Lec 6 28/02/20 (Fri)

③ endpoint case $p=2$

$$dY_t = f(Y)\,dX_t$$

- can solve this via Young's integration theory for $X \in V^p$, $p<2$
- will solve this for $X \in U^p$, $p \le 2$ (in particular $p=2$).

$$U^p = \text{predual of } V^{p'}, \text{ i.e. } (U^p)^* = V^{p'}$$

introduced by Tataru, Koch-Tataru in early 00's.
in the study of nonlinear dispersive PDEs.

- Lec note by Koch (pp. 28-52) Hadac-Herr-Koch 2009 AIHP Analyse Nonlineare
  Also, errata.

- We say that a function f is a <u>ruled function</u> if at every point (including endpts which may be $\pm\infty$) left and right limit exist.

(a,b)
$a = t_0$ may be $-\infty$
$b = t_m$ may be $+\infty$

$\mathcal{R}$ = collection of ruled function
Banach space under L^∞-norm.

R_{rc} = subspace of R of right-conti functions.

- Define $V^p((a,b))$ by

$$\|v\|_{V^p((a,b))} = \max\{\|v\|_{L^\infty}, \underbrace{\omega_p(v,(a,b))}_{\sup_P \left(\sum_{j=0}^{n-1}|v(t_{j+1})-v(t_j)|^p\right)^{1/p}}\}$$

<u>FACT</u> (see p.31)

- $V^p(I)$ is a Banach subspace of R.

  ($\Leftarrow$ $\omega_p(v,(a,b)) < \infty$ implies that v has left and right limits at every pt.

- $V^p_{rc}(I)$ is a closed subspace.

- $V^\infty = R$ with $\|\cdot\|_{V^\infty} = \|\cdot\|_{L^\infty}$

- X_j, Banach space

  $T : X_1 \times X_2 \to X_3$, bdd bilinear op

$1 \le p \le \infty$ $\Longrightarrow$ For $v \in V^p(I;X_1)$, $w \in V^p(I;X_2)$, we have $T(v,w) \in V^p(I;X_3)$

with

$$\|T(v,w)\|_{V^p(I;X_3)} \le 2\|T\|\,\|v\|_{V^p(I;X_1)}\|w\|_{V^p(I;X_2)}$$

Weak spaces:

weak ℓ^p: $\ell^p_w = \ell^{p,\infty}$

$$\|\{a_j\}\|_{\ell^p_w} = \sup_{\lambda>0} \lambda\left(\#\{j : |a_j| > \lambda\}\right)^{1/p}$$

Recall $L^{p,\infty}$

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \left|\{x : |f(x)| > \lambda\}\right|^{1/p}$$

- V^p_w, $1 \le p < \infty$ $\quad (\ell^\infty_w = \ell^\infty)$

$$\|v\|_{V^p_w} = \max\left\{ \|v\|_{L^\infty}, \ \sup_P \left\|\{v(t_{j+1}) - v(t_j)\}_{0\le j\le n-1}\right\|_{\ell^p_w} \right\}$$

- By Chebyshev,

$$\|v\|_{V^p_w} \le \|v\|_{V^p}$$

- B_t = Brownian motion

$$B \notin V^p \quad \text{a.s.} \quad p \le 2.$$

Thm 4.30 on p.47.

but

$$B \in V^2_w([0,1]) \quad \text{a.s.}$$

Def: A p-atom a is a step function in S_{rc}

$$a(t) = \sum_{j=0}^{n-1} \phi_j \chi_{[t_j, t_{j+1})}(t)$$

S = step functions

where $t_n = b$ and $\sum |\phi_j|^p \leq 1$.

- A p-atom a is called a strict p-atom if

$$\left(\max_j |\phi_j|\right)(\# P)^{1/p} \leq 1.$$

$P = t_0, t_1, \cdots, t_n$

Convention: Take $a < t_0$ s.t. an atom vanishes near the left endpt.
(a = left endpt)

- U^p = collection of functions s.t.

$$\|u\|_{U^p} = \inf \left\{ \sum |\lambda_j| : u = \sum \lambda_j a_j, \{a_j\} \text{ seq of p-atoms} \right\}$$

over all possible representations of u

- U^p_{strict} is defined similarly but with strict p-atoms.

