Lec 1 15/01/20 (Wed)

Rough path theory & pathwise well-posedness of stochastic PDEs

$$(*) \qquad dY_t = f(Y_t)\,dX_t$$

ODE, SPE } RDE = rough diff. eqn.

dX_t : given input source

ex: $\dot{X} = \xi =$ white noise, $\dot{X} = dX$

i.e. $X = B =$ Brownian motion

- By Itô integral / stochastic integration theory, we can solve $(*)$ but the soln map $\Phi : (Y_0, B) \mapsto Y$ (Itô map) lacks continuity, in general, due to roughness of BM.

Ref:

- Friz-Hairer : Intro to rough paths
- Friz-Victoir : Multidim'l stoch. processes as rough paths (Ch 1, 5, 6)
  (Friz' website : check for errata
- Baudoin : Lecture note.

FACT (Prop 1.1 in [FH]) $\exists$ separable Banach space $W \subset C([0,1])$

s.t. (i) sample paths of BM lie in W a.s.

$$B(\cdot\,;\omega) \in W, \quad \text{a.s.}$$

(ii) map $(f,g) \longmapsto \int_0^t f(t)\,dg(t)$ (or $\int_0^t f(t)\dot g(t)\,dt$),

a priori well defined on smooth functions,

extends to a continuous map from $W \times W$ into $C([0,1])$

- Given two indep BM's B^1 and B^2,

$$B = (B_1, B_2) \longmapsto \int_0^{\cdot} B^1(t)\,\underbrace{\dot B^2(t)\,dt}_{= dB^2(t)}$$

may still make sense of the integral but this map is not conti

$Y \in \mathbb{R}^2$,

$$\dot Y^1 = \dot B^1 \qquad\qquad Y^1 = B^1$$
$$\dot Y^2 = Y^1 \dot B^2 \quad \Longrightarrow \quad Y^2 = \int_0^t B^1\,dB^2.$$

$\Rightarrow$ Itô map for (*) is <u>not</u> continuous.

<u>Goal</u>: Give a meaning to the soln theory for ⊛ when X is rough.

(X as rough as BM is already non-trivial and important

- Rough path theory introduced by Terry Lyons '98 (S. Davie)
- Controlled paths by Gubinelli '04.

<u>Idea</u>: <u>Factorize</u> the "ill-posed" soln map $\Phi : B \mapsto Y$

- $$B(\omega) \xmapsto{\ \Psi\ } \underbrace{(B(\omega), \mathbb{B}(\omega))}_{\text{enhanced data set}} \xmapsto{\ \hat{\Phi}\ } Y(\omega)$$

measurable universal lift

continuous soln map (Itô-Lyons map) to a RDE (where we view $(B(\omega), \mathbb{B}(\omega))$ as deterministic data for fixed ω.)

$$B^{i,j}(s,t) = \int_s^t (B^i(r) - B^i(s))\, dB^j(r)$$

Kuo: Intro to stoch. integ. (Chap. 9) 4

iterated Itô-Wiener integral

Itô or Stratonovich ⇐ freedom to choose interpretation of stoch integral.

In general, given rough X, we do not have enough regularity / information

We need to <u>enhance</u> the data to $(X, \mathbb{X})$.

$$\mathbb{X}^{i,j}_{s,t} = X^{i,j}(s,t) = \int_s^t X^i_{s,r}\, dX^j_r \qquad F_{s,t} = F_t - F_s$$

Here, LHS defines RHS !!

- Gubinelli's controlled paths give a meaning to $\int_0^t Y_s\, dX_s$

  ⇐ can do this if Y is <u>controlled by X</u>.

$$Y_{s,t} = Y'_s X_{s,t} + R_{s,t} \qquad Y' = \text{Gubinelli derivative}$$

smoother

⇐ local behavior / fluctuation of Y is essentially given by that of X.

We want to solve $dY = f(Y)\,dX$.

① Riemann-Stieltjes integral & ODE $\qquad V^1 = BV,\ \ \mathrm{Lip}$

② Young integral, $\quad V^p,\quad C^{1/p},\ 1<p<2.$

Hölder space $\qquad$ · predual of $C^{1/2}$?

③ endpoint case $p=2$. $\qquad V^2,\ U^2 \leftarrow$ predual of V^2.

introduced by Tataru / Koch-Tataru.

④ controlled rough paths $\quad V^p,\ C^{1/p},\ 2 \le p < 3.$

In general, for $k \le p < k+1$, we need higher order rough paths

$$(X, \underbrace{\mathbb{X}^2, \mathbb{X}^3, \cdots, \mathbb{X}^k}_{\text{define higher order iterated integrals}})$$

· ④ with $2 \le p < 3$ is most important

since $BM \in V^p,\ C^{1/p},\quad p = 2+$, a.s.

5 stochastic PDE

smoothing op in x

$\partial_t u = \Delta u + N(u) + \sigma(u)\,\phi\,\xi$ heat eqn

$i\,\partial_t u = \Delta u + N(u) + u\,\phi\,\xi$ Schrödinger eqn.

space-time white noise

$\Leftarrow$ solve

- $L^2(\Omega)$: using stochastic integral theory
- pathwise (much harder)

There is also a notion of martingale soln but we are concerned with stronger notions of solns

ex: $(\partial_t^2 - \partial_x^2)\,u = u^3 + u\,\xi$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$\frac{1}{2}-, \frac{1}{2}-$ (space, time) $\quad -\frac{1}{2}-, -\frac{1}{2}-$ $\Longrightarrow$ cannot make sense of the product.

- $L^2(\Omega)$-theory : easy. see a note on my website.
- pathwise : open.

• Functions of finite p-variations and Hölder functions

$[0,T]$, $T>0$, fixed.

$$\|X\|_{V^p} = \sup_{\mathcal{P}} \left(\sum_{j=1}^{n-1} |X_{t_{j+1}} - X_{t_j}| \right)^{1/p}$$

$\mathcal{P}$: partition of $[0,T]$

$t_0 = 0 < t_1 < \cdots < t_n = T$

Banach space with a norm $|X_0| + \|X\|_{V^p}$

• $\|\cdot\|_{V^p}$ is a semi-norm

• $V^p = V^p([0,T])$, $p>0$

• $V^p_c = V^p \cap C([0,T])$ $\quad$ ($= C^{p\text{-var}}([0,T])$ in [FV])

• $V^p_{rc} = \{ X \in V^p : X$ is right continuous$\}$

• Hölder space: Hölder continuous functions $\quad \|X\|_{C^\alpha} = \sup_{0 \le s < t \le T} \dfrac{|X_t - X_s|}{|t-s|^\alpha}$

• $\alpha = 1$, Lipschitz conti func.

• $C^\alpha_{\text{Hölder}} = C^\alpha_{\text{Hölder}}([0,T]) = \begin{cases} C^{\alpha\text{-Höl}}([0,T]) & \text{in [FV]} \\ \Lambda^\alpha & \text{in Grafakos} \end{cases}$

(i) $V^p_c([0,T])$, $C^\alpha_{\text{Hölder}}([0,T])$

- Banach space
- <u>not</u> separable

(Thm 5.25, ex 5.26 in [FV].

(ii) $C^\alpha_{\text{Hölder}} \subset V^p_c$, $\alpha \geq \frac{1}{p}$

<u>Pf:</u> $$\underbrace{\left(\sum |X_{t_{j+1}} - X_j|^p\right)^{1/p}}_{\text{main building block of } V^p_c\text{-norm}} \leq \|X\|_{C^\alpha_{\text{Hölder}}} \left(\sum |t_{j+1} - t_j|^{\overbrace{p\alpha}^{\geq 1}}\right)^{1/p} \lesssim \|X\|_{C^\alpha_{\text{Hölder}}}$$

(iii) For $p<1$, V^p_c; $\alpha > 1$, $C^\alpha_{\text{Hölder}}$ = const func.

(We will drop the subscript "Hölder" since we only use $0<\alpha<1$, and use Lip when $\alpha = 1$

<u>Pf:</u> By (ii), suffices to show $V^p_c = \{\text{const func}\}$, $p<1$

$$|X_t - X_0| \leq \sum_{j=1}^{n-1} |X_{t_{j+1}} - X_{t_j}|$$

$$\leq \underbrace{\sup_j |X_{t_{j+1}} - X_{t_j}|^{1-p}}_{\to 0 \text{ by taking } |\mathcal{P}| \to 0, \text{ and using } \underline{\text{unif conti}} \text{ of } X \text{ on } [0,T].} \|X\|^p_{V^p}$$

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$$V^p_c([0,T]) = V^p([0,T]) \cap C([0,T])$$

norm : $|X_0| + \|X\|_{V^p}$ (or $\|X\|_{L^\infty} + \|X\|_{V^p}$)

(iv) · For $p_1 \le p_2$, $\quad V^{p_1} \subset V^{p_2}$

· For $\alpha_1 \ge \alpha_2$ $\quad C^{\alpha_1}_{\text{Hölder}} \subset C^{\alpha_2}_{\text{Hölder}}$

(v) Interpolation : $1 \le p_1 < p_2 < \infty$.

$$\|X\|_{V^{p_2}} \le \|X\|_{V^{p_1}}^{p_1/p_2} \left(\sup_{t,s} |X_t - X_s| \right)^{1 - p_1/p_2}$$

($\Leftarrow$ Hölder) Prop 5.5 in [FV]

(vi) $X^n \to X$ ptwise on $[0,T]$

$$\|X\|_{V^p} \le \liminf_{n\to\infty} \|X^n\|_{V^p}$$

(vii)

$$\lim_{p' \searrow p} \|X\|_{V^{p'}} = \|X\|_{V^p}$$

Lem 5.13 in [FV].

