Lec 4 14/02/20 (Fri)

Young's differential equation : $dY_t = V(Y)dX_t$

Thm 7: $X \in V_c^p([0,T])$, $1 < p < 2$

V, Lipschitz conti s.t. $|V(Y^1) - V(Y^2)| \le K|Y^1 - Y^2|$

Then, given Y_0, $\exists!$ soln Y to

$$Y_t = Y_0 + \int_0^t V(Y_s)\,dX_s \quad \text{on} \quad [0,T]$$

Moreover, $Y \in V_c^p([0,T])$

Pf: Let $\Gamma(Y)_t = Y_0 + \int_0^t V(Y_s)\,dX_s.$

Then, $\Gamma(Y)_0 = Y_0$ $\quad (V(Y_s) - V(0)) + V(0)$

and

$$\|\Gamma(Y)\|_{V^p([0,T])} \overset{\text{Prop6}}{\le} |Y_0| + C\|X\|_{V^p([0,T])}\Big(K(\|Y\|_{V^p([0,T])} + |Y_0|) + |V(0)|\Big)$$

So, for $Y \in \overline{B_R} \subset V_c^p([0,\tau])$

$$\|\Gamma(Y)\|_{V_c^p([0,\tau])} \overset{\text{def}}{=} |Y_0| + \|\Gamma(Y)\|_{V^p([0,\tau])}$$
$$\leq R$$

by choosing $R = 2\|Y\|_{V_c^p([0,\tau])}$

and $\tau > 0$ suff. small s.t. $C\|X\|_{V^p([0,\tau])} \ll_{R,V} 1$.

Similarly, for $Y^1, Y^2 \in \overline{B_R} \subset V_c^p([0,\tau])$,

$$\|\Gamma(Y^1) - \Gamma(Y^2)\|_{V_c^p([0,\tau])} \leq C\|X\|_{V^p([0,\tau])}\Big(K\|Y^1 - Y^2\|_{V^p([0,\tau])} + K|Y_0^1 - Y_0^2|\Big).$$
$$\leq \frac{1}{2}\|Y^1 - Y^2\|_{V_c^p([0,\tau])}$$

by choosing $\tau > 0$ small s.t. $C\|X\|_{V^p([0,\tau])} \ll 1$.

$\Rightarrow \exists!$ soln Y on $[0,\tau]$.

· For $t > \tau$, write

view as new initial data

$$Y_t = Y_0 + \int_0^t V(Y_s)\,dX_s = Y_\tau + \int_\tau^t V(Y_s)\,dX_s \Rightarrow \text{repeat}.$$

$\square$

- As before, $Y_0 \to Y$ is C^1.
- Also, Y depends continuously on X.

  i.e. if $X^n \to X$ in $V^p_c([0,T])$.

  Then, $$Y^n_t = Y_0 + \int_0^t V(Y^n_s)\, dX^n_s$$

  converges to
  $$Y_t = Y_0 + \int_0^t V(Y_s)\, dX_s$$
  in $V^p_c([0,T])$.

<u>Pf:</u> Let $0 \le s \le t \le T$. Then,

$$\|Y - Y^n\|_{V^p([s,t])} = \left\| \int_0^\cdot V(Y)\,dX - \int_0^\cdot V(Y^n)\,dX^n \right\|_{V^p([s,t])}$$

$$\le \left\| \int_0^\cdot (V(Y) - V(Y^n))\,dX \right\|_{V^p([s,t])} + \left\| \int_0^\cdot V(Y^n)\, d(X - X^n) \right\|_{V^p([s,t])}$$

$$\overset{\text{Prop 6}}{\le} \underbrace{C\|X\|_{V^p([s,t])}}_{\text{make this small}} K \|Y - Y^n\|_{V^p([s,t])}$$

$$+ C\|X - X^n\|_{V^p([s,t])} \left(K\|Y^n\|_{V^p([s,t])} + |V(0)| \right)$$

$$\text{①} \quad \Rightarrow \quad \|Y - Y^n\|_{V^p([s,t])} \le \frac{C\left(K\|Y^n\|_{V^p([s,t])} + |V(0)|\right)}{1 - cK\|X\|_{V^p([s,t])}} \|X - X^n\|_{V^p([s,t])}$$

