Lec 5 19/02/20 (Wed)

- Sewing map $\Lambda : C_3^{1+} \cap \delta C_2 \to C_2^{1+}$

  s.t. $\delta \Lambda = Id_{C_3}$

  $\|\Lambda \delta A\|_{C_2^\alpha(I)} \lesssim \|\delta A\|_{C_3^\alpha(I)}$ for any $\alpha > 1$.

provides a "correction" to a given 2-increment $A \in C_2$ to make it closed ($\delta \hat{A} = 0$) and hence exact ($\hat{A} = \delta B$)

exactness of the cochain complex

$$A \rightsquigarrow \delta A \rightsquigarrow A - \Lambda \delta A.$$

Then, $\delta(A - \Lambda\delta A) = \delta A - \overset{Id}{\overset{\shortparallel}{(\delta\Lambda)}}\delta A = 0.$

$\Rightarrow \exists!\ I \in C_1$ s.t. $\delta I = A - \underbrace{\Lambda \delta A}_{R}$

Pf of Sewing lemma cont'd:

Goal: Construct Λ s.t. $\delta\Lambda = Id_{C_3}$ and

$$|(\Lambda\delta A)(s,t)| \le c\,|t-s|^{\rho}, \quad \forall (s,t) \in \Delta_2$$

for some $\rho > 1$

Fix smooth $Q : \mathbb{R} \to \mathbb{R}_+$ with cpt supp in $\mathbb{R}_+$

$\int Q\,dx = 1$

and set $Q_\sigma(x) = \frac{1}{\sigma} Q\left(\frac{x}{\sigma}\right)$

Given $A \in C_2(\Delta_2; V)$, extend A to $\mathbb{R}\times\mathbb{R}$ by setting

$$A(s,t) = A(J_{0,1}(s), J_{0,1}(t))$$

$\Delta_2 = \{0 \le s \le t \le 1\}$

where $J_{a,b}(r) = \max(a, \min(r, b))$

$$A_\sigma(s,t) = -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\, A(s, r+r')$$

$\leftarrow$ smoothing only in the 2nd var.

$$\underset{\text{IBP}}{=} +\int_{\mathbb{R}} dr'\, Q_\sigma(r')\,\big(A(s, t+r') - A(s, s+r')\big)$$

$$\Longleftarrow \quad A_\sigma(s,s) = 0.$$

- Also, $A_\sigma \to A$ ptwise (b/c $A(s,s) = 0$)

- $\delta A_\sigma(s,u,t) = \int_{\mathbb{R}} dr'\, Q_\sigma(r')\big(\delta A(s,u,t+r') - \delta A(s,u,u+r')\big)$

  and $\delta A_\sigma \to \delta A$ ptwise

Now, define

$$(\mathcal{R} A_\sigma)(s,t) = \int_s^t \partial_2 A_\sigma(r,r)\, dr \overset{\text{F.T.C.}}{=} -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r') A(r, r+r'')$$

and set $\boxed{\Lambda \delta A_\sigma = A_\sigma - \mathcal{R} A_\sigma}$

from

Then, $(\Lambda \delta A_\sigma)(s,t) = -\int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\big(A(s,r+r') - A(r,r+r')\big)$

$$= \int_s^t dr \int_{\mathbb{R}} dr'\, Q'_\sigma(r')\, \delta A(s,r,r+r') \qquad \checkmark \quad -A(s,r)\underbrace{\int_{\mathbb{R}} dr'\, Q'_\sigma(r')}_{=0}$$

Check $(\delta \Lambda \delta A_\sigma))(s,u,t) = \delta A_\sigma (s,u,t).$

$\Leftarrow$ follows from $\Lambda \delta A_\sigma = A_\sigma - RA_\sigma$

and $\underline{\delta R A_\sigma = 0}$

This is clear since RA_σ is just an Integral with bdry pts s and t.

Now, we need to check the regularity of $\Lambda \delta A_\sigma$

Given small $\sigma > 0$, consider $|t-s| > \sigma$.

Recall

$$(\Lambda \delta A_\sigma)(s,t) = \sum_{j=0}^{2^n-1} (\Lambda \delta A_\sigma)(t_j^n, t_{j+1}^n) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \delta A_\sigma (t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1})$$

· choose n s.t. $|t-s| \cdot 2^{-n} \le \sigma < |t-s| 2^{-n+1}$

$$|(\Lambda \delta A_\sigma)(t_j^n, t_{j+1}^n)| \lesssim \underline{|t-s| 2^{-n}} \, \sigma^{\alpha-1} \lesssim |t-s|^\alpha 2^{-n\alpha}.$$

$\underline{|t_j^n - t_{j+1}^n| = 2^{-n} |t-s|}$ $\qquad$ $\delta A \in C_3^\alpha$ and $|r'| \lesssim \sigma$

and for $k < n$, we have

$$|\delta A_\sigma(t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1})|$$

$$\lesssim (|t-s|2^{-k} + \sigma)^\alpha \lesssim 2^{-\alpha k}|t-s|^\alpha(1+2^{k-n})^\alpha$$

$\sim 2^{-n}|t-s|$

$$\lesssim 2^{-\alpha k}|t-s|^\alpha$$

$\delta A \in C_3^\alpha$

for δA_σ, there is a "spread" of size $\sim \sigma$.

