Lec 11 : 14 / 04 / 21 (Wed) ①

(2.2) 2-d stoch. NLH $\quad p$ = polynomial.

= parabolic $P(\Phi)_2$ - model.

(or Φ_2^{k+1} - model):

$$\partial_t u + (1-\Delta) u + u^k = \sqrt{2}\, \xi.$$

$$\left(\partial_t u + \tfrac{1}{2}(1-\Delta) u + u^k = \xi \right.$$

Basic space : $C^s(\mathbb{T}^2) = B^s_{\infty,\infty}(\mathbb{T}^2)$.

$\Rightarrow$ stoch convolution

$$\Psi = \Psi_{\text{heat}} = \int_0^t P(t-t')\, dW(t'), \quad P(t) = e^{t(\Delta-1)}$$

belongs to $C_t C_x^s(\mathbb{T}^2)$, $\forall s<0$, a.s.

(also in $L^p(\Omega)$.)

As in the case of 2-d SNLW,

we use the first order expansion: Da Prato - Debussche '03

$$\boxed{u = \Psi + v}$$

and solve the equation for $\underline{v = u - \Psi}$:

$$\underline{\partial_t v + (1-\Delta) v + (v + \Psi)^k = 0}$$

- Since Ψ is not a function (i.e. is only a distribution-valued function), $\underset{j \geq 2}{\Psi^j}$ does not make sense and thus we consider the renormalized version:

$$(\mathrm{SNLH}_v) \quad \partial_t v + (1-\Delta) v + \sum_{j=0}^{k} \binom{k}{j} \underbrace{:\Psi^j:}_{\underset{N \to \infty}{\lim} :(P_{\leq N} \Psi)^j:} v^{k-j} = 0$$

In the following, we only consider $(SNLH_v)$.

but we can also show $v_N \to v$

where v_N satisfies

$$\partial_t v_N + (1-\Delta) v_N + \sum_{j=0}^{k} \binom{k}{j} :\Psi_N^j: v_N^{k-j} = 0$$

$$\Psi_N = P_{\leq N} \Psi$$

and hence $u_N = \Psi_N + v_N \longrightarrow u = \Psi + v$

Main tool: Paraproduct decomposition (Bony '81)

f, g on $\mathbb{T}^d$ (or $\mathbb{R}^d$) of regularities s_1 and s_2

$$\boxed{\begin{aligned} fg &= f \circledless g + f \circleddash g + f \circledgtr g \\ &= \sum_{j<k-2} P_j(f) P_k(g) + \sum_{|j-k|\leq 2} P_j(f) P_k(g) + \sum_{k<j-2} P_j(f) P_k(g) \end{aligned}}$$

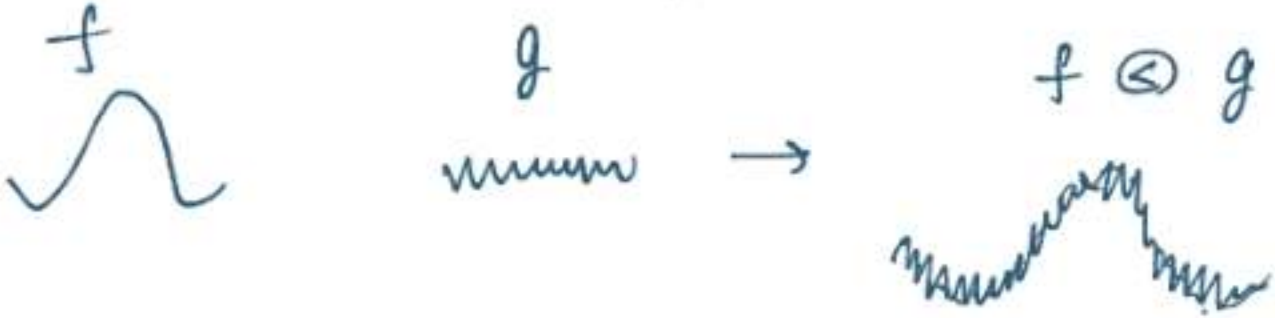
P_j = LP projection onto $\{|n| \sim 2^j\}$.

($j=0$: $\{|n| \lesssim 1\}$

• $f \mathbin{\text{\textcircled{$<$}}} g$ = paraproduct of g by f.

($f \prec g$, $\Pi_f(g)$, etc.