Similarly, we obtain

$$\text{②} \quad \|Y^n\|_{V^p([s,t])} \le \frac{C|V(0)|}{1 - cK\|X^n\|_{V^p([s,t])}}$$

$$\le \frac{C|V(0)|}{1/2 - cK\|X\|_{V^p([s,t])}} \quad \text{for any } N \gg 1.$$

From ① & ②,

$$\|Y - Y^n\|_{V^p([s,t])} \le C\left(\|X\|_{V^p([s,t])}, V\right) \|X - X^n\|_{V^p([s,t])}$$

$$\to 0.$$

$\square$

· Another view on Young's integral

(based on Gubinelli's lec note 1).

$$(f, g) \longmapsto I(f,g) = \int_0^{\cdot} f(s)\, \partial_s g(s)\, ds$$

$$\begin{array}{ccc} \cap & & \\ C^{\alpha} \times C^{\beta} & \longrightarrow & C^{\gamma} \end{array}$$

- <u>differential calculus viewpt</u>: $I(f,g)$ is the unique soln to

$$\partial_t I(f,g)(t) = f(t)\, \partial_t g(t), \quad I(0) = 0.$$

- <u>finite increment</u>:

(*) $$I(t) - I(s) = f(s)(g(t) - g(s)) + \underbrace{o_{\text{unif}}}_{\text{unif in } s,t}(|t-s|), \quad I(0) = 0$$ small o.

for any $0 \le s \le t \le 1$

$\Leftarrow$ This property is clearly satisfied if g is at least C^1.

$$I(t) - I(s) - f(s)(g(t) - g(s)) = \int_s^t \underbrace{(f(u) - f(s))}_{\underset{\text{unif}}{\to} 0} \partial_u g(u)\, du.$$

since f unif conti on $[0,1]$.

- $\circledast$ gives a characterization of the integral

i.e. if J satisfies $\circledast$, then with $D := I - J$, we have

$$D(t) - D(s) = o_u(|t-s|)$$

$$\Rightarrow D(t) = D(s) \text{ for any } 0 \le s \le t \le 1$$

but $D(0) = 0 \Rightarrow D(t) \equiv 0$.

- I is the only function whose increment match the "germ" $f(s)(g(t) - g(s))$ modulo a negligible error

- For $n \ge 1$, $C_n(V) = C(\Delta_n ; V)$, $\Delta_n = \Delta_n(0,1)$ $\quad$ $\mathbb{R}$

$$\Delta_n(s,t) = \{(s_1, \cdots, s_n) : s \le s_1 \le \cdots \le s_n \le t\}.$$

  - n-cochain is an element in $C_n(V)$
  - coboundary operator $\delta : C_n(V) \longrightarrow C_{n+1}(V)$ given by

$$\delta f(s_1, \cdots, s_{n+1}) = \sum_{k=1}^{n+1} (-1)^{n+1-k+1} f(s_1, \cdots, \cancel{s_k}, \cdots, s_{n+1})$$

ex: $\delta f(s,t) = f(t) - f(s)$, $\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u)$

FACT: $\delta \circ \delta = 0$.

Cochain complex: $\mathbb{R} \to C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \xrightarrow{\delta}$

(cohomology $H^m = \dfrac{\mathrm{Ker}\,\delta^m}{\mathrm{Im}\,\delta^{m-1}}$

FACT: This complex is exact.

If $\delta f = 0$, then $f = \delta g$ for some g.

ex: $f \in C_2$ and $\delta f = 0$.

i.e. $\delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u) \equiv 0$

Set $g(t) = f(0,t)$

Then, $\delta g(s,t) = g(t) - g(s) = f(0,t) - f(0,s) = f(s,t)$

Let $A(s,t) = f(s)\,\delta g(s,t) = f(s)\,(g(t) - g(s))$

Then, we have, from ⊛,

$$A = \delta I + R, \quad \text{where } R(s,t) = o_u(|t-s|)$$

$$(\Rightarrow \delta A = \delta R).$$

sewing map allows us to recover R from $\delta A \in C_3$.

Topology on C_m: We say $f \in C_m^\alpha$ if

$$\|f\|_{C_m^\alpha([s,t])} = \sup_{\Delta_m(s,t)} \frac{|f(s_1, \cdots, s_m)|}{|s_m - s_1|^\alpha} < \infty.$$

Set $C_m^{\alpha+} = \bigcup_{\beta > \alpha} C_m^\beta$

<u>Rmk</u>: $\delta C_1 \cap C_2^{1+} = \{0\}$. $\left(\Leftarrow \quad C^{1+}_{\text{Hölder}} = \{\text{const}\} \right.$.