(viii) <u>Compactness</u> : $\{X^n\} \subset C([0,T])$.

$X^n \to X$ uniformly.

- Suppose $\sup_n \|X^n\|_{V^p} < \infty$. Then, $X^n \to X$ in $V^{p'}$, $\forall p' > p$.

((vi) $\Rightarrow$ $X \in V^p$. Then apply interpolation (v))

- $V^p_\infty = \overline{C^\infty}^{\|\cdot\|_{V^p}}$ $(= C^{0,p\text{-var}})$

$C^\alpha_\infty = \overline{C^\infty}^{\|\cdot\|_{C^\alpha}}$ $(= C^{0,\alpha\text{-Hölder}})$

(VMO / CMO $= \overline{C^\infty_c}^{\|\cdot\|_{BMO}}$

- closed subspaces, Banach, separable (Prop 5.36 in [FV])

$\to$ Polish

V^p_c, $C^\alpha_{\text{Hölder}}$ not separable

- <u>Reparametrization</u> : (Prop 1.21, Prop 5.14 in [FV])

$X \in V^p_c([0,T])$ iff $\exists$ conti. increasing function h from $[0,T]$ to $[0,1]$ and $C^{1/p}_{\text{Hölder}}$-function Y s.t.

$$X = Y \circ h.$$

① Riemann - Stieljes integral & ODE. ($p=1$)

<u>Prop 1:</u> $X \in V^1_c([0,T])$, Y on $[0,T]$, piecewise continuous.

Then, the Riemann-Stieljes integral $\int_0^T Y\,dX$ exists,

linear in X and Y with the bound:

(*) $$\left| \int_0^T Y\,dX \right| \le \|Y\|_{L^\infty([0,T])} \|X\|_{V^1([0,T])}$$ Prop 2.2 in [FV]

<u>Idea:</u> ① dX - integrable functions if Riemann sum converges

lin space, (*) holds.

② Y^n, dX - integrable, $Y^n \to Y$ in $L^\infty([0,T])$

Use (*) to show $\int_0^T Y\,dX = \lim_{n\to\infty} \int_0^T Y^n\,dX$

etc.

<u>Rmk:</u> $Z(t) = \int_0^t Y\,dX \Rightarrow |Z(t_2) - Z(t_1)| = \left|\int_{t_1}^{t_2} Y\,dX\right| \le \|Y\|_{L^\infty} \|X\|_{V^1([t_1,t_2])}$

$$\Rightarrow \|Z\|_{V^1([0,T])} \le \|Y\|_{L^\infty} \|X\|_{V^1([0,T])}$$

Lemma 2: (Gronwall's inequality)

$X \in V^1_c([0,T])$. $\Phi : [0,T] \to \mathbb{R}_+$, measurable, bdd.

Suppose

(**) $$\Phi(t) \le A + \int_0^t \Phi(s)\,|dX(s)|, \quad \forall\, 0 \le t \le T$$

for some $A, B \ge 0$. Then,

$$\Phi(t) \le A \exp\left(B\, \|X\|_{V^1([0,T])} \right), \quad \forall\, 0 \le t \le T.$$

Here, $\int_0^t \cdots |dX(s)|$ = integration w.r.t. a V^1-path $\ell(t) = \|X\|_{V^1([0,t])}$.

Pf: By iterating (**) n times, we get

$$\Phi(t) \le A + \sum_{k=1}^{n} A B^k \underbrace{\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} |dX(t_k)| \cdots |dX(t_1)|}_{= \frac{\|X\|^k_{V^1([0,t])}}{k!}}$$

$$+ \underbrace{\text{remainder}}_{\sim \frac{\|\Phi\|_{L^\infty}}{(n+1)!}}$$

□

Thm 3: $X \in V_c^1([0,T])$.

V, Lipschitz

Then, $\forall\, Y_0$, $\exists!$ soln to $\Longleftrightarrow$ $dY = V(Y)\,dX$

$$Y(t) = Y_0 + \int_0^t V(Y(s))\,dX(s), \quad 0 \le t \le T.$$

Moreover, $Y \in V_c^1([0,T])$.

Pf: $$\Gamma(Y)(t) = Y_0 + \int_0^t V(Y(s))\,dX(s)$$

① work on $[0,\tau]$, $0<\tau \ll 1$.

$$\|\Gamma(Y)\|_{V^1([0,\tau])} \le |Y_0| + \underbrace{\|V(Y) - V(0)\|_{L^\infty([0,\tau])}}_{\le K\|Y\|_{L^\infty([0,\tau])}} \|X\|_{V^1([0,\tau])}$$

$$+ |V(0)|\,\|X\|_{V^1([0,\tau])}$$

$\le R$ by choosing $R = 2|Y_0|$ and τ small.

$\forall\, Y \in B_R \subset V_c^1([0,T])$ $\left(\Rightarrow \|Y\|_{L^\infty([0,T])} \le R\right)$

$$\| \Gamma(Y^1) - \Gamma(Y^2) \|_{V^1([0,\tau])} \le K \| Y^1 - Y^2 \|_{L^\infty([0,\tau])} \| X \|_{V^1([0,\tau])}$$
$$\le \frac{1}{2} \| Y^1 - Y^2 \|_{V^1([0,\tau])}$$

$\Rightarrow$ Γ is a contraction on B_R

$\Rightarrow$ $\exists !\ Y \in B_R$ s.t. $Y = \Gamma(Y)$

$\boxtimes$

<u>Rmk:</u> • By Gronwall's inequality,

$$Y^j(t) = Y^j(0) + \int_0^t V(Y^j(s))\, dX(s), \quad j = 1, 2. \qquad X \in V^1_c([0,T])$$

$$\Rightarrow \| Y^1 - Y^2 \|_{L^\infty([0,T])} \le | Y^1(0) - Y^2(0) | \exp\left(K \| X \|_{V^1([0,T])} \right)$$

• soln map : $Y_0 \to Y$ is in fact C^1.

$$\Phi(t, Y_0) = Y(t)$$

Let $J_t = \dfrac{\partial \Phi(t, Y_0)}{\partial Y_0}$. Then,

$$J_t = Id + \int_0^t DV(\Phi(s, Y_0))\, J_s\, dX(s)$$

② Young's integral $\frac{1}{p} + \frac{1}{q} > 1$

- A map $\omega : \{0 \le s \le t \le T\} \to \mathbb{R}_+$ is called superadditive if

$$\omega(s,t) + \omega(t,u) \le \omega(s,u), \quad \forall\ 0 \le s \le t \le u.$$

- We say ω is a control if it is superadditive, conti, $\omega(t,t) = 0$.
- We say a path $X : [0,T] \to \mathbb{R}$ is controlled by a control ω if $\exists\ c > 0$ s.t. $|X_t - X_s| \le C\,\omega(s,t)$, $\forall\ 0 \le s \le t \le T$.

Prop 5.8 in [FV] (i) $X \in V^p_c([0,T])$, Then, $\omega(s,t) = \|X\|^p_{V^p([s,t])}$ is a control s.t.

$$|X_t - X_s| \le \omega(s,t)^{1/p}. \quad \forall\ 0 \le s \le t \le T$$

Prop 5.10 in [FV] (ii) $X \in V^p_c([0,T])$, $\omega : \{0 \le s \le t \le T\} \to \mathbb{R}_+$, control s.t.

$$|X_t - X_s| \le c\,\omega(s,t)^{1/p}, \quad \forall\ 0 \le s \le t \le T.$$

Then,

$$\|X\|_{V^p([s,t])} \le C\,\omega(s,t)^{1/p}.$$

Back to V^p_c, $V^p_\infty = \overline{C^\infty}^{\|\cdot\|_{V^p}}$

· $p > 1$

Thm 5.31 in [FV]

$$X \in V^p_\infty \text{ iff } \lim_{\delta \to 0} \sup_{|P| \leq \delta} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^p = 0$$

$$\left(\text{Also, } \lim_{\delta \to 0} \sup_{|P| \leq \delta} \sum_{k=0}^{n-1} \|X\|^p_{V^p([t_k, t_{k+1}])} = 0 \right.$$

Pf: $\Rightarrow$ Fix $\varepsilon > 0$ and smooth path Y

s.t. $\|X - Y\|^p_{V^p} \leq \varepsilon / 2^p$.

Then, $\displaystyle\sum_P |X_{t_{k+1}} - X_{t_k}|^p \leq 2^{p-1} \sum |Y_{t_{k+1}} - Y_{t_k}|^p + 2^{p-1} \|X - Y\|^p_{V^p}$

Since Y is smooth, $\exists\ \delta = \delta(n) > 0$ s.t. $\forall\ |P| \leq \delta$, we have

$$\sum_P |Y_{t_{k+1}} - Y_{t_k}|^p \leq \|Y'\|_{L^\infty} \sum_P (t_{k+1} - t_k)^p$$
$$\leq \delta^{p-1} T \|Y'\|_{L^\infty} < \varepsilon / 2^p \quad \text{for } p > 1$$

$\Rightarrow$ Putting together, we obtain $\displaystyle\sum_P |X_{t_{k+1}} - X_{t_k}|^p < \varepsilon, \ \forall |P| \leq \delta$

$\square$

Cor: $1 \le p < q$

$$V_c^p([0,T]) \subset V_\infty^q([0,T]) \subset V_c^q([0,T])$$

- *Young's integral.*

Thm 4: $X, Y \in V_c^1([0,T])$

Let $p, q \ge 1$ s.t. $\theta = \frac{1}{p} + \frac{1}{q} > 1$. Then, we have

Prop 6.4 in [FV]

$$\underbrace{\left| \int_s^t Y_u \, dX_u - \underbrace{Y_s X_{s,t}}_{\overset{\shortparallel}{Y(s)(X(t)-X(s))}} \right|}_{= \int_s^t (Y_u - Y_s)\, dX_u} \le \frac{1}{1-2^{1-\theta}} \|X\|_{V^p([s,t])} \|Y\|_{V^q([s,t])}$$

$\Leftarrow$ Young–Loève estimate.

