Singular stochastic dispersive PDEs. 17/02/21 (Wed) ①

· Dispersive PDEs

nonlinear Schrödinger eqn : u, $\mathbb{C}$-valued

(NLS) $i\partial_t u - \Delta u \pm |u|^{k-1} u = 0$, $k \in 2\mathbb{N}+1$

nonlinear wave eqn : u, $\mathbb{R}$-valued

(NLW) $\partial_t^2 u - \Delta u \pm u^k = 0$

+ sign : defocusing case ($k \in 2\mathbb{N}+1$)

− sign : focusing case

Main goal: Understand how given initial data is propagated by nonlinear dynamics

1st question: Well-posedness [existence, uniqueness, stability under perturbation

local well-posedness. i.e for short times.

$\xrightarrow{LWP}$ Q: global well-posedness or finite time blowup?

Stochastic dispersive PDEs

SNLS $\quad i\partial_t u - \Delta u + |u|^{k-1}u = \phi\,\xi$ ← stoch forcing

bdd op on L^2 (smoothing operator in x)

SNLW $\quad \partial_t^2 u - \Delta u + u^k = \xi$

SdNLW (stochastic damped NLW):

$$\partial_t^2 u + \underline{\partial_t u} - \Delta u + u^k = \xi$$

In terms of LWP, SNLW and SdNLW are the "same".

ξ = space-time white noise $\quad \xi(t,x)$

(Gaussian)

$$\mathbb{E}[\xi(t_1,x_1)\,\xi(t_2,x_2)] = \delta(t_1-t_2)\,\delta(x_1-x_2)$$

⇐ random behavior at diff pts are <u>indep</u>

⇒ very rough: reg $-\frac{d}{2}-\varepsilon$ in x

$\qquad\qquad\qquad\quad -\frac{1}{2}-\varepsilon$ in t.

$\Rightarrow$ makes the relevant analysis challenging.

① analytically challenging (need analysis, PDEs, stoch analysis)

<u>SNLS</u>: 1-d cubic, $\phi = \mathrm{Id}$ (i.e. space-time white noise)

LWP: OPEN, critical

<u>SNLW</u>: 3-d cubic is open.

② SdNLW formally preserves the <u>Gibbs measure</u>

measure on functions/distributions

Stochastic nonlinear heat eqn
(reaction-diffusion eqn)

$$\partial_t u - \Delta u + u^k = \xi.$$

formally preserves Φ_d^{k+1}-measure : $d\rho = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1}dx}$

$$\times \underbrace{e^{-\frac{1}{2}\int |\nabla u|^2 dx} du}_{\text{Gaussian free field}}$$

$\Leftarrow$ Euclidean QFT (quantum field theory)
constructive QFT

'70 – '80 : construction of Φ_d^{k+1} – measures

$k+1$ = deg of nonlinearity
d = spatial dimension

$\underline{d=1,2}$; any $k \in 2\mathbb{N}+1$

$-\frac{1}{k+1}\int u^{k+1}dx$, good sign

$d=2$ requires renormalization

$\underline{d=3}$: $\underline{k=3}$ only Φ_3^4 – measure.

· stochastic quatization: Introduce a stoch PDE which preserves Φ_d^{k+1} – measure. (Parisi – Wu)

$\Rightarrow$ SNLH (LWP) : $d=1$: easy
$d=2$: Da Prato – Debussche '03

$\underline{d=3}$ $(k=3)$ SQE = stoch quantization eqn

$$\partial_t u + (1-\Delta)u + u^3 \underbrace{- \infty \cdot u}_{\text{renormalization}} = \xi$$

- $\underline{\text{Hairer 2014}}$: LWP of dynamical Φ^4_3-model (parabolic Φ^4_3-model)
  - regularity structures
- $\underline{\text{Gubinelli - Imkeller - Perkowski 2015}}$
  - paracontrolled distributions
- $\underline{\text{Kupiainen 2016}}$
  - RG method (renormalization group method)

$\underline{\text{Wave analogue:}}$ $\partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 - \infty \cdot u = \xi$

hyperbolic Φ^4_3-model

$(u, \partial_t u) \sim \underset{u}{\Phi^4_3\text{-measure}} \otimes \underset{\partial_t u}{\text{white noise}}$ formally invariant

So far, we discussed results on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$\Leftarrow$ spatial roughness is the issue.

On $\mathbb{R}^d$: noise does not decay as $|x| \to \infty$.

soln, not integrable in the $W^{s,p}$-sense.

heat: Mourrat - Weber '17 A.P.

Hofmanova - Gubinelli

SNLW: 2-d cubic. Tolomeo A.P.

<u>SNLS</u>: $i\partial_t u - \Delta u + |u|^{k-1}u = \Phi\xi$

<u>Ito</u>: $i\,du = (\Delta u - |u|^{k-1}u)\,dt + \Phi\,dW$

$$\boxed{\begin{aligned} W(t,x) &= L^2\text{- cylindrical Wiener process.} \\ &= \sum_{n\in\mathbb{Z}^d} \beta_n(t)\, e^{in\cdot x} \end{aligned}}$$

$\{\beta_n\}_{n\in\mathbb{Z}^d}$ = independent standard $\mathbb{C}$-valued B.M.