" freq of g $\gg$ freq of f. "



– always makes sense (for any $s_1, s_2 \in \mathbb{R}$.)

as a distribution. with regularity $\boxed{\min(\underbrace{s_2}_{\text{reg of } g},\ s_1+s_2)}$

- same for $f \circledcirc g \sim \min(s_1, s_1+s_2)$.

- $f \ominus g$: Resonant product of f and g.

  ($f \odot g$, $R(f,g)$, etc.

  - may not make sense as a distribution.
  - In general,

    $s_1 + s_2 > 0 \Rightarrow f \ominus g$ makes sense

    $\sim s_1 + s_2$.

(Recall the product estimate (ii) from GKO, where
$f \ominus g$ makes sense for $s_1 + s_2 = 0$.
in terms of Sobolev spaces.

- In studying nonlinear PDEs, the main task is to make sense of the nonlinearity, say u^k

  (or give a meaning.)

- Since the paraproduct $f \,\textcircled{<}\, g$ and $f \,\textcircled{>}\, g$ always make sense, the main job is to make sense of the resonant product $f \,\textcircled{=}\, g$.

  (more in the parabolic thinking.)

- When there is an issue in making sense of u^k, we overcome this issue by imposing a structure on u

  - $u = \Psi + v$.
  - $u \in X^{s,b}$

Lemma: (Paraproduct estimates)

$s_1, s_2 \in \mathbb{R}, \quad 1 \le p, p_1, p_2, q \le \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

① $\| f \circledless g \|_{B^{s_2}_{p,q}} \lesssim \| f \|_{L^{p_1}} \| g \|_{B^{s_2}_{p_2,q}}.$

f is a function ($s_1 \ge 0$)

$s_1 < 0$

② $\| f \circledless g \|_{B^{s_1+s_2}_{p,q}} \lesssim \| f \|_{B^{s_1}_{p_1,q}} \| g \|_{B^{s_2}_{p_2,q}}$

① & ② : $f \circledless g \sim \min(s_2, s_1+s_2)$

③ $s_1 + s_2 > 0$

$\| f \circleddash g \|_{B^{s_1+s_2}_{p,q}} \lesssim \| f \|_{B^{s_1}_{p_1,q}} \| g \|_{B^{s_2}_{p_2,q}}$

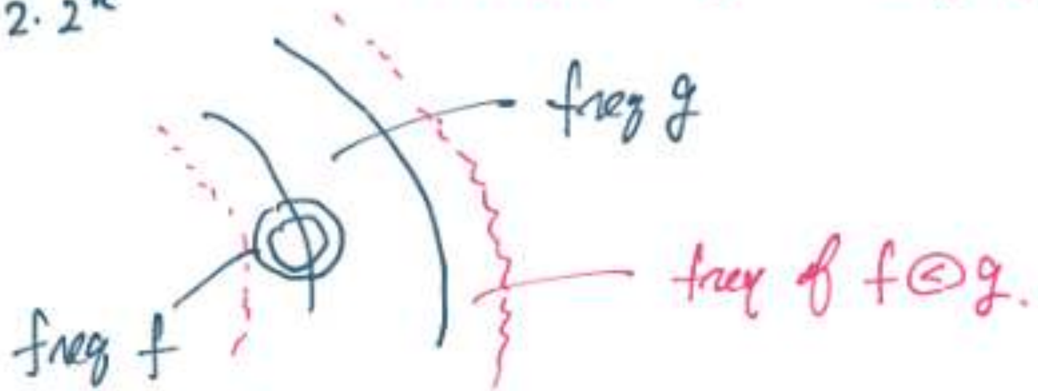
Pf: ① $\| f \circledcirc g \|_{B^{s_2}_{p,q}} \sim \Big\| 2^{s_2 k} \| P_k (f \circledcirc g) \|_{L^p_x} \Big\|_{\ell^q_k(\mathbb{Z}_{\geq 0})}$

$$\boxed{P_k (f \circledcirc g) = \sum_{i=-2}^{2} \sum_{j<k+i-2} P_j(f)\, P_{k+i}(g)}$$

freq $\sim 2^k$

$\frac{1}{2} \cdot 2^k \leq \cdot \leq 2 \cdot 2^k$

freq of $f \circledcirc g$ ~ freq of g



Let $S_k(f) = \sum_{j<k} P_j(f)$: f projected onto freq $\{|n| \lesssim 2^k\}$

$\Rightarrow P_k (f \circledcirc g) = \sum_{i=-2}^{2} S_{k+i-2}(f)\, P_{k+i}(g)$.