Putting together,

$$|(\Lambda \delta A_\sigma)(s,t)| \lesssim |t-s|^\alpha \left(2^{n(1-\alpha)} + \sum_{k=0}^{n-1} 2^{k(1-\alpha)}\right)$$

$$\lesssim |t-s|^\alpha. \quad (\text{for } \alpha > 1) \quad \text{unif in } n$$

i.e. in σ small.

- By showing "$RA_\sigma \to RA$" ptwise,

we obtain

$$|(\Lambda \delta A)(s,t)| = \lim_{\sigma \to 0} |(\Lambda \delta A_\sigma)(s,t)| \lesssim |t-s|^\alpha$$

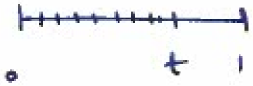
$\boxed{/}$

Alternative proof of Sewing Lemma

via discrete approximations.

· $\{u_j^n\}$, dyadic partition of $[0,1]$ of size 2^{-n}.

Set
$$S_n A(t) = \sum_{j=0}^{2^n-1} \mathbb{1}_{t \ge u_j^n} A(u_j^n, u_{j+1}^n \wedge t)$$



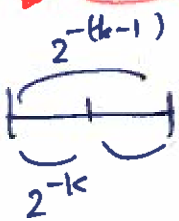
$$\Rightarrow |\delta S_n A(s,t) - A(s,t)|$$
$$\le \sum_{k=\ell+1}^{n} |\delta S_k A(s,t) - \underbrace{\delta S_{k-1} A(s,t)|}_{} + \|\delta A\|_{C_3^\alpha} |t-s|^{-\alpha}$$

where ℓ is the greatest integer s.t. $2^{-\ell} \ge |t-s|$.

last term
$$\delta S_\ell A(s,t)$$
$$= A(u_j^\ell, t) - A(u_j^\ell, s)$$
$$= A(s,t) + \underbrace{\delta A(u_j^\ell, s, t)}_{\le \|\delta A\|_{C_3^\alpha} 2^{-\ell\alpha}}$$

· $|\delta S_k A(s,t) - \delta S_{k-1} A(s,t)|$
$$\lesssim 2^{k-\ell}\, 2^{-\alpha k} \|\delta A\|_{C_3^\alpha}$$

# of intervals of length $2^{-(k-1)}$ in $[s,t] \sim 2^{-\ell}$



$$\Rightarrow \left| \delta S_n A(s,t) - A(s,t) \right|$$

$$\lesssim \|\delta A\|_{C_3^\alpha} 2^{-\ell} \sum_{k=\ell+1}^{\infty} 2^{k(1-\alpha)} + \|\delta A\|_{C_3^\alpha} |t-s|^\alpha$$

$$\lesssim \|\delta A\|_{C_3^\alpha} |t-s|^\alpha$$

Now, set $\Lambda \delta A = A - \delta S_n A$ where $n = n(s,t) > \ell = \ell(s,t)$

$\boxtimes$

<u>Cor:</u> Let $\delta A \in C_3^{1+} \cap \delta C_2$ and $\delta I = A - \Lambda \delta A$.

Then,

$$\underset{\substack{\uparrow \\ \text{partition of } [s,t]}}{S_{P(s,t)}}(A) = \sum_j A(t_j, t_{j+1}) \xrightarrow[\text{as } |P| \to 0]{} I(t) - I(s).$$

<u>Pf:</u>

$$\sum_j A(t_j, t_{j+1}) = \sum_j \delta I(t_j, t_{j+1}) + \underline{\sum_j \boxed{\Lambda \delta A(t_j, t_{j+1})}} \quad \lesssim |t_{j+1} - t_j|^\alpha$$

$$= \delta I(s,t) + |P|^{\alpha-1} O(|t-s|)$$

$$\to \delta I(s,t) \quad \text{as } |P| \to 0$$

$\boxtimes$

Young's integral:

Thm: $0 < \alpha, \beta < 1$ s.t. $\alpha + \beta > 1$.

Then, the integral map $I(f,g) = \int_0^{\cdot} f(s)\, \partial_s g(s)\, ds$
has a conti extension: $C^\alpha \times C^\beta \to C^\beta$ s.t.

$$|\delta I(f,g)(s,t) - f(s)\, \delta g(s,t)| \lesssim_{\alpha+\beta} |t-s|^{\alpha+\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta}$$

and

$$\|I(f,g)\|_{C^\beta([s,t])} \leq \left(\|f\|_{L^\infty([s,t])} + \|f\|_{C^\alpha([s,t])} \right) \|g\|_{C^\beta([s,t])}$$

for any $0 \leq s \leq t \leq 1$.

Moreover, $$I(f,g)(t) = \lim_{|P(0,t)| \to 0} \sum_{j=0}^{n-1} f(t_j)\left(g(t_{j+1}) - g(t_j)\right)$$

"Pf": $A(s,t) = f(s)\, \delta g(s) = f(s)\left(g(t) - g(s)\right)$

$\Rightarrow \delta A(s,u,t) = \delta f(s,u)\, \delta g(u,t)$

$\Rightarrow \|\delta A\|_{C_3^{\alpha+\beta}} \leq \|f\|_{C^\alpha} \|g\|_{C^\beta}$ $\qquad \|\delta f\|_{C_2^\alpha} = \|f\|_{C^\alpha}$

Then, with $R = \Lambda \delta A$, set

$$\delta I(f,g)(s,t) = A(s,t) - R(s,t)$$

$\boxtimes$