

- Hairer '14
- Catellier-Chouk AP
  (⇐ Gubinelli-Imkeller-Perkowski '15
- Mourrat-Weber GWP '17.
- Kupiainen AHP '16.

- Construction of stoch objects
  Mourrat-Weber-Xu.

- Lecture note from my informal course.

- <u>1st order expansion:</u> $u = v + \overset{\bullet}{|}$

Tree notation : $\bullet = \xi$

$| = \text{edge} = (\partial_t + 1 - \Delta)^{-1}$

<u>ex:</u> $\overset{\bullet}{|} = \text{stoch conv}$

$$(\partial_t + 1 - \Delta) v = -\left(v + \overset{\bullet}{|}\right)^3$$

$$= -v^3 - 3v^2\overset{\bullet}{|} - \underline{\underline{3v\overset{\bullet\,\bullet}{\vee}}} - \overset{\bullet\bullet\bullet}{\Psi} \qquad \underset{\displaystyle \| }{(-\tfrac{1}{2}-)} + (-\tfrac{1}{2}-)$$

$\underline{\underline{\overset{\bullet}{|}^2}} \rightsquigarrow \overset{\bullet\,\bullet}{\vee} \leftarrow$ renormalized versions : $\overset{\bullet\,\bullet}{\vee} \sim -1-$

$\underline{\underline{\overset{\bullet}{|}^3}} \rightsquigarrow \overset{\bullet\bullet\bullet}{\Psi} \swarrow$ $\qquad \overset{\bullet\bullet\bullet}{\Psi} \sim -\frac{3}{2}-$

do not make sense

- worst term. $\overset{\bullet\bullet\bullet}{\Psi} \sim -\frac{3}{2}- \Rightarrow v = (-\frac{3}{2}-) + \underset{\substack{\uparrow \\ \text{really } 2-}}{\overset{\substack{\text{Schauder} \\ \downarrow}}{2}} = \frac{1}{2}-$

$$\Rightarrow \quad v \ \overset{\bullet\,\bullet}{\vee}$$

$$(\tfrac{1}{2}-) + (-1-) = -\tfrac{1}{2}- < 0 \ \Rightarrow \ \text{NOT well defined}$$

- <u>2nd order expansion</u>: $u = v + \overset{\bullet}{|} - \overset{\bullet\,\bullet\,\bullet}{\Psi}$

$$\overset{\bullet\,\bullet\,\bullet}{\Psi} = (\partial_t + 1 - \Delta)^{-1}\, \overset{\bullet\,\bullet\,\bullet}{\psi} \sim \left(-\tfrac{3}{2}-\right) + 2 = \tfrac{1}{2}-$$

$$(\partial_t + 1 - \Delta)\, v = \overset{\bullet\,\bullet\,\bullet}{\psi} - \left(v + \overset{\bullet}{|} - \overset{\bullet\,\bullet\,\bullet}{\Psi}\right)^3$$

$$= \cancel{\overset{\bullet\,\bullet\,\bullet}{\psi}} - \underset{\underset{(\overset{\bullet}{|})^3}{\uparrow}}{\cancel{\overset{\bullet\,\bullet\,\bullet}{\psi}}} - \left(v - \overset{\bullet\,\bullet\,\bullet}{\Psi}\right)^3 - 3\left(v - \overset{\bullet\,\bullet\,\bullet}{\Psi}\right)^2 \overset{\bullet}{|} - \underset{\underset{(\overset{\bullet}{|})^2}{\uparrow}}{\underline{3\left(v - \overset{\bullet\,\bullet\,\bullet}{\Psi}\right) \overset{\bullet\,\bullet}{\vee}}}$$

<u>WORST term</u>: $\left(v - \overset{\bullet\,\bullet\,\bullet}{\Psi}\right) \varolessthan \overset{\bullet\,\bullet}{\vee} \sim -1-$.

$$\Rightarrow v \sim (-1-) + 2 = 1- \ \Rightarrow \ \underset{(1-)+(-1-)\,<\,0}{v\, \overset{\bullet\,\bullet}{\vee}} \quad \text{NOT well defined.}$$

Note: higher order expansion would not help since the worst term involves the unknown.

- Idea: impose a structure on v.

  Paracontrolled ansatz.