Take L^p_x-norm and apply Hölder's ineq.

$$\|P_k(f \circledcirc g)\| \lesssim \sum_{i=-2}^{2} \underbrace{\|S_{k+i-2}(f)\|_{L^{p_1}}}_{\lesssim \|f\|_{L^{p_1}} \text{ unif in } k,i} \|P_{k+i}(g)\|_{L^{p_2}}$$

$2^{s_2 k}$ $\qquad\qquad\qquad\qquad$ $\sim 2^{s_2(k+i)}$

- Take ℓ^q_k $(\sim \ell^q_{k+i})$

$$\|f \circledcirc g\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{s_2}_{p_2,q}}.$$

(2)

$$\underbrace{P_k(f \circledS g)}_{2^{(s_1+s_2)k}} \sim \sum_{i=-2}^{2} \underbrace{\sum_{0\le j<k+i-2}}_{2^{-s_1 j}\,2^{s_1 k}} \overbrace{P_j(f)}^{2^{s_1 j}}\, 2^{s_2 k} P_{k+i}(g)$$

Take L^p_x-norm and apply Hölder's ineq:

$s_1<0$

$$\lesssim \sum_i \underbrace{\sum_{0\le j\le k+i-2} 2^{s_1(k-j)} \cdot \underbrace{\left(2^{s_1 j}\|P_j(f)\|_{L^{p_1}}\right)}_{\ell^q_j}}_{\text{Hölder in } j.} \left(2^{s_2(k+i)}\|P_{k+i}(g)\|_{L^{p_2}}\right).$$

$$\lesssim \sum_i \|f\|_{B^{s_1}_{p_1,q}} \left(2^{s_2(k+i)}\|P_{k+i}(g)\|_{L^{p_2}}\right)$$

$\Rightarrow$ Now, take ℓ^q_k $(\sim \ell^q_{k+i})$.

③ $$\underbrace{P_k(f \circledcirc g)}_{2^{(s_1+s_2)k}} = \sum_{i=-2}^{2} \boxed{\underline{\underline{\sum_{j \geq k-10}}}\ \underbrace{P_k}_{2^{(s_1+s_2)(k-j)}} \big(\underbrace{P_j(f)}_{2^{s_1 j}}\ \underbrace{P_{j+i}(g)}_{2^{s_2 j}}\big)}$$

Take ℓ^q_k-norm.

View this as a convolution of

$$\underbrace{2^{(s_1+s_2)j}\,\mathbb{1}_{j\leq 10}}_{\in\ \ell^1_j(\mathbb{Z})} \quad \text{and} \quad \underbrace{P_j(f)P_{j+i}(g)\cdot \mathbb{1}_{j\geq 0}}_{\text{in } \ell^q_j(\mathbb{Z})}$$

$\Leftarrow$ Need $s_1+s_2>0$

Hölder in j.

$\Rightarrow$ By Young's ineq,

$$\| f \circledcirc g \|_{B^{s_1+s_2}_{p,q}} \lesssim \| f \|_{B^{s_1}_{p_1,q}} \| g \|_{B^{s_2}_{p_2,\infty}}$$

$\square$

Cor: Let $s>0$. Then, $C^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$ is an algebra.

with $\|fg\|_{C^s} \lesssim \|f\|_{C^s}\|g\|_{C^s}$

· Back to $(SNLH_v)$.

Duhamel formulation:

$$v(t) = \Gamma v(t) = P(t)u_0 - \sum_{j=0}^{k}\binom{k}{j}\int_0^t P(t-t')\overbrace{\underbrace{:\Psi^j:}_{\text{in } C_T C_x^{-\varepsilon}}\, v^{k-j}}^{\text{put this in } C^{-\varepsilon}}(t')\,dt'.$$

(can replace this by deterministic $\Xi_j \in C([0,1]; C_x^{-\varepsilon})$)

· $s>0$ $\quad s=2\varepsilon.$

$$\|\Gamma v\|_{C_T C_x^s} \lesssim \|u_0\|_{C^s} + \sum_{j=0}^{k}\binom{k}{j}\int_0^t \underbrace{(t-t')^{-\frac{3}{2}\varepsilon}}_{\text{Schauder}}\|:\Psi^j:\, v^{k-j}(t')\|_{C^{-\varepsilon}}\,dt'.$$

$$\| :\Psi^j: v^{k-j} \|_{C^{-\varepsilon}}$$

$$\lesssim \| :\Psi^j: \|_{C^{-\varepsilon}} \| v \|_{C^{2\varepsilon}}^{k-j}$$

$\Leftarrow$ Moral: In order to make sense of the product, the sum of the regularities > 0.