$\operatorname{Var} \beta_n(t) = t$ $\qquad\qquad$ $\beta_n = \operatorname{Re}\beta_n + i\operatorname{Im}\beta_n$

indep $\mathbb{R}$-valued B.M.

SNLW, SNLH.
impose $\beta_{-n} = \overline{\beta_n}$

<u>Mild formulation</u>: (Duhamel formulation)

$$u(t) = S(t)u_0 - \int_0^t S(t-t')|u|^{k-1}u(t')\,dt' + \boxed{\int_0^t S(t-t')\,\Phi\,dW(t')}$$

$S(t) = e^{-it\Delta}$ = linear Schrö propagator $\qquad$ stoch. convolution

· Stoch convolution (for SNLS)

$$\Psi(t) = \int_0^t S(t-t')\,\phi\, dW(t')$$

$$W = \sum_n \beta_n e_n$$

$$e_n(x) = e^{in\cdot x}$$

$$= \sum_{n\in\mathbb{Z}^d} e_n \underbrace{\int_0^t e^{i(t-t')|n|^2} \phi_n \, d\beta_n(t')}_{\text{Wiener integral}}$$

$$S(t) = e^{-it\Delta}$$

$$\widehat{S(t)f}(n) = e^{it|n|^2}\hat{f}(n)$$

if ϕ is "diagonal"

$$\phi(e_n) = \phi_n e_n$$

i.e. Fourier multiplier

· <u>Wiener integral:</u> ($\mathbb{R}$-valued case) Kuo: Intro to stoch integ.

Given $f \in L^2([a,b])$, f, deterministic

$$I(f) = \int_a^b f(t)\, dB(t) \qquad B = \text{B.M.}$$

$\Leftarrow$ • mean 0 Gaussian r.v ($\mathbb{E}[I(f)] = 0$)

• $\mathrm{Var}(I(f)) = \|f\|^2_{L^2([a,b])}$

$$\mathrm{Var}(I(f)) = \mathbb{E}\left[|I(f)|^2\right]$$

($\Leftarrow$ check for step functions and approximate f. in $L^2(\Omega)$

- $B(0)=0$
- indep increment over disjoint intervals
- $B(t_2)-B(t_1) \sim N(0, t_2-t_1)$

- If f is C^1, we can define it as a Paley–Wiener–Zygmund integral

$$I(f) = \int_a^b f \, dB = -\int_a^b f'(t) B(t) \, dt \quad \text{pathwise}$$

- I is an isometry from $L^2([a,b])$ onto its image in $L^2(\Omega)$

- Kolmogorov continuity criterion : $\{X_t\}$ with values in a metric space

Suppose $\mathbb{E}[d(X_s, X_t)^p] \leq C_0 |s-t|^{1+\alpha}$ for some $p, \alpha > 0$

Then,

$$P\left(\sup_{s \neq t} \frac{d(X_s, X_t)}{|s-t|^{\alpha/p - \varepsilon}} \geq \lambda\right) \leq \frac{C_1}{\lambda^p} \quad \forall \, 0 < \varepsilon < \frac{\alpha}{p}$$

i.e. X_t is a.s. $(\frac{\alpha}{p} - \varepsilon)$-Hölder continuous.

ex: B.M.

$$\mathbb{E}\left[|B(t_2) - B(t_1)|^2\right] = t_2 - t_1$$

$$\Rightarrow \mathbb{E}\left[|B(t_2) - B(t_1)|^p\right] \sim |t_2 - t_1|^{p/2} \overset{//}{} 1+\alpha$$

$p/2$

$$\frac{\alpha}{p} - = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} - \to \frac{1}{2} - \quad \text{as } p \to \infty$$

i.e. BM is a.s. $(\frac{1}{2} - \varepsilon)$- Hölder.

Prop: $\Phi \in HS(L^2; \underline{\underline{H^s}})$. $\qquad \|\Phi\|_{HS(X;Y)} = \left(\sum_{n \in \mathbb{Z}^d} \|\Phi(e_n)\|_Y^2\right)^{1/2}$.

Then, $\Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}(\mathbb{T}^d)$ slight loss of regularity $\qquad \{e_n\}$ O.N.B. of X

a.s. $\quad r \leq \infty$, $\alpha > 0$

Banach setting:
γ- radonifying op.
Appendix: Forlano-Oh-Wang on SNLS.

Also,
$\underline{r=2}$: $\quad \underline{\Psi \in C_t H_x^s, \text{ a.s.}}$

$\frac{\alpha}{2} - = \frac{\alpha}{2} - \varepsilon$ for any small $\varepsilon > 0$

Sobolev spaces (Bessel potential space L^p_s)

$s \in \mathbb{R}$
$1 \le p \le \infty$

$$\|f\|_{W^{s,p}} = \| \langle \nabla \rangle^s f \|_{L^p} \qquad \langle \nabla \rangle = \sqrt{1-\Delta}$$

$$= \| \mathcal{F}^{-1}(\langle n \rangle^s \hat{f}(n)) \|_{L^p}. \qquad \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$$

Japanese bracket.

$\underline{p=2}$: $W^{s,2} = H^s$

$$\|f\|_{H^s} = \Big(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2 \Big)^{1/2}$$

Hölder - Besov space : $\mathcal{C}^s = B^s_{\infty,\infty}$

Besov space : $B^s_{p,q}$