Pf: For $0 \le s \le t \le T$, define

$$\Gamma_{s,t} = \int_s^t Y_u \, dX_u - Y_s X_{s,t}.$$

Then, for $s \le t \le u$, we have

$$\Gamma_{s,u} - \Gamma_{s,t} - \Gamma_{t,u} = -Y_s (X_u - \cancel{X_s}) + Y_s (X_t - \cancel{X_s}) + Y_t (X_u - X_t)$$

$$= (Y_s - Y_t)(X_t - X_u)$$

$$\Rightarrow \quad |\Gamma_{s,u}| \le |\Gamma_{s,t}| + |\Gamma_{t,u}| + \|X\|_{V^p([t,u])} \|Y\|_{V^q([s,t])}$$

· Now, set $\omega(s,t) = \|X\|_{V^p([s,t])}^{1/\theta} \|Y\|_{V^q([s,t])}^{1/\theta}$

Claim: ω is a control.

· conti, $\omega(t,t) = 0$, ✓

· For $s \le t \le u$,

$$\omega(s,t) + \omega(t,u) = \|X\|_{V^p([s,t])}^{1/\theta} \|Y\|_{V^q([s,t])}^{1/\theta} + \|X\|_{V^p([t,u])}^{1/\theta} \|Y\|_{V^q([t,u])}^{1/\theta}$$

$1 = \frac{1}{p\theta} + \frac{1}{q\theta} \longrightarrow$ Hölder

$$\le \left(\|X\|_{V^p([s,t])}^p + \|X\|_{V^p([t,u])}^p \right)^{\frac{1}{p\theta}} \left(\|Y\|_{V^q([s,t])}^q + \|Y\|_{V^q([t,u])}^q \right)^{1/q\theta}$$

$$\underset{=}{\le} \|X\|_{V^p([s,u])}^{1/\theta} \|Y\|_{V^q([s,u])}^{1/\theta} = \omega(s,u)$$

(*) • We have $|\Gamma_{s,u}| \le |\Gamma_{s,t}| + |\Gamma_{t,u}| + \omega(s,u)^\theta$

Given $\varepsilon > 0$, consider the control:

$$\omega_\varepsilon(s,t) = \omega(s,t) + \varepsilon\left(\|X\|_{V^1([s,t])} + \|Y\|_{V^1([s,t])}\right)$$

Also, define

$$\Psi(r) = \sup_{\substack{s,u \\ \omega_\varepsilon(s,u) \le r}} |\Gamma_{s,u}|$$

• By conti, $\exists\ t \in [s,u]$ s.t. $\omega_\varepsilon(s,t) = \omega_\varepsilon(t,u) \le \frac{1}{2} r$.

$$\underset{(*)}{\Longrightarrow} \quad |\Gamma_{s,u}| \le 2\Psi\left(\tfrac{r}{2}\right) + r^\theta.$$

$$\Longrightarrow \quad \Psi(r) \le 2\Psi\left(\tfrac{r}{2}\right) + r^\theta. \quad \left(\le 2^2\Psi\left(\tfrac{r}{2^2}\right) + 2\left(\tfrac{r}{2}\right)^\theta + r^\theta\right)$$

iterate n times

(**) $$\Longrightarrow \quad \Psi(r) \le 2^n \Psi\left(\frac{r}{2^n}\right) + \underbrace{\sum_{k=0}^{n-1} 2^{k(1-\theta)} r^\theta}$$

claim (see next page)

$$0 \qquad\qquad = \frac{1}{1 - 2^{1-\theta}} \cdot r^\theta.$$

- $$\begin{aligned}|\Gamma_{s,t}| &= \left|\int_s^t (Y_u - Y_s)\,dX_u\right| \\ &\le \|X\|_{V^1([s,t])}\,\|Y_\cdot - Y_s\|_{L^\infty([s,t])} \\ &\le \left(\|X\|_{V^1([s,t])} + \|Y\|_{V^1([s,t])}\right)^2 \\ &\le \frac{1}{\varepsilon^2}\,\omega_\varepsilon^2(s,t)\end{aligned}$$

$$\Longrightarrow \lim_{n\to\infty} 2^n \Psi\left(\frac{r}{2^n}\right) \le \lim_{n\to\infty} 2^n\left(\frac{1}{\varepsilon^2}\frac{r^2}{2^{2n}}\right) = 0$$

Hence, from (**), we obtain

$$\Psi(r) \le \frac{1}{1-2^{1-\theta}}\, r^\theta$$

$$\Longrightarrow |\Gamma_{s,u}| \le \frac{1}{1-2^{1-\theta}}\,\omega_\varepsilon(s,u)^\theta.$$

(For given s, u, X, Y. $\omega_\varepsilon(s,u) = r$

Finally, send $\varepsilon \to 0$

$\boxtimes$

<u>Prop 5</u>: $X \in V_c^p([0,T])$, $Y \in V_c^q([0,T])$, $\theta = \frac{1}{p} + \frac{1}{q} > 1$

Suppose $\exists \{X^n\}, \{Y^n\} \subset V_c^1([0,T])$

s.t. $X^n \to X$ in V_c^p

$Y^n \to Y$ in V_c^q.

Then, $\forall\ 0 \le s \le t \le T$,

Thm 6.8 in [FV]

$\int_s^t Y_u^n \, dX_u^n$ converges to a <u>limit</u>

Young's integral $\int_s^t Y_u \, dX_u$.

- The limit is indep of the choice of approx seq $\{X^n\}, \{Y^n\}$.
- For $0 \le s \le t \le T$,

$$\left| \int_s^t Y_u \, dX_u - Y_s (X_t - X_s) \right| \le \frac{1}{1-2^{1-\theta}} \|X\|_{V^p([s,t])} \|Y\|_{V^q([s,t])}$$

Rmk: ① $\overline{V_c^1}^{\|\cdot\|_{V^p}} = V_\infty^p$

but we have

$$V_c^p \subset V_\infty^{p+\varepsilon}$$

i.e. Given $X \in V_c^p$, $Y \in V_c^q$,

use Prop 5. with $\frac{1}{p+\varepsilon} + \frac{1}{q+\varepsilon} = \theta_\varepsilon > 1$ ($\nearrow \theta$)

since $X \in V_\infty^{p+\varepsilon}$. $Y \in V_\infty^{q+\varepsilon}$.

· Recalling $\lim_{p' \searrow p} \|X\|_{V^{p'}} = \|X\|_{V^p}$,

we conclude that the Young-Loève estimate holds for $X \in V_c^p$, $Y \in V_c^q$.

② Riemann sum $\sum_{k=0}^{n-1} Y_{t_k}(X_{t_{k+1}} - X_{t_k})$ converges to $\int_s^t Y_u \, dX_u$

use $X \in V_\infty^{p+\varepsilon}$, $Y \in V_\infty^{q+\varepsilon}$ and the result from page ② to the Young-Loève estimate on $[t_k, t_{k+1}] \subset [s,t]$

3 **Prop 6**: $X \in V^p_c$, $Y \in V^q_c$, $\frac{1}{p}+\frac{1}{q}>1$.

$$\left\| \int_0^{\cdot} Y_u \, dX_u \right\|_{V^p([s,t])} \le C \|X\|_{V^p([s,t])} \left(\|Y\|_{V^q([s,t])} + \|Y\|_{L^\infty([s,t])} \right)$$

$$\le 2 \|X\|_{V^p([s,t])} \left(\|Y\|_{V^q([s,t])} + |Y_0| \right).$$

Prop 6.11 in [FV]

- $t \longmapsto \int_0^t Y_u \, dX_u$ is conti (with values in V^p_c).
- $(X, Y) \longmapsto \int_0^{\cdot} Y \, dX$ is bilinear & conti.

$$V^p_c \times V^q_c \longrightarrow V^p_c$$

Lipschitz conti & Fréchet smooth

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Young's differential equation : $dY_t = V(Y)\, dX_t$

Thm 7: $X \in V^p_c([0,T])$, $1 < p < 2$

V, Lipschitz conti s.t. $|V(Y^1) - V(Y^2)| \le K|Y^1 - Y^2|$

Then, given Y_0, $\exists!$ soln Y to

$$Y_t = Y_0 + \int_0^t V(Y_s)\, dX_s \quad \text{on} \quad [0,T]$$

Moreover, $Y \in V^p_c([0,T])$

Pf: Let $\Gamma(Y)_t = Y_0 + \int_0^t V(Y_s)\, dX_s$.

Then, $\Gamma(Y)_0 = Y_0$ $\quad (V(Y_s) - V(0)) + V(0)$

and

$$\|\Gamma(Y)\|_{V^p([0,T])} \overset{\text{Prop6}}{\le} |Y_0| + C\|X\|_{V^p([0,T])}\Big(K\big(\|Y\|_{V^p([0,T])} + |Y_0|\big) + |V(0)|\Big)$$

So, for $Y \in \overline{B_R} \subset V^p_c([0,\tau])$ ②

$$\|\Gamma(Y)\|_{V^p_c([0,\tau])} \overset{\text{def}}{=} |Y_0| + \|\Gamma(Y)\|_{V^p([0,\tau])}$$
$$\leq R$$

by choosing $R = 2\|Y\|_{V^p_c([0,\tau])}$

and $\tau > 0$ suff. small s.t. $C\|X\|_{V^p([0,\tau])} \ll_{R,V} 1$.

