Lec 10   09/04/21 (Wed)

First order decomposition : $u = \Psi + v$
$\uparrow$ smoother

- Da Prato – Debussche '02.
- McKean '95, Bourgain '96.

Consider

$$(\text{SNLW}'_v) \quad \begin{cases} (\partial_t^2 + 1 - \Delta) v + \sum_{j=0}^{k} \binom{k}{j} \Xi_j v^{k-j} = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1) \end{cases} \qquad \Xi_0 = 1.$$

Enhanced data set: $\Xi = (u_0, u_1, \Xi_1, \cdots, \Xi_k)$

$$\boxed{\Xi = (u_0, u_1, \Xi_1, \cdots, \Xi_k) \in \mathcal{H}^s \times \big(L^\infty([0,1]; W^{-\varepsilon,\infty}(\mathbb{T}^2))\big)^{\otimes k}}$$

$$\|\Xi\|_{\mathcal{X}^s} = \|(u_0, u_1)\|_{\mathcal{H}^s} + \sum_{j=1}^{k} \|\Xi_j\|_{L^\infty([0,T]; W_x^{-\varepsilon,\infty})}$$ (2)

<u>Prop:</u> $\exists\ \varepsilon_k > 0$ small s.t. for $0<\varepsilon<\varepsilon_k$,

(SNLW$_v'$) is locally well-posed:

- existence, uniqueness,
- $\Xi \longmapsto \underset{\substack{\cap \\ C([0,T]; H^{1-\varepsilon})}}{v}$ is continuous.

$T = T(\|\Xi\|_{\mathcal{X}^s}) > 0$

- <u>Case 1:</u> $s = 1-\varepsilon$.

Duhamel formulation

$$v(t) = \Gamma v(t) = \partial_t S(t) u_0 + S(t) u_1 - \sum_{j=0}^{k} \binom{k}{j} \int_0^t S(t-t')(\Xi_j v^{k-j})(t')dt'$$

$$S(t) = \frac{\sin t\langle\nabla\rangle}{\langle\nabla\rangle}$$

$\underline{j=0}$: $\displaystyle \left\| \int_0^t S(t-t')\, v^k(t')\,dt' \right\|_{C_T H_x^{1-\varepsilon}}$

$T \leq 1$

$$\lesssim T \, \| v^k \|_{C_T H_x^{-\varepsilon}} \qquad \frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}$$

Sobolev

$$\lesssim T \, \| v^k \|_{C_T L_x^{2/1+\varepsilon}} \qquad \frac{\varepsilon}{2} = \frac{1+\varepsilon}{2} - \frac{1}{2}$$

$$= T \, \| v \|^k_{C_T L_x^{2k/1+\varepsilon}}$$

Sobolev

$$\lesssim T \, \| v \|^k_{C_T H_x^{1-\varepsilon}} \qquad \frac{1-\varepsilon}{2} \geq \frac{1}{2} - \frac{1+\varepsilon}{2k}$$

$$\left(\Leftarrow 0 < \varepsilon \leq \frac{1}{k-1} \right)$$

$\underline{1 \leq j < k}$: $\displaystyle \left\| \int_0^t S(t-t') \left(\Xi_j v^{k-j} \right)(t')\,dt' \right\|_{C_T H_x^{1-\varepsilon}}$

$$\lesssim T \, \| \Xi_j v^{k-j} \|_{C_T H_x^{-\varepsilon}}$$

only a distribution (in x)

$$= T \, \| \langle \nabla \rangle^{-\varepsilon} (\Xi_j \, v^{k-j}) \|_{C_T L^2_x}$$

$$\lesssim T \, \| \langle \nabla \rangle^{-\varepsilon} \Xi_j \|_{L^\infty_T L^{2/\varepsilon}_x} \, \| \langle \nabla \rangle^{\varepsilon} (v^{k-j}) \|_{L^\infty_T L^2_x}$$

frac. leib. rule

$$\lesssim T \, \| \Xi \|_{\mathcal{X}^{1-\varepsilon}} \, \| \langle \nabla \rangle^{\varepsilon} v \|^{k-j}_{L^\infty_T L^{2(k-j)}_x} \qquad \frac{1}{p} + \frac{1}{2} \le \frac{1}{2} + \frac{\varepsilon}{2}$$

Sobolev

$$\lesssim T \, \| \Xi \|_{\mathcal{X}^{1-\varepsilon}} \, \| v \|^{k-j}_{C_T H^{1-\varepsilon}_x} \qquad \frac{1-2\varepsilon}{2} \ge \frac{1}{2} - \frac{1}{2(k-j)}$$

$$\left(\Leftarrow \quad 0 < \varepsilon \le \frac{1}{2(k-1)} \right.$$

$\underline{j = k:}$

$$\left\| \int_0^t S(t-t') \, \Xi_k(t') \, dt' \right\|_{C_T H^{1-\varepsilon}_x}$$

$$\lesssim T \, \| \Xi_k \|_{C_T H^{-\varepsilon}_x}$$

$$\lesssim T \, \| \Xi \|_{\mathcal{X}^{1-\varepsilon}} .$$

$\Rightarrow$ Putting together, we get

$$\|\Gamma v\|_{C_T H_x^{1-\varepsilon}} \le C_1 \|(u_0,u_1)\|_{\mathcal{H}^{1-\varepsilon}} + C_2 \underline{T\,\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}} \left(1+\|v\|_{C_T H_x^{1-\varepsilon}}\right)^{k-1} + C_3 T \|v\|^k_{C_T H_x^{1-\varepsilon}}$$

A similar estimate holds for the difference $\Gamma(v_1)-\Gamma(v_2)$.