<u>Thm 8</u>: <u>Sewing Lemma</u>: $\exists !$ map $\Lambda : C_3^{1+} \cap \delta C_2 \to C_2^{1+}$

s.t. $\delta \Lambda = Id_{C_3}$

and

$$\|\Lambda \delta A\|_{C_2^\alpha(I)} \leq \frac{2^\alpha}{1-2^\alpha} \|\delta A\|_{C_3^\alpha(I)}$$

for any $\alpha > 1$ and any closed interval $I \subset \mathbb{R}$.

Dyadic partition of $[s,t]$: Given $n \geq 0$,

$$t_j^n = s + \frac{t-s}{2^n} j, \quad j = 0, \cdots, 2^n$$

Then, we have

$$(**) \quad A(s,t) = \sum_{j=0}^{2^n-1} A(t_j^n, t_{j+1}^n) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k-1} \delta A(t_{2j}^{k+1}, t_{2j+1}^{k+1}, t_{2j+2}^{k+1}) \quad (= \mathrm{I} + \mathrm{II})$$

for any $n \geq 0$.

- $n = 0$: ✓

$s=0,\ t=1$

- $n = 1$: $\mathrm{I} = A(0,\tfrac{1}{2}) + A(\tfrac{1}{2},1)$

$$\left.\begin{aligned} \mathrm{I} &= A(0,\tfrac{1}{2}) + A(\tfrac{1}{2},1) \\ \mathrm{II} &= A(0,\tfrac{1}{2},1) = A(0,1) - A(0,\tfrac{1}{2}) - A(\tfrac{1}{2},1) \end{aligned}\right\} = A(0,1).$$

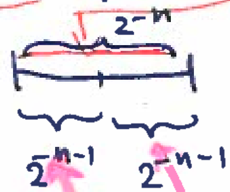
general case by induction.

Assume (**) for n.

$$\mathrm{I}_n = \underbrace{\sum_{j=0}^{2^{n+1}-1} A(t_j^{n+1}, t_j^{n+1})}_{= \mathrm{I}_{n+1}} + \boxed{\sum_{j=0}^{2^n-1} A(t_j^n, t_{j+1}^n) - \sum_{j=0}^{2^{n+1}-1} A(t_j^{n+1}, t_j^{n+1})}$$

$$\| $$

$$A(t_{2j}^{n+1}, t_{2j+2}^{n+1})$$



$$\sum_{j=0}^{2^n-1} \delta A(t_{2j}^{n+1}, t_{2j+1}^{n+1}, t_{2j+2}^{n+1}).$$

$\Rightarrow$ obtain (**) for $n+1$.

Pf of Sewing lemma:

Suppose that $\exists\, \Lambda$ s.t. $\delta\Lambda = \mathrm{Id}_{C_3}$ and

$$|(\Lambda\delta A)(s,t)| \le c\,|t-s|^{\rho}, \quad \forall (s,t) \in \Delta_2$$

for some $\rho > 1$ (this assumption can be weakened)

Use (**) for $\Lambda\delta A$. and $\delta(\Lambda\delta A) = (\delta\Lambda)\delta A = \delta A$.

$$\Rightarrow |(\Lambda\delta A)(s,t)| \le C\, 2^{n} 2^{-m\rho} + \|\delta A\|_{C_3^{\alpha}(I)} \sum_{k=0}^{n-1} 2^{k} 2^{-k\alpha} |t-s|^{\alpha}$$

$$\Big(|t^n_{j+1} - t^n_j| = 2^{-n} \underbrace{|t-s|}_{\le 1} \Big)$$

$$\le C\, 2^{m(1-\rho)} + \|\delta A\|_{C_3^{\alpha}(I)} \frac{1}{1-2^{1-\alpha}} |t-s|^{\alpha}$$

Take $m \to \infty$. $(\rho > 1)$

$$\Rightarrow |(\Lambda\delta A)(s,t)| \le \frac{2^{\alpha}}{2^{\alpha}-1} \|\delta A\|_{C_3^{\alpha}(I)} |t-s|^{\alpha}.$$

$\Rightarrow$ claimed estimate.