Similarly, for $Y^1, Y^2 \in \overline{B_R} \subset V^p_c([0,\tau])$,

$$\|\Gamma(Y^1) - \Gamma(Y^2)\|_{V^p_c([0,\tau])} \leq C\|X\|_{V^p([0,\tau])} \left(K\|Y^1 - Y^2\|_{V^p([0,\tau])} + K|Y^1_0 - Y^2_0| \right).$$
$$\leq \frac{1}{2}\|Y^1 - Y^2\|_{V^p_c([0,\tau])}$$

by choosing $\tau > 0$ small

s.t. $C\|X\|_{V^p([0,\tau])} \ll 1$.

$\Rightarrow \exists!$ soln Y on $[0,\tau]$.

- For $t > \tau$, write

view as new initial data

$$Y_t = Y_0 + \int_0^t V(Y_s)\,dX_s = Y_\tau + \int_\tau^t V(Y_s)\,dX_s \Rightarrow \text{repeat}.$$

$\boxtimes$

- As before, $Y_0 \to Y$ is C^1.
- Also, Y depends continuously on X.

  i.e. if $X^n \to X$ in $V^p_c([0,T])$.

  Then, $Y^n_t = Y_0 + \int_0^t V(Y^n_s)\, dX^n_s$

  converges to

  $$Y_t = Y_0 + \int_0^t V(Y_s)\, dX_s$$

  in $V^p_c([0,T])$.

<u>Pf:</u> Let $0 \le s \le t \le T$. Then,

$$\|Y - Y^n\|_{V^p([s,t])} = \left\| \int_0^\cdot V(Y)\, dX - \int_0^\cdot V(Y^n)\, dX^n \right\|_{V^p([s,t])}$$

$$\le \left\| \int_0^\cdot (V(Y) - V(Y^n))\, dX \right\|_{V^p([s,t])} + \left\| \int_0^\cdot V(Y^n)\, d(X - X^n) \right\|_{V^p([s,t])}$$

$$\overset{\text{Prop 6}}{\le} \underbrace{C \|X\|_{V^p([s,t])}}_{\text{make this small}} K \|Y - Y^n\|_{V^p([s,t])}$$

$$+ C \|X - X^n\|_{V^p([s,t])} \left(K \|Y^n\|_{V^p([s,t])} + |V(0)| \right)$$

① $\Rightarrow \quad \|Y - Y^n\|_{V^p([s,t])} \leq \dfrac{C\left(K\|Y^n\|_{V^p([s,t])} + |V(0)|\right)}{1 - CK\|X\|_{V^p([s,t])}} \|X - X^n\|_{V^p([s,t])}$

Similarly, we obtain

$$\|Y^n\|_{V^p([s,t])} \leq \frac{C|V(0)|}{1 - cK\|X^n\|_{V^p([s,t])}}$$

② $$\leq \frac{C|V(0)|}{1/2 - cK\|X\|_{V^p([s,t])}} \quad \text{for any } N \gg 1.$$

From ① & ②,

$$\|Y - Y^n\|_{V^p([s,t])} \leq C\left(\|X\|_{V^p([s,t])}, V\right) \|X - X^n\|_{V^p([s,t])}$$

$$\to 0.$$

$\square$

· Another view on Young's integral

(based on Gubinelli's lec note 1)

$$(f, g) \longmapsto I(f,g) = \int_0^{\cdot} f(s)\, \partial_s g(s)\, ds$$

$$\begin{array}{ccc} \cap & & \\ C^{\alpha} \times C^{\beta} & \longrightarrow & C^{\gamma} \end{array}$$

- <u>differential calculus viewpt</u>: $I(f,g)$ is the unique soln to

$$\partial_t I(f,g)(t) = f(t)\, \partial_t g(t), \quad I(0) = 0.$$

- <u>finite increment</u>:

small o.

$$(*) \qquad I(t) - I(s) = f(s)\big(g(t) - g(s)\big) + o_{\text{unif}}(|t-s|), \qquad I(0) = 0$$

for any $0 \le s \le t \le 1$

unif in s, t.

$\Leftarrow$ This property is clearly satisfied if g is at least C^1

$$I(t) - I(s) - f(s)\big(g(t) - g(s)\big) = \int_s^t \underbrace{\big(f(u) - f(s)\big)}_{\substack{\to 0 \\ \text{unif}}} \partial_u g(u)\, du.$$

since f unif conti on $[0,1]$.

- ⊛ gives a characterization of the integral

i.e. if J satisfies ⊛, then with $D := I - J$, we have

$$D(t) - D(s) = o_u(|t-s|)$$

$$\Rightarrow D(t) = D(s) \text{ for any } 0 \le s \le t \le 1$$

but $D(0) = 0 \Rightarrow D(t) \equiv 0$.

- I is the only function whose increment match the "germ" $f(s)(g(t)-g(s))$ modulo a negligible error

- For $n \geq 1$, $C_n(V) = C(\Delta_n; V)$, $\Delta_n = \Delta_n(0,1)$
$\mathbb{R}$

$$\Delta_n(s,t) = \{(s_1, \cdots, s_n) : s \le s_1 \le \cdots \le s_n \le t\}.$$

  - n-cochain is an element in $C_n(V)$
  - coboundary operator $\delta : C_n(V) \longrightarrow C_{n+1}(V)$ given by

$$\delta f(s_1, \cdots, s_{n+1}) = \sum_{k=1}^{n+1} (-1)^{n+1-k+1} f(s_1, \cdots, \cancel{s_k}, \cdots, s_{n+1})$$

ex: $\delta f(s,t) = f(t) - f(s)$, $\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u)$

FACT: $\delta \circ \delta = 0$.

Cochain complex: $\mathbb{R} \to C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \xrightarrow{\delta}$

(cohomology $H^m = \dfrac{\mathrm{Ker}\, \delta^m}{\mathrm{Im}\, \delta^{m-1}}$

FACT: This complex is exact.

If $\delta f = 0$, then $f = \delta g$ for some g.

ex: $f \in C_2$ and $\delta f = 0$.

i.e. $\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u) \equiv 0$

Set $g(t) = f(0,t)$

Then, $\delta g(s,t) = g(t) - g(s) = f(0,t) - f(0,s) = f(s,t)$

Let $A(s,t) = f(s)\, \delta g(s,t) = f(s)\, (g(t) - g(s))$

Then, we have, from ⊛,

$$A = \delta I + R, \quad \text{where } R(s,t) = o_u(|t-s|)$$

$$(\Rightarrow \delta A = \delta R).$$

sewing map allows us to recover R from $\delta A \in C_3$.

Topology on C_m: We say $f \in C_m^\alpha$ if

$$\|f\|_{C_m^\alpha([s,t])} = \sup_{\Delta_m(s,t)} \frac{|f(s_1,\cdots,s_m)|}{|s_m - s_1|^\alpha} < \infty.$$

Set $C_m^{\alpha+} = \bigcup_{\beta>\alpha} C_m^\beta$

Rmk: $\delta C_1 \cap C_2^{1+} = \{0\}$ $\left(\Leftarrow C^{1+}_{\text{Hölder}} = \{\text{const}\}\right.$.

Thm 8: Sewing Lemma: $\exists!$ map $\Lambda : C_3^{1+} \cap \delta C_2 \to C_2^{1+}$

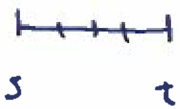
s.t. $\delta \Lambda = Id_{C_3}$

and
$$\|\Lambda \delta A\|_{C_2^\alpha(I)} \leq \frac{2^\alpha}{1-2^\alpha} \|\delta A\|_{C_3^\alpha(I)}$$

for any $\alpha > 1$ and any closed interval $I \subset \mathbb{R}$.

Dyadic partition of $[s,t]$: Given $n \geq 0$,

$$t_j^n = s + \frac{t-s}{2^n} j, \quad j = 0, \cdots, 2^n$$



Then, we have

$$(**) \qquad \underline{A(s,t) = \sum_{j=0}^{2^n-1} A(t_j^n, t_{j+1}^n) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \delta A(t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1})} \quad (= \mathrm{I} + \mathrm{II})$$

for any $n \geq 0$.

- $\underline{n=0}$: ✓

$s=0, t=1$
- $\underline{n=1}$: $\mathrm{I} = A(0,\frac{1}{2}) + A(\frac{1}{2},1)$

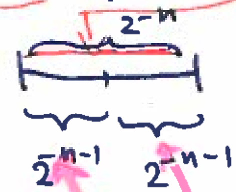
$$\left.\begin{array}{l} \mathrm{I} = A(0,\frac{1}{2}) + A(\frac{1}{2},1) \\ \mathrm{II} = A(0,\frac{1}{2},1) = A(0,1) - A(0,\frac{1}{2}) - A(\frac{1}{2},1) \end{array}\right\} = A(0,1).$$

general case by induction.

Assume $(**)$ for n.

$$\mathrm{I}_n = \underbrace{\sum_{j=0}^{2^{n+1}-1} A(t_j^{n+1}, t_j^{n+1})}_{= \mathrm{I}_{n+1}} + \boxed{\sum_{j=0}^{2^n-1} A(t_j^n, t_{j+1}^n) - \sum_{j=0}^{2^{n+1}-1} \underline{A(t_j^{n+1}, t_j^{n+1})}}$$

$$\sum_{j=0}^{2^n-1} A(t_j^n, t_{j+1}^n) = \sum_{j=0}^{2^n-1} A(t_{2j}^{n+1}, t_{2j+2}^{n+1})$$



$$\sum_{j=0}^{2^n-1} \delta A(t_{2j}^{n+1}, t_{2j+1}^{n+1}, t_{2j+2}^{n+1}).$$

$\Rightarrow$ obtain $(**)$ for $\underline{n+1}$.