$\Rightarrow$ Γ is a contraction on a ball of radius

$$\sim \|(u_0,u_1)\|_{\mathcal{H}^{1-\varepsilon}} + \|\Xi_k\|_{C([0,1];W_x^{-\varepsilon,\infty})}$$

by choosing $T = T(\|\Xi\|_{\mathcal{X}^{1-\varepsilon}}) > 0$ sufficiently small.

Summary :

Known : 2-d NLW is ill-posed in negative Sobolev spaces
(Christ - Colliander - Tao '03, Oh-Okamoto-Tzvetkov,
Forlano - Okamoto DPDE '20

$\Rightarrow$ 2-d SNLW :

$(u_0, u_1, \xi) \longrightarrow u$ ill-posed.

Idea: Decompose the ill-defined soln map.

① Use stochastic analysis to construct an enhanced data set
only in this step.

② use deterministic analysis to prove LWP for the remainder term $v = u - \Psi$.

continuous map

$$\boxed{(u_0, u_1, \xi) \overset{①}{\longmapsto} \Xi = (u_0, u_1, \Psi, :\Psi^2:, \cdots, :\Psi^k:) \overset{②}{\longmapsto} v \rightarrow u = \Psi + v}$$

NOT conti

Thm: $\exists\, \varepsilon_k > 0$ small s.t. for $0 < \varepsilon < \varepsilon_k$,
the renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $\mathcal{H}^{1-\varepsilon}(\mathbb{T}^2)$.
Moreover, the soln u_N to the truncated renormalized
SNLW : $(\partial_t^2 + 1 - \Delta) u_N + :u_N^k: = \xi_N$
converges to $u = \Psi + v$. in $C([0, T_\omega]; H^{-\varepsilon}(\mathbb{T}^2))$, a.s.

- First order expansion $u = \underbrace{\Psi}_{\text{rough}} + \underbrace{v}_{\text{smoother}}$
    - trick to solve the equation
    - also gives a description
        ex: in small scales, u "behaves like" Ψ.
- Without renormalization, no non-trivial limit exists.
  ($u_N \to 0$ or lin soln.)

Oh - Okamoto - Robert
SPA '20.

$\Leftarrow$ Triviality

- **Regularization on ξ:**
  - by $P_{\le N}$
  - by mollification (spatial / space-time). $\xi_\delta = \gamma_\delta * \xi$ (conv in prob.)

$\Rightarrow$ same limit

Q: Rougher initial data?

NLW: $\partial_t^2 u - \Delta u + u^k = 0$ on $\mathbb{R}^d$.

- Scaling invariance $\Rightarrow$ Scaling critical regularity
$$s_{\text{scaling}} = \frac{d}{2} - \frac{2}{k-1}.$$
- Lorentz invariance (Conformal) $\Rightarrow$ $s_{\text{conf}} = \frac{d+1}{4} - \frac{1}{k-1}$

$$s_{crit} = \max(s_{scaling}, s_{conf}, 0) \underset{d=2}{=} \max\left(1 - \frac{2}{k-1}, \frac{3}{4} - \frac{1}{k-1}, 0\right).$$

Thm: GKO TAMS '18

(i) $k = 2,3$: $s > s_{crit}$ (ii) $k \geq 4$: $s \geq s_{crit}$.

The renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$.

for initial data

$\boxed{\text{Strichartz estimates}}$: $0 \leq s \leq 1$

(q, r) s-admissible

$(\tilde{q}, \tilde{r})$ dual s-admissible

if $1 \leq \tilde{q} \leq 2 \leq q \leq \infty$
$1 \leq \tilde{r} \leq 2 \leq r \leq \infty$

scaling: $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} - 2$

$(q, r, d) \neq (2, \infty, 3)$

$(\tilde{q}, \tilde{r}, d) \neq (2, 1, 3)$

admissibility: $\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}$, $\frac{1}{\tilde{q}'} + \frac{d-1}{2\tilde{r}'} \leq \frac{d-1}{4}$

$(\tilde{q}, \tilde{r})$ dual s-admissible

$\iff (\tilde{q}', \tilde{r}')$ is $(1-s)$-admissible

Then, $\begin{cases} (\partial_t^2 + 1 - \Delta)\, u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$ on $\mathbb{T}^d$.

$$\Rightarrow \boxed{\|(u, \partial_t u)\|_{L_T^\infty \mathcal{H}_x^s} + \|u\|_{L_T^q L_x^r} \lesssim \|(u_0, u_1)\|_{\mathcal{H}_x^s} + \|F\|_{L_T^{\tilde{q}'} L_x^{\tilde{r}'}}} \qquad 0 \le T \le 1$$

Also, $\boxed{(\text{LHS}) \lesssim \|(u_0, u_1)\|_{\mathcal{H}_x^s} + \|F\|_{L_T^1 H_x^{s-1}}}$

· Strichartz on $\mathbb{R}^d$: Vişan's Oberwolfach note.

and finite speed of propagation

- We only consider $k=3$. $\Rightarrow$ $s_{crit} = 1/4$.