BUT: the resulting regularity of the product is given by one of the paraproducts.

ex) $s_1 < 0 < s_2$. Need $s_1 + s_2 > 0$

but $fg \sim s_1$ coming from $f \circledcirc g$

$$\Rightarrow \|\Gamma v\|_{C_T C^s_x} \lesssim \|u_0\|_{C^s} + \sum_{j=0}^{k} \binom{k}{j} T^{\theta} \|:\Psi^j:\|_{C_T C_x^{-\varepsilon}} \|v\|^{k-j}_{C_T C^s}$$

for $s > 0$ (s.t. $s + \varepsilon > 0$)

$\Rightarrow$ LWP of SNLH_ν in $C^s(\mathbb{T}^2)$, $s > 0$.

· <u>Rougher initial data?</u> $u_0 \in C^s(\mathbb{T}^2)$, $s < 0$.

$$\|u\|_{Y(T)} = \sup_{0<t<T} t^{\theta} \|u(t)\|_{C^\sigma_x} \qquad \sigma > 0. \quad \theta > 0$$

$$\Rightarrow t^{\theta} \|\Gamma v(t)\|_{\cancel{C^\sigma_x}} \lesssim t^{\theta}\, \overbrace{t^{\frac{s-\sigma}{2}}}^{<0} \|u_0\|_{C^s} \qquad \theta = \frac{\sigma - s}{2}, \quad \underline{\sigma = 2\varepsilon}$$

$C^{2\varepsilon}_x$

$$+ \sum_{j=0}^{k} \binom{k}{j} t^{\theta} \int_0^t \underline{(t-t')^{-\frac{3}{2}\varepsilon}} \|:\Psi^j: v^{k-j}(t')\|_{C_x^{-\varepsilon}}\, dt'$$

$$\|:\Psi^j: v^{k-j}(t')\|_{C_x^{-\varepsilon}}$$

$$\lesssim \|:\Psi^j:\|_{C_T C_x^{-\varepsilon}} \underbrace{\left((t')^\theta \|v(t')\|_{C_x^{2\varepsilon}}\right)^{k-j}}_{\le \|v\|_{Y(T)}^{k-j}} \underline{(t')^{-(k-j)\theta}}$$

Need to control

$$t^\theta \int_0^t (t-t')^{-\frac{3}{2}\varepsilon} \underline{(t')^{-(k-j)\theta}} \, dt' \qquad t \le T.$$

Beta function

$$\underline{\text{FACT:}} \quad t^{\alpha_1} \int_0^t (t-t')^{\alpha_2} (t')^{\alpha_3} \, dt' = B(\alpha_2+1, \alpha_3+1) < \infty$$

$$\alpha_1 + \alpha_2 + \alpha_3 = -1, \quad \alpha_2, \alpha_3 > -1$$

$$B(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1} \, dt, \quad \mathrm{Re}\, x, \mathrm{Re}\, y > 0.$$

Need
$$\theta + \left(-\tfrac{3}{2}\varepsilon\right) + (-k\theta) \geq -1.$$

$$\theta = \frac{\sigma - s}{2} = \varepsilon - \frac{s}{2}$$

$$\Rightarrow \quad s > -\frac{2}{k-1}.$$

- $\alpha_2 > -1 \Rightarrow \varepsilon \ll 1$
- $\alpha_3 > -1 \qquad -k\theta > -1 \Rightarrow \underset{\varepsilon - \frac{s}{2}}{\overset{\shortparallel}{\theta}} < \frac{1}{k}$

$$\Rightarrow \underline{\underline{s > -\frac{2}{k}}} \quad \left(> -\frac{2}{k-1} \right)$$

↑ scaling crit regularity

(When $\tilde{j} = 0$: $t^{\theta} \int_0^t (t')^{-k\theta} \, dt' < \infty \Rightarrow -k\theta > -1$.

- By a contraction argument, construct $v \in Y(T)$

$\Rightarrow v \in C([0,T]; C^{\sigma}_{x}(\mathbb{T}^2))$, $\sigma = 2\varepsilon > 0$.

- A posteriori, show $v \in C([0,T]; C^{s}_{x}(\mathbb{T}^2))$, $(s<0)$

$\Rightarrow$ LWP of $(SNLH_v)$ in $C^s(\mathbb{T}^2)$, $s > -\frac{2}{k}$.