Basic properties:

(i) For a p-atom a, we have $\|a\|_{U^p} \le 1$.

(may be $\|a\|_{U^p} < 1$ but difficult to determine the norm of an atom.

(ii) functions in U^p are right-conti.

and $\lim_{t \downarrow a} u(t) = 0$.

(iii) U^p is closed under the $\|\cdot\|_{U^p}$-norm.

$U^p \subset R_{rc}$ with $\|\cdot\|_{L^\infty} \le \|\cdot\|_{U^p}$.

(iv) $p < q$. Then,

$$\|u\|_{U^q} \le \|u\|_{U^p}$$

(v) $1 \le p < \infty$.

$$\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}$$

$1 \le p < q < \infty$

$U^p \subset V^p_{rc} \subset U^q \subset L^\infty$

(vi) $T : S_{rc} \to Y$, lin op (Y, Banach space)

with $\|Ta\|_Y \le C$ for every p-atom a.

Then, T has a unique extension to a bdd. lin op from U^p to Y

s.t.

$$\|Tu\|_Y \le C \|u\|_{U^p}$$

(vii) $T : X_1 \times X_2 \to X_3$, bdd bilin op.

For $v \in U^p(I; X_1)$, $w \in U^p(I; X_2)$, we have $T(v,w) \in U^p(I; X_3)$ with

$$\| T(v,w) \|_{U^p(I;X_3)} \le 2 \|T\| \, \|v\|_{U^p(I;X_1)} \|w\|_{U^p(I;X_2)}$$

- $U^q \subsetneq V^q_{rc}$.

Let ϕ be a smooth func with cpt supp

Lemma 4.15 in [K] Then, for $1 < q < \infty$,

See Ex 5.26 in [FV]

$$\phi(x) \sum_{j=1}^{\infty} 2^{-j/q} \cos(2^j t) \in V^q_{rc} \setminus U^q$$

In PDE application, we often need to estimate $\| \int f \, dt \|_X$ $\quad X = H^b, B^b_{p,q}$, etc.

$$\| \int f \, dt \|_{H^{1/2+}} \sim \|f\|_{H^{-1/2+}} = \sup_{\|g\|_{H^{1/2-}}=1} \left| \int f g \, dt \right|$$

"Duality"

$$\begin{array}{ccccc} H^{1/2+} & \subset & B^{1/2}_{2,1} & \subset & U^2 \\ \updownarrow \varepsilon \text{ gap of reg} & & \updownarrow \log \text{ gap} & & \cap \\ H^{1/2-} & \supset & B^{1/2}_{2,\infty} & \supset & V^2 \end{array}$$

In PDE theory, we want a space which scales like L^∞_t

ex: $\dot{H}^{1/2}_t \leftarrow$ NOT good b/c $\dot{H}^{1/2} \not\subset C_t$

<u>Thm</u> : $\exists !$ continuous bilinear map

$$B : \mathcal{V}^q(X) \times V^p(X^*) \to \mathbb{R} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

which satisfies (with $t_0 = a$, $u(t_0) = 0$
$v(b) = 0$

$$B(u,v) = \sum_{j=1}^{n} v(t_j)\left(u(t_j) - u(t_{j-1})\right)$$

i.e. partition of the step func u

for $v \in V^p$ and $u \in S_{rc}$ with associated partition $P = \{t_0 < t_1 < \cdots < t_n\}$ with the bound:

$$|B(u,v)| \le \|u\|_{\mathcal{V}^q(X)} \|v\|_{V^p(X^*)}.$$

The map

$$v \in V^p(X^*) \longmapsto \left(u \mapsto B(u,v)\right) \in \left(\mathcal{V}^q(X)\right)^*$$

is a surjective isometry if $1 \le q < \infty$.