Pf of Sewing lemma:

Suppose that $\exists \Lambda$ s.t. $\delta\Lambda = Id_{C_3}$ and

$$|(\Lambda\delta A)(s,t)| \le c\,|t-s|^{\rho}, \quad \forall (s,t) \in \Delta_2$$

for some $\rho > 1$ (this assumption can be weakened)

Use (✱✱) for $\Lambda\delta A$. and $\delta(\Lambda\delta A) = (\delta\Lambda)\delta A = \delta A$.

$$\Rightarrow |(\Lambda\delta A)(s,t)| \le C\, 2^{n}\, 2^{-m\rho} + \|\delta A\|_{C_3^\alpha(I)} \sum_{k=0}^{n-1} 2^{k}\, 2^{-k\alpha} |t-s|^{\alpha}$$

$$\left(\quad |t^n_{j+1} - t^n_j| = 2^{-n} \underbrace{|t-s|}_{\le 1} \right)$$

$$\le C\, 2^{m(1-\rho)} + \|\delta A\|_{C_3^\alpha(I)} \frac{1}{1-2^{1-\alpha}} |t-s|^{\alpha}$$

Take $m \to \infty$. ($\rho > 1$)

$$\Rightarrow |(\Lambda\delta A)(s,t)| \le \frac{2^{\alpha}}{2^{\alpha}-1} \|\delta A\|_{C_3^\alpha(I)} |t-s|^{\alpha}.$$

$\Rightarrow$ claimed estimate.

Lec 5 19/02/20 (Wed)

- Sewing map $\Lambda : C_3^{1+} \cap \delta C_2 \to C_2^{1+}$

s.t. $\delta \Lambda = Id_{C_3}$

$$\|\Lambda \delta A\|_{C_2^\alpha(I)} \lesssim \|\delta A\|_{C_3^\alpha(I)} \quad \text{for any } \alpha > 1.$$

provides a "correction" to a given 2-increment $A \in C_2$ to make it closed ($\delta \hat{A} = 0$) and hence exact ($\hat{A} = \delta B$)

exactness of the cochain complex

$$A \rightsquigarrow \delta A \rightsquigarrow A - \Lambda \delta A.$$

Then, $\delta(A - \Lambda\delta A) = \delta A - \underbrace{(\delta\Lambda)}_{= Id}\delta A = 0.$

$$\Rightarrow \exists!\ I \in C_1 \text{ s.t. } \delta I = A - \underbrace{\Lambda \delta A}_{R}$$

Pf of Sewing lemma cont'd:

Goal: Construct Λ s.t. $\delta\Lambda = Id_{C_3}$ and

$$|(\Lambda\delta A)(s,t)| \le c\,|t-s|^{\rho}, \quad \forall (s,t)\in\Delta_2$$

for some $\rho > 1$

Fix smooth $Q : \mathbb{R} \to \mathbb{R}_+$ with cpt supp in $\mathbb{R}_+$

$\int Q\,dx = 1$

and set $Q_\sigma(x) = \frac{1}{\sigma} Q\left(\frac{x}{\sigma}\right)$

Given $A \in C_2(\Delta_2; V)$, extend A to $\mathbb{R}\times\mathbb{R}$ by setting

$$A(s,t) = A(J_{0,1}(s), J_{0,1}(t)) \qquad \Delta_2 = \{0 \le s \le t \le 1\}$$

where $J_{a,b}(r) = \max(a, \min(r,b))$

$$A_\sigma(s,t) = -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\, A(s, r+r') \qquad \leftarrow \text{smoothing only in the } 2^{nd} \text{ var.}$$

$$\underset{IBP}{=} + \int_{\mathbb{R}} dr'\, Q_\sigma(r') \big(A(s, t+r') - A(s, s+r') \big)$$

$\Longleftarrow \quad A_\sigma(s,s) = 0.$

- Also, $A_\sigma \to A$ ptwise (b/c $A(s,s)=0$)
- $\delta A_\sigma(s,u,t) = \int_{\mathbb{R}} dr'\, Q_\sigma(r')\big(\delta A(s,u,t+r') - \delta A(s,u,u+r')\big)$

  and $\delta A_\sigma \to \delta A$ ptwise

Now, define

$$(RA_\sigma)(s,t) = \int_s^t \partial_2 A_\sigma(r,r)\,dr \overset{\text{F.T.C.}}{=} -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r') A(r, r+r'')$$

and set $\boxed{\Lambda \delta A_\sigma = A_\sigma - RA_\sigma}$

from

Then, $(\Lambda \delta A_\sigma)(s,t) = -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\big(A(s,r+r') - A(r,r+r')\big)$

$$= \int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\, \delta A(s,r,r+r')$$

$-A(s,r)\underbrace{\int_{\mathbb{R}} dr'\, Q'_\sigma(r')}_{=0}$

Check $(\delta \Lambda\, \delta A_\sigma)(s,u,t) = \delta A_\sigma(s,u,t)$.

$\Leftarrow$ follows from $\Lambda \delta A_\sigma = A_\sigma - RA_\sigma$

and $\underline{\delta R A_\sigma = 0}$

$\uparrow$

This is clear since RA_σ is just an integral with bdry pts s and t.

Now, we need to check the regularity of $\Lambda \delta A_\sigma$

Given small $\sigma > 0$, consider $|t-s| > \sigma$.

Recall

$$(\Lambda \delta A_\sigma)(s,t) = \sum_{j=0}^{2^n-1} (\Lambda \delta A_\sigma)(t_j^n, t_{j+1}^n) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \delta A_\sigma(t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1})$$

· choose n s.t. $|t-s| \cdot 2^{-n} \le \sigma < |t-s| 2^{-n+1}$

$$|(\Lambda \delta A_\sigma)(t_j^n, t_{j+1}^n)| \lesssim \underline{|t-s| 2^{-n}}\, \sigma^{\alpha-1} \lesssim |t-s|^\alpha 2^{-n\alpha}.$$

$\uparrow$ $\underline{|t_j^n - t_{j+1}^n| = 2^{-n}|t-s|}$

$\uparrow$ $\delta A \in C_3^\alpha$ and $|r'| \lesssim \sigma$

and for $k < n$, we have

$$|\delta A_\sigma(t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1})|$$

$\sim 2^{-n}|t-s|$

$$\lesssim (|t-s|2^{-k} + \sigma)^\alpha \lesssim 2^{-\alpha k}|t-s|^\alpha (1+2^{k-n})^\alpha$$

$$\lesssim 2^{-\alpha k}|t-s|^\alpha$$

$\delta A \in C_3^\alpha$

for δA_σ, there is a "spread" of size $\sim \sigma$.

Putting together,

$$|(\Lambda \delta A_\sigma)(s,t)| \lesssim |t-s|^\alpha \left(2^{m(1-\alpha)} + \sum_{k=0}^{m-1} 2^{k(1-\alpha)}\right)$$

$$\lesssim |t-s|^\alpha. \quad (\text{for } \alpha > 1) \quad \text{unif in } n$$

i.e. in σ small.

- By showing "$RA_\sigma \to RA$" ptwise,

we obtain

$$|(\Lambda \delta A)(s,t)| = \lim_{\sigma \to 0} |(\Lambda \delta A_\sigma)(s,t)| \lesssim |t-s|^\alpha$$

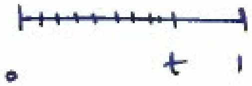
$\square$

Alternative proof of Sewing Lemma

via discrete approximations.

· $\{u_j^n\}$, dyadic partition of $[0,1]$ of size 2^{-n}.

Set
$$S_n A(t) = \sum_{j=0}^{2^n-1} 1_{t \ge u_j^n} A(u_j^n, u_{j+1}^n \wedge t)$$



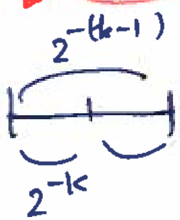
$$\Rightarrow \quad |\delta S_n A(s,t) - A(s,t)|$$
$$\le \sum_{k=l+1}^{n} |\delta S_k A(s,t) - \delta S_{k-1} A(s,t)| + \underbrace{\|\delta A\|_{C_3^\alpha} |t-s|^{-\alpha}}$$

↳ last term
$$\delta S_l A(s,t)$$
$$= A(u_j^l, t) - A(u_j^l, s)$$
$$= A(s,t) + \underbrace{\delta A(u_j^l, s, t)}_{\le \|\delta A\|_{C_3^\alpha} 2^{-l\alpha}}$$

where l is the greatest integer s.t. $2^{-l} \ge |t-s|$.

· $|\delta S_k A(s,t) - \delta S_{k-1} A(s,t)|$
$$\lesssim 2^{k-l} \, \underline{2^{-\alpha k} \|\delta A\|_{C_3^\alpha}}$$

\# of intervals of length $2^{-(k-1)}$ in $[s,t]$ $\sim 2^{-l}$



$$\Rightarrow |\delta S_n A(s,t) - A(s,t)|$$
$$\lesssim \|\delta A\|_{C_3^\alpha} 2^{-\ell} \sum_{k=\ell+1}^{\infty} 2^{k(1-\alpha)} + \|\delta A\|_{C_3^\alpha} |t-s|^\alpha$$
$$\lesssim \|\delta A\|_{C_3^\alpha} |t-s|^\alpha$$

Now, set $\Lambda \delta A = A - \delta S_n A$ where $n = n(s,t) > \ell = \ell(s,t)$

$\boxtimes$

<u>Cor:</u> Let $\delta A \in C_3^{1+} \cap \delta C_2$ and $\delta I = A - \Lambda \delta A$.

