Lec 7 13 / 03 / 20 (Fri)

4 rough paths & controlled rough paths

· Friz - Hairer

$$X : [0,T] \to V \quad (\cong \mathbb{R}^d \text{ or Banach space})$$

If X does not have suff regularity (e.g. $X \notin C^{\frac{1}{2}+\varepsilon}$)

then $\int X \, dX$ does not make sense

Idea: introduce a second order object $\mathbb{X} = \mathbb{X}(s,t) = \mathbb{X}_{s,t}$

and set

$$\int_s^t X_{s,r} \otimes dX_r := \mathbb{X}_{s,t} \quad \Leftarrow \text{RHS defines LHS}$$

$$\parallel$$

$$\int_s^t \int_s^r dX_u \otimes dX_r$$

Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \underbrace{X_{s,u}}_{= X_u - X_s} \otimes X_{u,t}$$

If $\mathbb{X}_{s,t} = \int_s^t X_{s,r}\, dX_r$ (interpreted as a R-S or Young integral for smooth X)

$$\mathrm{LHS} = \int_s^t \cancel{X_{s,r}}\, dX_r - \int_s^u \cancel{X_{s,r}}\, dX_r - \underbrace{\int_u^t X_{u,r}\, dX_r}$$

$$= - \int_u^t \cancel{X_{s,r}}\, dX_r + \underbrace{\int_u^t X_{s,u}\, dX_r}_{= \mathrm{RHS}}$$

Def: $\frac{1}{3} < \alpha \le \frac{1}{2}$.

α - Hölder rough path (over V)

$\mathbf{X} = (X, \mathbb{X})$ $\quad X : [0,T] \to V$ $\quad$ satisfying Chen's relation.

$\quad \mathbb{X} : [0,T]^2 \to V \otimes V$

$$\left.\begin{aligned} \|X\|_\alpha &= \sup_{s \neq t \in [0,T]} \frac{|X_{s,t}|}{|t-s|^\alpha} < \infty \\ \|\mathbb{X}\|_{2\alpha} &= \sup_{s \neq t} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty \end{aligned}\right\} \Rightarrow \mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T]; V)$$

$\Updownarrow$

$\mathbb{X} \in C_2^{2\alpha}$

Rmk: $\alpha > 1$. $X \in C^\alpha \Rightarrow X \equiv \text{const}$.

but $\exists$ non-trivial element in C_2^α.

- Natural dilation for $\mathscr{C}^\alpha$

$$(X, \mathbb{X}) \to (\lambda X, \lambda^2 \mathbb{X})$$

NOT a norm but good for our application

$\Rightarrow$ α - Hölder rough path "norm"

$$|||\mathbf{X}|||_\alpha = \|X\|_\alpha + \sqrt{\|\mathbb{X}\|_{2\alpha}}$$

- From the def of $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, we introduce the α - Hölder rough path metric ρ_α

$$\rho_\alpha(\mathbf{X}, \mathbf{Y}) = \sup_{s \neq t} \frac{|X_{s,t} - Y_{s,t}|}{|s-t|^\alpha} + \sup_{s \neq t} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|s-t|^{2\alpha}}$$

for $\mathbf{X}$ and $\mathbf{Y} = (Y, \mathbb{Y})$ in $\mathscr{C}^\alpha$

- Convergence w.r.t. ρ_α $\Leftarrow$ interpolation EX 2.9
  - ptwise convergence
  - unif rough path bound $\|X\|_{\alpha+} + \|\mathbb{X}\|_{2\alpha+} \leq C$.

Chen's relation does not determine $\mathbb{X}$ uniquely.
(given X)

· $\tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + F_t - F_s \qquad F \in C^{2\alpha}$

does not change LHS of Chen's relation.

$\Rightarrow$ $\mathbb{X}$ is in general determined only up to the increment of some func $F \in C^{2\alpha}$.

NO canonical choice
(unless it is defined for a specific application)

· For $\alpha > 1/2$, and $X \in C^{\alpha}$ we can define $\mathbb{X}_{s,t} = \int_s^t X_{s,r} dX_r$ as a Young integral.

uniquely determined.

For $\alpha \le 1/2$, $\quad F \in C^{2\alpha}([0,T]; V \otimes V)$ and set $\mathbb{X}_{s,t} = F_t - F_s$

Then, $\quad (0, \mathbb{X}) \in \mathscr{C}^{\alpha}$

smooth

· <u>geometric rough path</u> ($\Leftarrow$ Stratonovich)

(2.5) $$\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$$

$\Leftarrow$ satisfied for $\mathbb{X}^{ij}_{s,t} = \int_s^t X^i_{s,r}\, dX^j_r$ for smooth X

$$\mathbb{X}^{ij}_{s,t} + \mathbb{X}^{ji}_{s,t} = \int_s^t X^i_{s,r}\, dX^j_r + \int_s^t X^j_{s,r}\, dX^i_r$$

$$= \underbrace{\int_s^t d(X^i X^j)_r}_{=(X^i X^j)_{s,t}} - X^i_s X^j_{s,t} - X^j_s X^i_{s,t}$$

$$= X^i_{s,t} X^j_{s,t}$$

α-Hölder geometric rough paths $\mathscr{C}^\alpha_g \subset \mathscr{C}^\alpha$

satisfying (2.5) for every $s,t \in [0,T]$

· rough paths as Lie-group valued paths

$$V = \mathbb{R}^d, \qquad X : [0,T] \to \mathbb{R}^d, \ \mathbb{X} : [0,T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

$$\mathbf{X}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t}) \in \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) = T^{(2)}(\mathbb{R}^d).$$