Moreover, $$\|v\|_{V^p(X^*)} = \sup_{\|u\|_{\mathcal{V}^q(X)} = 1} B(u,v) = \sup_{a,\ q\text{-atom}} B(a,v)$$

Pf: Let $v \in V^p$.

$$F_v(u) = \sum_{j=1}^{n} v(t_j)\big(u(t_j) - u(t_{j-1})\big) \overset{\text{SBP}}{=} -\sum_{j=0}^{n}\big(v(t_{j+1}) - v(t_j)\big)\, u(t_j)$$

- well defined for $v \in V^p$ and $u \in S_{rc}$.

$b = t_{n+1}$
$v(b) = 0$

- map is linear in v and u.
- For a q-atom a, we have

$$|F_v(a)| \le \sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}\, \|a(t_j)\|_X$$

$$\le \Big(\sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}^p\Big)^{1/p} \underbrace{\Big(\sum_{j=1}^{n} \|a(t_j)\|_X^q\Big)^{1/q}}_{\le 1}$$

$$\le \|v\|_{V^p(X^*)}$$

by (vi) $\Longrightarrow$ F_v extends uniquely to a bdd lin op on $U^q(X)$

s.t. $|F_v(u)| \le \|v\|_{V^p(X^*)}\, \|u\|_{U^q(X)}$.

- $B : V^p \longrightarrow (U^q)^*$ with norm at most 1.
- Let us show that it is an isometry.

Fix $v \in V^p$, $\varepsilon > 0$, and $P = \{t_0 < t_1 < \cdots < t_n\}$ s.t.

$$(*) \qquad \|v\|_{V^p} \leq \left(\sum_{j=1}^{n} \| v(t_{j+1}) - v(t_j) \|_{X^*}^p \right)^{1/p} + \varepsilon$$

Choose $x_j \in X$ with norm 1 s.t.

$$(**) \qquad (v(t_{j+1}) - v(t_j))(x_j) \geq (1-\varepsilon) \| v(t_{j+1}) - v(t_j) \|_{X^*}.$$

and let $\phi_j = \mu \| v(t_{j+1}) - v(t_j) \|_{X^*}^{p-1} x_j$.

where $\mu = \|v\|_{V^p}^{1-p}$.

$$p' = \frac{p}{p-1}$$

Then, $$\sum_{j=1}^{n} \|\phi_j\|_X^{p'} \leq \|v\|_{V^p}^{-p} \sum_{j=1}^{n} \| v(t_{j+1}) - v(t_j) \|_{X^*}^p \leq 1.$$

$$\Rightarrow \quad a = \sum_{j=1}^{n} \phi_j \chi_{[t_j, t_{j+1})} \text{ is a } p'\text{-atom.}$$

and

$$\|v\|_{V^p} \le B(a,v) - C\varepsilon$$

$\Leftarrow$ Take ($*$) to the power p.

$$\|v\|_{V^p} \le \sum_{j=1}^{n} \mu \|v(t_{j+1}) - v(t_j)\|_{X^*}^{p-1} \underbrace{\|v(t_{j+1}) - v(t_j)\|_{X^*}}_{\text{use } (**)} + \varepsilon^p$$

i.e. B is an isometry.

· Surjectivity: Let $F \in (U^q(X))^*$.

Define an element $v(t) \in X^*$ by

$$v(t)(x) = F(x \chi_{[t,b)}), \quad x \in X.$$

Let a be a q-atom.

$$F(a) = \sum_{j=1}^{n} F(\phi_j \chi_{[t_j,b)}) - F(\phi_j \chi_{[t_{j+1},b)})$$

$$= -\sum_{j=1}^{n} (v(t_{j+1}) - v(t_j))(\phi_j)$$

$$= \sum_{j=1}^{n} v(t_j)(a(t_j) - a(t_{j-1})) = B(a,v)$$

We also have

$$\|v\|_{V^p} \leq \|F\|_{(U^q)^*}$$

$\Rightarrow F(u) = B(u,v)$ on $U^q(X)$.