Then,

$$\underbrace{S_{P(s,t)}}_{\text{partition of } [s,t]}(A) = \sum_j A(t_j, t_{j+1}) \xrightarrow[\text{as } |P| \to 0]{} I(t) - I(s).$$

<u>Pf:</u>
$$\sum_j A(t_j, t_{j+1}) = \sum_j \delta I(t_j, t_{j+1}) + \sum_j \boxed{\Lambda \delta A(t_j, t_{j+1})} \lesssim |t_{j+1} - t_j|^\alpha$$
$$= \delta I(s,t) + |P|^{\alpha-1} O(|t-s|)$$
$$\to \delta I(s,t) \quad \text{as} \quad |P| \to 0$$

$\boxtimes$

Young's integral:

Thm: $0 < \alpha, \beta < 1$ s.t. $\alpha + \beta > 1$.

Then, the integral map $I(f,g) = \int_0^\cdot f(s)\, \partial_s g(s)\, ds$ has a conti extension: $C^\alpha \times C^\beta \to C^\beta$ s.t.

$$|\delta I(f,g)(s,t) - f(s)\,\delta g(s,t)| \lesssim_{\alpha+\beta} |t-s|^{\alpha+\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta}$$

and

$$\|I(f,g)\|_{C^\beta([s,t])} \leq \left(\|f\|_{L^\infty([s,t])} + \|f\|_{C^\alpha([s,t])}\right) \|g\|_{C^\beta([s,t])}$$

for any $0 \leq s \leq t \leq 1$.

Moreover,

$$I(f,g)(t) = \lim_{|P(0,t)| \to 0} \sum_{j=0}^{n-1} f(t_j)\left(g(t_{j+1}) - g(t_j)\right)$$

"Pf": $A(s,t) = f(s)\,\delta g(s) = f(s)\,(g(t) - g(s))$

$\Rightarrow \delta A(s,u,t) = \delta f(s,u)\, \delta g(u,t)$

$\Rightarrow \|\delta A\|_{C_3^{\alpha+\beta}} \leq \|f\|_{C^\alpha} \|g\|_{C^\beta}$ $\qquad \|\delta f\|_{C_2^\alpha} = \|f\|_{C^\alpha}$

Then, with $R = \Lambda \delta A$, set

$$\delta I(f,g)(s,t) = A(s,t) - R(s,t)$$

$\square$

③ endpoint case $p = 2$

$$dY_t = f(Y)\, dX_t$$

- can ~~solve~~ this via Young's integration theory for $X \in V^p$, $p < 2$
- will solve this for $X \in U^p$, $p \le 2$ (in particular $p = 2$).

U^p = predual of $V^{p'}$, i.e. $(U^p)^* = V^{p'}$

introduced by Tataru, Koch-Tataru in early 00's.

in the study of nonlinear dispersive PDEs.

- Lec note by Koch (pp. 28-52) Hadac-Herr-Koch 2009 AIHP Analyse Nonlineare
  Also, errata

(a, b)

$a = t_0$ maybe $-\infty$

$b = t_n$ may be $+\infty$

- We say that a function f is a <u>ruled function</u> if at every point (including endpts which may be $\pm\infty$) left and right limit exist.

$\mathcal{R}$ = collection of ruled function

Banach space under L^∞-norm.

R_{rc} = subspace of R of right-conti functions.

- Define $V^p((a,b))$ by

$$\|v\|_{V^p((a,b))} = \max\{\|v\|_{L^\infty}, \underbrace{\omega_p(v,(a,b))}_{=\sup_P \left(\sum_{j=0}^{n-1}|v(t_{j+1})-v(t_j)|^p\right)^{1/p}}\}$$

<u>FACT</u> (see p.31)

- $V^p(I)$ is a Banach subspace of R.

  ($\Leftarrow$ $\omega_p(v,(a,b)) < \infty$ implies that v has left and right limits at every pt.

- $V^p_{rc}(I)$ is a closed subspace.

- $V^\infty = R$ with $\|\cdot\|_{V^\infty} = \|\cdot\|_{L^\infty}$

- X_j, Banach space

  $T: X_1 \times X_2 \to X_3$, bdd bilinear op

$1 \le p \le \infty$ $\Longrightarrow$ For $v \in V^p(I;X_1)$, $w \in V^p(I;X_2)$, we have $T(v,w) \in V^p(I;X_3)$

with $$\|T(v,w)\|_{V^p(I;X_3)} \le 2\|T\|\,\|v\|_{V^p(I;X_1)}\|w\|_{V^p(I;X_2)}$$

Weak spaces:

weak ℓ^p: $\ell^p_w = \ell^{p,\infty}$

$$\|\{a_j\}\|_{\ell^p_w} = \sup_{\lambda>0} \lambda \left(\#\{j : |a_j| > \lambda\}\right)^{1/p}$$

Recall $L^{p,\infty}$

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \,|\{x : |f(x)| > \lambda\}|^{1/p}$$

- V^p_w, $1 \le p < \infty$ $\quad (\ell^\infty_w = \ell^\infty)$

$$\|v\|_{V^p_w} = \max\left\{ \|v\|_{L^\infty}, \sup_P \left\|\{v(t_{j+1}) - v(t_j)\}_{0\le j\le n-1}\right\|_{\ell^p_w} \right\}$$

- By Chebyshev,

$$\|v\|_{V^p_w} \le \|v\|_{V^p}$$

- B_t = Brownian motion

$$B \notin V^p \text{ a.s. } p \le 2.$$

Thm 4.30 on p.47.

but

$$B \in V^2_w([0,1]) \text{ a.s.}$$

Def: A p-atom a is a step function in S_{rc}

$$a(t) = \sum_{j=0}^{n-1} \phi_j \chi_{[t_j, t_{j+1})}(t)$$

S = step functions

where $t_n = b$ and $\sum |\phi_j|^p \le 1$.

- A p-atom a is called a strict p-atom if

$$\left(\max_j |\phi_j|\right)(\# P)^{1/p} \le 1.$$

$P = t_0, t_1, \cdots, t_n$

Convention: Take $a < t_0$ s.t. an atom vanishes near the left endpt.
left endpt

- U^p = collection of functions s.t.

$$\|u\|_{U^p} = \inf \left\{ \sum |\lambda_j| : u = \sum \lambda_j a_j,\ \{a_j\} \text{ seq of p-atoms} \right\}$$

over all possible representations of u

- U^p_{strict} is defined similarly but with strict p-atoms.

Basic properties:

(i) For a p-atom a, we have $\|a\|_{V^p} \le 1$.

(may be $\|a\|_{V^p} < 1$ but difficult to determine the norm of an atom.

(ii) functions in U^p are right-conti.

and $\lim_{t \downarrow a} u(t) = 0$.

(iii) U^p is closed under the $\|\cdot\|_{V^p}$-norm.

$U^p \subset R_{rc}$ with $\|\cdot\|_{L^\infty} \le \|\cdot\|_{V^p}$.

(iv) $p < q$. Then,

$$\|u\|_{V^q} \le \|u\|_{U^p}$$

(v) $1 \le p < \infty$.

$$\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}$$

$1 \le p < q < \infty$

$U^p \subset V^p_{rc} \subset U^q \subset L^\infty$

(vi) $T : S_{rc} \to Y$, lin op (Y, Banach space)

with $\|Ta\|_Y \le C$ for every p-atom a.

Then, T has a unique extension to a bdd. lin op from U^p to Y

s.t.

$$\|Tu\|_Y \le C \|u\|_{V^p}$$

(vii) $T : X_1 \times X_2 \to X_3$, bdd bilin op.

For $v \in U^p(I; X_1)$, $w \in U^p(I; X_2)$, we have $T(v,w) \in U^p(I; X_3)$ with

$$\| T(v,w) \|_{U^p(I;X_3)} \leq 2 \|T\| \, \|v\|_{U^p(I;X_1)} \, \|w\|_{U^p(I;X_2)}$$

- $U^q \subsetneq V^q_{rc}$.

Let ϕ be a smooth func with cpt supp.

Lemma 4.15 in [K] Then, for $1 < q < \infty$,

See Ex 5.26 in [FV]

$$\phi(x) \sum_{j=1}^{\infty} 2^{-j/q} \cos(2^j t) \in V^q_{rc} \setminus U^q$$

In PDE application, we often need to estimate $\| \int f \, dt \|_X$ $X = H^b, B^b_{p,q}$, etc.

$$\| \int f \, dt \|_{H^{\frac{1}{2}+}} \sim \| f \|_{H^{-\frac{1}{2}+}} = \sup_{\|g\|_{H^{\frac{1}{2}-}}=1} \left| \int f g \, dt \right|$$

"Duality"

$$\begin{array}{ccccc} H^{\frac{1}{2}+} & \subset & B^{\frac{1}{2}}_{2,1} & \subset & U^2 \\ \updownarrow \varepsilon \text{ gap of reg} & & \updownarrow \text{log gap} & & \cap \\ H^{\frac{1}{2}-} & \supset & B^{\frac{1}{2}}_{2,\infty} & \supset & V^2 \end{array}$$

In PDE theory, we want a space which scales like L^∞_t

ex: $\dot{H}^{\frac{1}{2}}_t$ ← NOT good b/c $\dot{H}^{\frac{1}{2}} \not\subset C_t$

Thm : $\exists!$ continuous bilinear map

$$B : U^q(X) \times V^p(X^*) \to \mathbb{R} \qquad \frac{1}{p}+\frac{1}{q}=1$$

which satisfies (with $t_0 = a$, $u(t_0)=0$
$v(b)=0$

$$B(u,v) = \sum_{j=1}^{n} v(t_j)\left(u(t_j) - u(t_{j-1})\right)$$

i.e. partition of the step func u

for $v \in V^p$ and $u \in S_{rc}$ with associated partition $P = \{t_0 < t_1 < \cdots < t_n\}$ with the bound:

$$|B(u,v)| \le \|u\|_{U^q(X)} \|v\|_{V^p(X^*)}.$$

The map

$$v \in V^p(X^*) \longmapsto \left(u \mapsto B(u,v)\right) \in \left(U^q(X)\right)^*$$

is a surjective isometry if $1 \le q < \infty$.