$T^{(2)}(\mathbb{R}^d)$ is a vector space

also a non-commutative algebra with unit element $(1,0,0)$ under

$$(a,b,c) \otimes (a',b',c') = (aa', ab' + a'b, ac' + a'c + b \otimes b')$$

$$\left.\begin{array}{l} X_{s,t} = X_{s,u} + X_{u,t} \\ \text{Chen's relation} \end{array}\right\} \Rightarrow \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}.$$

Set $T_a^{(2)}(\mathbb{R}^d) = \{(a,b,c) : b \in \mathbb{R}^d, c \in \mathbb{R}^d \otimes \mathbb{R}^d\}.$

· $T_1^{(2)}(\mathbb{R}^d)$ is a Lie group:

$$(1,b,c) \otimes (1, -b, -c + b \otimes b) = (1, 0, 0)$$

Lie group valued path : $t \longmapsto \mathbb{X}_{0,t}$

increment $\mathbb{X}_{s,t} = \mathbb{X}_{0,s}^{-1} \otimes \mathbb{X}_{0,t}$

Identify

$$1 \rightsquigarrow (1,0,0)$$
$$b \leftrightsquigarrow (0,b,0)$$
$$c \leftrightsquigarrow (0,0,c)$$

and write

$$(1,b,c) = 1 + b + c.$$

$$(1+x)^{-1} = 1 - x + x^2 - \cdots$$

No term of "deg" > 2.

$$\Rightarrow (1+b+c)^{-1} = 1 - (b+c) + (b+c)\otimes(b+c)$$
$$= 1 - b - c + b\otimes b$$

$\Leftarrow$ inverse of $(1,b,c)$.

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usual power series

$$\log(1+b+c) \overset{\text{def}}{=} b + c - \frac{1}{2} b\otimes b$$
$$\exp(b+c) \overset{\text{def}}{=} 1 + b + c + \frac{1}{2} b\otimes b.$$

$$\Longrightarrow \quad T_0^{(2)}(\mathbb{R}^d) \cong \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$$
$$T_1^{(2)}(\mathbb{R}^d) = \exp(\mathbb{R}^d \oplus \mathbb{R}^{d\times d})$$

Lie algebra

· Brownian motion as rough paths

Consider random $X(\omega) : [0,T] \to V$
$\mathbb{X}(\omega) : [0,T]^2 \to V \otimes V$ satisfying Chen's relation

ex: B = Brownian motion

$$\mathbb{B}_{s,t} = \underbrace{\int_s^t B_{s,r} \otimes dB_r}_{\text{interpreted in either Ito or Stratonovich sense.}} \Leftarrow \text{RHS defines LHS}$$

$\mathbb{B}^{Ito}$ $\qquad$ $\mathbb{B}^{Strat}$.

$B \in C^\alpha, \quad \alpha < 1/2$

Q: $\mathbb{B} \in C^{2\alpha}$?

Thm 3.1 (Kolmogorov criterion for rough paths) $q \geq 2$, $\beta > 1/q$. suppose

$$\| X_{s,t} \|_{L^q(\Omega)} \leq C |t-s|^\beta, \quad \| \mathbb{X}_{s,t} \|_{L^{q/2}(\Omega)} \leq C |t-s|^{2\beta}, \quad \forall\ s,t$$

Then, for any $\alpha \in [0, \beta - 1/q)$, we have

$$|X_{s,t}| \leq \underbrace{K_\alpha(\omega)}_{\in L^q(\Omega)} |t-s|^\alpha, \quad |\mathbb{X}_{s,t}| \leq \underbrace{\mathbb{K}_\alpha(\omega)}_{\in L^{q/2}(\Omega)} |t-s|^{2\alpha}$$

In particular, if $\beta - \frac{1}{q} > 1/3$, then

$$(X, \mathbb{X}) \in \mathscr{C}^\alpha, \text{ a.s. for any } \alpha \in (\tfrac{1}{3}, \beta - \tfrac{1}{3}).$$

See also Thm 3.3. for Kolmogorov criterion for rough path distances

$$\mathbf{X} = (X, \mathbb{X}), \quad \tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}).$$

- Ito Brownian motion (Left endpt.)

$$\mathbb{B}^{\text{Ito}}_{s,t} = \int_s^t B_{s,r} \otimes dB_r \Leftarrow \text{iterated Ito-Wiener integral.}$$

Prop 3.4: $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $T > 0$. Then,

$$\mathbf{B} = (B, \mathbb{B}^{\text{Ito}}) \in \mathscr{C}^\alpha([0,T]; \mathbb{R}^d), \text{ a.s.}$$

($\Leftarrow$ follows from Kolmogorov conti criterion)

Rmk: $$\mathrm{Sym}(\mathbb{B}^{\text{Ito}}_{s,t}) = \frac{1}{2} B_{s,t} \otimes B_{s,t} - \frac{1}{2} I(t-s)$$

$$\neq \frac{1}{2} B_{s,t} \otimes B_{s,t}.$$

- Stratonovich Brownian motion mid pt rule

$$\mathbb{B}^{\text{Strat}}_{s,t} = \mathbb{B}^{\text{Ito}}_{s,t} + \frac{1}{2} I (t-s)$$

Prop 3.5: $\alpha \in (1/3, 1/2)$ and $T > 0$. Then,

$$\mathbf{B} = (B, \mathbb{B}^{\text{Strat}}) \in \mathscr{C}_g^{\alpha}([0,T]; \mathbb{R}^d)$$