$V^q_c = V^q \cap C$, $v(b)=0$

$U^p(X^*) \to (V^q_c(X))^*$

is a surjective isometry

Lemma 4.23

$\boxtimes$

Rmk: Cor 4.24

$$\|u\|_{U^p(X)} = \sup\{B(u,v) : v \in C_0^\infty(X),\ \|v\|_{V^q(X^*)} = 1\}$$

$$\|v\|_{V^p_{rc}(X)} = \sup\{B(u,v) : u \in C_0^\infty(X),\ \|u\|_{U^q(X^*)} = 1\}.$$

Back to $dY = f(Y)\,dX$. $\quad f$, Lipschitz.

(+) $$Y_t = Y_0 + \underbrace{\int_0^t f(Y)_s \, dX_s}_{\text{define this for } X \in U^2 \text{ (and } Y \in V^2)}$$

Bilinear form as integral:

Def 4.26
Lem 4.27

$v \in V^p((a,b))$, $u \in U^q((a,b))$ ++

integral is unchanged even if we replace u by $u - c$.

For $a \le s \le t \le b$, we define

$$\boxed{\int_s^t v\,du} = B_{(s,t)}(u - u(s), v) + (u(t) - u(t-))\,v(t)$$

In particular, the integral $\int_0^{\cdot} v\,du$ makes sense for $v \in V^2$, $u \in U^2$

(***)

$$\left.\begin{aligned} &\left\| \int_a^t v\,du \right\|_{U^q} \\ &\left\| \int_a^t v\,du \right\|_{V^p} \end{aligned}\right\} \le \|u\|_{U^q} \|v\|_{V^p}$$

Using (***), we can construct a unique soln $Y \in U^2$ to (+) for $X \in U^2$

$$\left\| \int_0^t f(Y)\,dX \right\|_{U^2(0,\tau)} \le \|X\|_{U^2(0,\tau)} \Big(|f(0)| + K \underbrace{\|Y\|_{V^2(0,\tau)}}_{\lesssim \|Y\|_{U^2(0,\tau)}} \Big)$$

$\Leftarrow$ Need $\|X\|_{U^2((0,\tau))} \ll 1$

- In general, $\|X\|_{U^2((0,\tau))} \not\to 0$ as $\tau \to 0$ since $U^2 \subset L^\infty$.

but in view of (++) on page (12), we can assume $X(0) = 0$.

$\| \leftarrow$ by right-conti.

$\lim_{t \downarrow 0} X(t)$

$\Rightarrow \|X\|_{U^2((0,\tau))} \to 0$ as $\tau \to 0+$

This follows from the definition of U^2:

Given $\varepsilon > 0$, write

$$u = \sum_{j=1}^\infty \lambda_j a_j \quad \text{s.t.} \quad \Big| \sum_{j=1}^\infty |\lambda_j| - \|u\|_{U^2} \Big| < \varepsilon.$$

Now, $\exists\, N = N(\varepsilon) \geq 1$ s.t.

$$\sum_{j=N+1}^\infty |\lambda_j| < \varepsilon.$$

By our convention, $a_j(0) = 0 \Rightarrow \sum_{j=1}^N \lambda_j a_j \equiv 0$ on $[0, t_*)$ for some $t_* > 0$

$$\Rightarrow \|u\|_{U^2([0,t_*))} < 2\varepsilon.$$

For $t \geq \tau$, we can repeat the analysis.

In particular, ++ on page 12 allows us to assume $X(\tau) = 0$

and thus $\|X\|_{V^2([\tau, \tau+\tau_1))} \ll 1$ by choosing $\tau_1 > 0$ suff. small.

$\Rightarrow$ Iterate this procedure as in $\underbrace{p=1}_{\text{Riemann-Stieljes}}$ and $\underbrace{p<2}_{\text{Young}}$ cases.