Moreover, $$\|v\|_{V^p(X^*)} = \sup_{\|u\|_{U^q(X)}=1} B(u,v) = \sup_{a,\ q\text{-atom}} B(a,v)$$

Pf: Let $v \in V^p$. (8)

$$F_v(u) = \sum_{j=1}^{n} v(t_j)\big(u(t_j) - u(t_{j-1})\big) \overset{\text{SBP}}{=} - \sum_{j=0}^{n} \big(v(t_{j+1}) - v(t_j)\big)\, u(t_j)$$

$b = t_{n+1}$

$v(b) = 0$

- well defined for $v \in V^p$ and $u \in S_{rc}$.
- map is linear in v and u.
- For a q-atom a, we have

$$\begin{aligned} |F_v(a)| &\le \sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}\, \|a(t_j)\|_X \\ &\le \Big(\sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}^p\Big)^{1/p} \underbrace{\Big(\sum_{j=1}^{n} \|a(t_j)\|_X^q\Big)^{1/q}}_{\le 1} \\ &\le \|v\|_{V^p(X^*)} \end{aligned}$$

by (vi)

$\Longrightarrow$ F_v extends uniquely to a bdd lin op on $U^q(X)$

s.t. $|F_v(u)| \le \|v\|_{V^p(X^*)} \|u\|_{U^q(X)}$.

- $B : V^p \longrightarrow (V^q)^*$ with norm at most 1.
- Let us show that it is an isometry.

Fix $v \in V^p$, $\varepsilon > 0$, and $P = \{t_0 < t_1 < \cdots < t_n\}$ s.t.

$$(*) \qquad \|v\|_{V^p} \le \left(\sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}^p \right)^{1/p} + \varepsilon$$

Choose $x_j \in X$ with norm 1 s.t.

$$(**) \qquad \big(v(t_{j+1}) - v(t_j)\big)(x_j) \ge (1-\varepsilon)\,\|v(t_{j+1}) - v(t_j)\|_{X^*}.$$

and let
$$\phi_j = \mu\, \|v(t_{j+1}) - v(t_j)\|_{X^*}^{p-1}\, x_j.$$

where $\mu = \|v\|_{V^p}^{1-p}$.

$$p' = \frac{p}{p-1}$$

Then,
$$\sum_{j=1}^{n} \|\phi_j\|_X^{p'} \le \|v\|_{V^p}^{-p} \sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}^p \le 1.$$

$$\Rightarrow \quad a = \sum_{j=1}^{n} \phi_j\, \chi_{[t_j, t_{j+1})} \quad \text{is a } p'\text{-atom.}$$

and

$$\|v\|_{V^p} \leq B(a,v) - C\varepsilon$$

$\Leftarrow$ Take ⊛ to the power p.

$$\|v\|_{V^p} \leq \sum_{j=1}^{n} \mu \, \|v(t_{j+1}) - v(t_j)\|_{X^*}^{p-1} \underbrace{\|v(t_{j+1}) - v(t_j)\|_{X^*}}_{\text{use } (**)} + \varepsilon^p$$

i.e. B is an isometry.

· <u>Surjectivity</u>: Let $F \in (U^q(X))^*$.

Define an element $v(t) \in X^*$ by

$$v(t)(x) = F(x \chi_{[t,b)}) , \quad x \in X.$$

Let a be a q-atom.

$$F(a) = \sum_{j=1}^{n} F(\phi_j \chi_{[t_j,b)}) - F(\phi_j \chi_{[t_{j+1},b)})$$

$$= - \sum_{j=1}^{n} \big(v(t_{j+1}) - v(t_j)\big)(\phi_j)$$

$$= \sum_{j=1}^{n} v(t_j)\big(a(t_j) - a(t_{j-1})\big) = B(a,v)$$

We also have

$$\|v\|_{V^p} \leq \|F\|_{(U^q)^*}$$

$$\Rightarrow F(u) = B(u,v) \text{ on } U^q(X).$$

$\boxtimes$

$V^q_c = V^q \cap C,\ v(b)=0$

$U^p(X^*) \to (V^q_c(X))^*$ is a surjective isometry

Lemma 4.23

Rmk: Cor 4.24

$$\|u\|_{U^p(X)} = \sup\{B(u,v) : v \in C_0^\infty(X),\ \|v\|_{V^q(X^*)} = 1\}$$

$$\|v\|_{V^p_{rc}(X)} = \sup\{B(u,v) : u \in C_0^\infty(X),\ \|u\|_{U^q(X^*)} = 1\}.$$

Back to $dY = f(Y)\,dX$. f, Lipschitz.

$$\textcircled{+} \qquad Y_t = Y_0 + \underbrace{\int_0^t f(Y)_s\, dX_s}_{}$$

define this for $X \in U^2$ (and $Y \in V^2$)

Bilinear form as integral:

$$v \in V^p((a,b)),\ u \in U^q((a,b))$$ ++

Def 4.26.
Lem 4.27

For $a \le s \le t \le b$, we define

$$\boxed{\int_s^t v\,du} = B_{(s,t)}(u - u(s), v) + \big(u(t) - u(t-)\big)v(t)$$

integral is unchanged even if we replace u by $u - c$.

In particular, the integral $\int_0^\cdot v\,du$ makes sense for $v \in V^2$, $u \in U^2$

(***)
$$\left.\begin{array}{l} \left\| \int_a^t v\,du \right\|_{U^q} \\ \left\| \int_a^t v\,du \right\|_{V^p} \end{array}\right\} \le \|u\|_{U^q} \|v\|_{V^p}$$

Using (***), we can construct a unique soln $Y \in U^2$ to (+) for $X \in U^2$

$$\left\| \int_0^t f(Y)\,dX \right\|_{U^2(0,\tau)} \le \|X\|_{U^2(0,\tau)} \Big(|f(0)| + K \underbrace{\|Y\|_{V^2(0,\tau)}}_{\lesssim \|Y\|_{U^2(0,\tau)}} \Big)$$

$\Leftarrow$ Need $\|X\|_{U^2((0,\tau))} \ll 1$

- In general, $\|X\|_{U^2((0,\tau))} \not\to 0$ as $\tau \to 0$ since $U^2 \subset L^\infty$.

but in view of (++) on page (12), we can assume $X(0) = 0$.

$\| \leftarrow$ by right-conti.

$\lim_{t \downarrow 0} X(t)$

$\Rightarrow \|X\|_{U^2((0,\tau))} \longrightarrow 0$ as $\tau \to 0+$

This follows from the definition of U^2:

Given $\varepsilon > 0$, write

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{s.t.} \quad \left| \sum_{j=1}^{\infty} |\lambda_j| - \|u\|_{U^2} \right| < \varepsilon.$$

Now, $\exists\ N = N(\varepsilon) \geq 1$ s.t.

$$\sum_{j=N+1}^{\infty} |\lambda_j| < \varepsilon.$$

By our convention, $a_j(0) = 0 \Rightarrow \sum_{j=1}^{N} \lambda_j a_j \equiv 0$ on $[0, t_*)$ for some $t_* > 0$

$\Rightarrow \|u\|_{U^2([0,t_*))} < 2\varepsilon$.

For $t \geq \tau$, we can repeat the analysis.

In particular, (††) on page 12 allows us to assume $X(\tau) = 0$

and thus $\|X\|_{U^2([\tau, \tau+\tau_1))} \ll 1$ by choosing $\tau_1 > 0$ suff. small.

$\Rightarrow$ Iterate this procedure as in $\underbrace{p=1}_{\text{Riemann-Stieltjes}}$ and $\underbrace{p<2}_{\text{Young}}$ cases.

Lec 7 13/03/20 (Fri)

④ rough paths & controlled rough paths

· Friz - Hairer

$$X : [0,T] \to V \quad (\cong \mathbb{R}^d \text{ or Banach space})$$

If X does not have suff regularity (e.g. $X \notin C^{\frac{1}{2}+\varepsilon}$)

then $\int X\, dX$ does not make sense

Idea: introduce a second order object $\mathbb{X} = \mathbb{X}(s,t) = \mathbb{X}_{s,t}$

and set

$$\int_s^t X_{s,r} \otimes dX_r := \mathbb{X}_{s,t} \quad \Leftarrow \text{RHS defines LHS}$$

$$\parallel$$

$$\int_s^t \int_s^r dX_u \otimes dX_r$$

Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \underset{\underset{X_u - X_s}{\parallel}}{X_{s,u}} \otimes X_{u,t}$$

If $\mathbb{X}_{s,t} = \int_s^t X_{s,r}\, dX_r$ (interpreted as a R-S or Young integral for smooth X)

$$\text{LHS} = \int_s^t \cancel{X_{s,r}}\, dX_r - \int_s^u \cancel{X_{s,r}}\, dX_r - \underbrace{\int_u^t X_{u,r}\, dX_r}$$

$$= -\int_u^t \cancel{X_{s,r}}\, dX_r + \underbrace{\int_u^t X_{s,u}\, dX_r}_{= \text{RHS}}$$

Def: $\frac{1}{3} < \alpha \le \frac{1}{2}$.