  (general case : GKO, Oh–Thomann, Tadase '20.

$$(q, r) = \left(\frac{12}{1+4\delta}, \frac{3}{1-2\delta}\right), \quad \left(\tfrac{1}{4}+\delta\right)\text{- admis.}$$

$$(\tilde{q}, \tilde{r}) = \left(\frac{12}{9+4\delta}, \frac{3}{3-2\delta}\right), \quad \text{dual } \left(\tfrac{1}{4}+\delta\right)\text{- admis.}$$

· <u>nonlinearity</u>: $\Xi_j \, v^{3-j}$ in $L^{\tilde{q}}_T L^{\tilde{r}}_x + L^1_T H^{s-1}_x$

$$\cdot \; \| v^3 \|_{L^{\tilde{q}}_T L^{\tilde{r}}_x} = \| v \|^3_{L_T^{36/9+4\delta} L_x^{9/3-2\delta}} \lesssim T^{\theta} \| v \|^3_{L^q_T L^r_x}$$

$$\frac{36}{9+4\delta} < q = \frac{12}{1+4\delta}$$

$$\frac{9}{3-2\delta} < r = \frac{3}{1-2\delta}$$

deterministic theory.

LWP of cubic NLW in $\mathcal{H}^s(\mathbb{T}^2)$, $s > \frac{1}{4}$.

$$\cdot\ \|\Xi_3\|_{L^1_T \mathcal{H}^{s-1}_x} \le T\,\|\Xi_3\|_{L^\infty_T W^{-\varepsilon,\infty}_x} \quad \text{if } s-1 \le -\varepsilon.$$

$$\le T\,\|\Xi\|_{\mathcal{X}^s}. \qquad \boxed{s = \tfrac14 + \delta}$$

$$\cdot\ \|\Xi_j v^{3-j}\|_{L^1_T H^{s-1}_x} \qquad s-1 = -\tfrac34 + \delta$$

Sobolev

$$\le \|\langle\nabla\rangle^{-\delta}(\Xi_j v^{3-j})\|_{L^1_T L^{8/7-8\delta}_x} \qquad \frac{\frac34 - 2\delta}{2} = \frac{7-8\delta}{8} - \frac12$$

$$\lesssim \|\langle\nabla\rangle^{-\delta}\Xi_j\|_{L^\infty_T L^{\infty-}_x} \underbrace{\|\langle\nabla\rangle^{\delta} v^{3-j}\|_{L^1_T L^{8/7-8\delta}_x}} \qquad \frac1p + \frac1q \le \frac1r + \frac{s}{d}$$

$$\frac{\delta}{2} + \frac{7-8\delta}{2} \le \frac{7-8\delta}{8} + \frac{\delta}{2}$$

replace by L^∞_x

$\underline{j=2}$: $$\lesssim T\,\|v\|_{L^\infty_T H^\delta_x} \le T\,\|v\|_{L^\infty_T H^s_x}$$

$\underline{j=1}$: $$\lesssim T\,\|\langle\nabla\rangle^{\delta} v\|_{L^\infty_T L^{16/7-8\delta}_x} \lesssim T\,\|v\|^2_{L^\infty_T H^{\frac{1+16\delta}{8}}}$$

frac Leib. Sobolev

$$\frac{1+8\delta}{8}\Big/2 = \frac12 - \frac{7-8\delta}{16}$$ and add δ from $\langle\nabla\rangle^\delta$

Let $\|\vec{u}\|_{Y^s(T)} = \|(u, \partial_t u)\|_{L^\infty_T \mathcal{H}^s_x} + \|u\|_{L^q_T L^r_x}$.

$$\Rightarrow \|\overrightarrow{\Gamma(v)}\|_{Y^s(T)} = \|(\Gamma(v), \partial_t \Gamma(v))\|_{Y^s(T)}$$

$$\leq C_1 \|(u_0, u_1)\|_{\mathcal{H}^s} + C_2 \left(\|v\|_{Y^s(T)} + \|\Xi\|_{\mathcal{Z}^{-\delta}}\right)^3$$

where $\|\Xi\|_{\mathcal{Z}^{-\delta}} = \sum_{j=1}^{3} \|\Xi_j\|_{L^\infty([0,1]; W_x^{-\delta,\infty})}$

A similar estimate holds for the difference.