α - Hölder rough path (over V)

$\mathbf{X} = (X, \mathbb{X})$ $\quad X : [0,T] \to V$ $\quad$ satisfying Chen's relation.

$\quad \mathbb{X} : [0,T]^2 \to V \otimes V$

$$\left.\begin{array}{l} \|X\|_\alpha = \sup\limits_{s \neq t \in [0,T]} \dfrac{|X_{s,t}|}{|t-s|^\alpha} < \infty \\[2ex] \|\mathbb{X}\|_{2\alpha} = \sup\limits_{s \neq t} \dfrac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty \end{array}\right\} \Rightarrow \mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T]; V)$$

$\Updownarrow$

$\mathbb{X} \in C_2^{2\alpha}$

Rmk: $\alpha > 1$. $X \in C^\alpha \Rightarrow X \equiv \text{const}$.

but $\exists$ non-trivial element in C_2^α.

- Natural dilation for $\mathscr{C}^\alpha$

$$(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$$

NOT a norm but good for our application

$\Rightarrow$ α - Hölder rough path "norm"

$$|||\mathbf{X}|||_\alpha = \|X\|_\alpha + \sqrt{\|\mathbb{X}\|_{2\alpha}}$$

- From the def of $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, we introduce the α - Hölder rough path metric ρ_α

$$\rho_\alpha(\mathbf{X}, \mathbf{Y}) = \sup_{s \neq t} \frac{|X_{s,t} - Y_{s,t}|}{|s-t|^\alpha} + \sup_{s \neq t} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|s-t|^{2\alpha}}$$

for $\mathbf{X}$ and $\mathbf{Y} = (Y, \mathbb{Y})$ in $\mathscr{C}^\alpha$

- Convergence w.r.t. ρ_α $\Leftarrow$ interpolation EX 2.9
  - ptwise convergence
  - unif rough path bound $\|X\|_{\alpha+} + \|\mathbb{X}\|_{2\alpha+} \le C$.

Chen's relation does not determine $\mathbb{X}$ uniquely.
(given X)

- $\tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + F_t - F_s \qquad F \in C^{2\alpha}$

does not change LHS of Chen's relation.

$\Rightarrow$ $\mathbb{X}$ is in general determined only up to the increment of some func $F \in C^{2\alpha}$.

NO canonical choice
(unless it is defined for a specific application)

- For $\alpha > 1/2$, and $X \in C^{\alpha}$ we can define $\mathbb{X}_{s,t} = \int_s^t X_{s,r} dX_r$ as a Young integral.

uniquely determined.

For $\alpha \le 1/2$, $\quad F \in C^{2\alpha}([0,T]; V \otimes V)$ and set $\mathbb{X}_{s,t} = F_t - F_s$

Then, $\quad (0, \mathbb{X}) \in \mathscr{C}^{\alpha}$

smooth

· <u>geometric rough path</u> ($\Leftarrow$ Stratonovich)

(2.5) $$\mathrm{Sym}\left(\mathbb{X}_{s,t}\right) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$$

$\Leftarrow$ satisfied for $\mathbb{X}^{ij}_{s,t} = \int_s^t X^i_{s,r}\, dX^j_r$ for smooth X

$$\mathbb{X}^{ij}_{s,t} + \mathbb{X}^{ji}_{s,t} = \int_s^t X^i_{s,r}\, dX^j_r + \int_s^t X^j_{s,r}\, dX^i_r$$

$$= \underbrace{\int_s^t d(X^i X^j)_r}_{=(X^i X^j)_{s,t}} - X^i_s X^j_{s,t} - X^j_s X^i_{s,t}$$

$$= X^i_{s,t}\, X^j_{s,t}$$

α-Hölder geometric rough paths $\mathscr{C}^\alpha_g \subset \mathscr{C}^\alpha$

satisfying (2.5) for every $s,t \in [0,T]$

· rough paths as Lie-group valued paths

$$V = \mathbb{R}^d, \qquad X : [0,T] \to \mathbb{R}^d, \ \mathbb{X} : [0,T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

$$\mathbf{X}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t}) \in \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) = T^{(2)}(\mathbb{R}^d).$$

$T^{(2)}(\mathbb{R}^d)$ is a vector space

also a non-commutative algebra with unit element $(1,0,0)$ under

$$(a,b,c) \otimes (a',b',c') = (aa', ab' + a'b, ac' + a'c + b \otimes b')$$

$$\left.\begin{array}{l} X_{s,t} = X_{s,u} + X_{u,t} \\ \text{Chen's relation} \end{array}\right\} \Rightarrow \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}.$$

Set $T_a^{(2)}(\mathbb{R}^d) = \{(a,b,c) : b \in \mathbb{R}^d, \ c \in \mathbb{R}^d \otimes \mathbb{R}^d\}.$

· $T_1^{(2)}(\mathbb{R}^d)$ is a Lie group:

$$(1,b,c) \otimes (1, -b, -c + b \otimes b) = (1, 0, 0)$$

Lie group valued path : $t \longmapsto \mathbb{X}_{0,t}$

increment $\mathbb{X}_{s,t} = \mathbb{X}_{0,s}^{-1} \otimes \mathbb{X}_{0,t}$

Identify

$$1 \leftrightsquigarrow (1,0,0)$$
$$b \leftrightsquigarrow (0,b,0)$$
$$c \leftrightsquigarrow (0,0,c)$$

and write

$$(1,b,c) = 1 + b + c.$$

$$(1+x)^{-1} = 1 - x + x^2 - \cdots$$

No term of "deg" > 2.

$$\Rightarrow (1+b+c)^{-1} = 1 - (b+c) + (b+c) \otimes (b+c)$$
$$= 1 - b - c + b \otimes b \quad \Leftarrow \text{inverse of } (1,b,c).$$

---

usual power series

$$\log(1+b+c) \overset{\text{def}}{=} b + c - \frac{1}{2} b \otimes b$$
$$\exp(b+c) \overset{\text{def}}{=} 1 + b + c + \frac{1}{2} b \otimes b.$$

$$\Longrightarrow \quad T_0^{(2)}(\mathbb{R}^d) \cong \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$$

Lie algebra

$$T_1^{(2)}(\mathbb{R}^d) = \exp(\mathbb{R}^d \oplus \mathbb{R}^{d \times d})$$

· Brownian motion as rough paths

Consider random $X(\omega) : [0,T] \to V$
$\mathbb{X}(\omega) : [0,T]^2 \to V \otimes V$ satisfying Chen's relation

ex: B = Brownian motion

$$\mathbb{B}_{s,t} = \underbrace{\int_s^t B_{s,r} \otimes dB_r}_{\text{interpreted in either Ito or Stratonovich sense.}} \quad \Leftarrow \text{ RHS defines LHS}$$

$\mathbb{B}^{\text{Ito}}$ $\qquad$ $\mathbb{B}^{\text{Strat}}$.

$B \in C^\alpha, \quad \alpha < 1/2$

Q: $\mathbb{B} \in C^{2\alpha}$?

Thm 3.1 (Kolmogorov criterion for rough paths) $q \geq 2$, $\beta > 1/q$. Suppose

$$\| X_{s,t} \|_{L^q(\Omega)} \leq C |t-s|^\beta, \quad \| \mathbb{X}_{s,t} \|_{L^{q/2}(\Omega)} \leq C |t-s|^{2\beta}, \quad \forall\ s,t$$

Then, for any $\alpha \in [0, \beta - 1/q)$, we have

$$|X_{s,t}| \leq \underbrace{K_\alpha(\omega)}_{\in L^q(\Omega)} |t-s|^\alpha, \quad |\mathbb{X}_{s,t}| \leq \underbrace{\mathbb{K}_\alpha(\omega)}_{\in L^{q/2}(\Omega)} |t-s|^{2\alpha}$$

In particular, if $\beta - \frac{1}{q} > 1/3$, then

$$(X, \mathbb{X}) \in \mathscr{C}^{\alpha}, \text{ a.s. for any } \alpha \in \left(\tfrac{1}{3}, \beta - \tfrac{1}{3}\right).$$

See also Thm 3.3. for Kolmogorov criterion for rough path distances

$$\mathbf{X} = (X, \mathbb{X}), \quad \tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}).$$

- Ito Brownian motion (Left endpt.)

$$\mathbb{B}^{Ito}_{s,t} = \int_s^t B_{s,r} \otimes dB_r \Leftarrow \text{ iterated Ito-Wiener integral.}$$

Prop 3.4: $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $T > 0$. Then,

$$\mathbf{B} = (B, \mathbb{B}^{Ito}) \in \mathscr{C}^{\alpha}([0,T]; \mathbb{R}^d), \text{ a.s.}$$

($\Leftarrow$ follows from Kolmogorov conti criterion)

Rmk: $\mathrm{Sym}(\mathbb{B}^{Ito}_{s,t}) = \frac{1}{2} B_{s,t} \otimes B_{s,t} - \frac{1}{2} I (t-s)$

$$\neq \frac{1}{2} B_{s,t} \otimes B_{s,t}.$$

- Stratonovich Brownian motion   mid pt rule

$$\mathbb{B}^{\text{Strat}}_{s,t} = \mathbb{B}^{\text{Ito}}_{s,t} + \frac{1}{2} I (t-s)$$

Prop 3.5: $\alpha \in (1/3, 1/2)$ and $T > 0$. Then,

$$\mathbf{B} = (B, \mathbb{B}^{\text{Strat}}) \in \mathscr{C}^{\alpha}_{g}([0,T]; \mathbb{R}^d)$$