Lec 3 31/01/20 (Fri)

2 Young's integral $\frac{1}{p}+\frac{1}{q}>1$

- A map $\omega : \{0 \le s \le t \le T\} \to \mathbb{R}_+$ is called superadditive if

$$\omega(s,t)+\omega(t,u) \le \omega(s,u), \quad \forall\ 0 \le s \le t \le u.$$

- We say ω is a control if it is superadditive, conti, $\omega(t,t)=0$.
- We say a path $X : [0,T] \to \mathbb{R}$ is controlled by a control ω if $\exists\ c>0$ s.t. $|X_t - X_s| \le c\,\omega(s,t)$, $\forall\ 0 \le s \le t \le T$.

Prop 5.8 in [FV] (i) $X \in V_c^p([0,T])$, Then, $\omega(s,t) = \|X\|^p_{V^p([s,t])}$ is a control s.t.

$$|X_t - X_s| \le \omega(s,t)^{1/p}. \quad \forall\ 0 \le s \le t \le T$$

Prop 5.10 in [FV] (ii) $X \in V_c^p([0,T])$, $\omega : \{0 \le s \le t \le T\} \to \mathbb{R}_+$, control s.t.

$$|X_t - X_s| \le c\,\omega(s,t)^{1/p}, \quad \forall\ 0 \le s \le t \le T.$$

Then,

$$\|X\|_{V^p([s,t])} \le c\,\omega(s,t)^{1/p}.$$

Back to V^p_c, $V^p_\infty = \overline{C^\infty}^{\|\cdot\|_{V^p}}$

- $p > 1$

Thm 5.31 in [FV]

$$X \in V^p_\infty \text{ iff } \lim_{\delta\to 0} \sup_{|P|\le\delta} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}|^p = 0$$

$$\left(\text{Also, } \lim_{\delta\to 0} \sup_{|P|\le\delta} \sum_{k=0}^{n-1} \|X\|^p_{V^p([t_k, t_{k+1}])} = 0 \right.$$

Pf: $\Rightarrow$ Fix $\varepsilon > 0$ and smooth path Y

s.t. $\|X - Y\|^p_{V^p} \le \varepsilon/2^p$.

Then, $\sum_P |X_{t_{k+1}} - X_{t_k}|^p \le 2^{p-1} \sum |Y_{t_{k+1}} - Y_{t_k}|^p + 2^{p-1} \|X - Y\|^p_{V^p}$

Since Y is smooth, $\exists\ \delta = \delta(n) > 0$ s.t. $\forall\ |P| \le \delta$, we have

$$\begin{aligned}\sum_P |Y_{t_{k+1}} - Y_{t_k}|^p &\le \|Y'\|_{L^\infty} \sum_P (t_{k+1} - t_k)^p \\ &\le \delta^{p-1} T \|Y'\|_{L^\infty} < \varepsilon/2^p \quad \text{for } p > 1\end{aligned}$$

$\Rightarrow$ Putting together, we obtain $\sum_P |X_{t_{k+1}} - X_{t_k}|^p < \varepsilon$, $\forall\ |P| \le \delta$

$\boxtimes$

Cor: $1 \le p < q$

$$V_c^p([0,T]) \subset V_\infty^q([0,T]) \subset V_c^q([0,T])$$

- Young's integral.

Thm 4: $X, Y \in V_c^1([0,T])$

Let $p, q \ge 1$ s.t. $\theta = \frac{1}{p} + \frac{1}{q} > 1$. Then, we have

Prop 6.4 in [FV]

$$\Bigg| \underbrace{\int_s^t Y_u \, dX_u - \underbrace{Y_s X_{s,t}}_{= Y(s)(X(t)-X(s))}}_{= \int_s^t (Y_u - Y_s) \, dX_u} \Bigg| \le \frac{1}{1-2^{1-\theta}} \|X\|_{V^p([s,t])} \|Y\|_{V^q([s,t])}$$

$\Leftarrow$ Young–Loève estimate.

<u>Pf</u>: For $0 \le s \le t \le T$, define

$$\Gamma_{s,t} = \int_s^t Y_u \, dX_u - Y_s X_{s,t}.$$

Then, for $s \le t \le u$, we have

$$\Gamma_{s,u} - \Gamma_{s,t} - \Gamma_{t,u} = -Y_s (X_u - \cancel{X_s}) + Y_s (X_t - \cancel{X_s}) + Y_t (X_u - X_t)$$

$$= (Y_s - Y_t)(X_t - X_u)$$

$$\Rightarrow \quad |\Gamma_{s,u}| \le |\Gamma_{s,t}| + |\Gamma_{t,u}| + \|X\|_{V^p([t,u])} \|Y\|_{V^q([s,t])}$$

- Now, set $\omega(s,t) = \|X\|^{1/\theta}_{V^p([s,t])} \|Y\|^{1/\theta}_{V^q([s,t])}$

<u>Claim</u>: ω is a control.

- conti, $\omega(t,t) = 0$, ✓

- For $s \le t \le u$,

$$\omega(s,t) + \omega(t,u) = \|X\|^{1/\theta}_{V^p([s,t])} \|Y\|^{1/\theta}_{V^q([s,t])} + \|X\|^{1/\theta}_{V^p([t,u])} \|Y\|^{1/\theta}_{V^q([t,u])}$$

$$1 = \frac{1}{p\theta} + \frac{1}{q\theta} \longrightarrow \overset{\text{Hölder}}{\le} \left(\|X\|^p_{V^p([s,t])} + \|X\|^p_{V^p([t,u])} \right)^{\frac{1}{p\theta}} \left(\|Y\|^q_{V^q([s,t])} + \|Y\|^q_{V^q([t,u])} \right)^{1/q\theta}$$

$$\le \|X\|^{1/\theta}_{V^p([s,u])} \|Y\|^{1/\theta}_{V^q([s,u])} = \omega(s,u)$$

(*) • We have $|\Gamma_{s,u}| \leq |\Gamma_{s,t}| + |\Gamma_{t,u}| + \omega(s,u)^\theta$

Given $\varepsilon > 0$, consider the control:

$$\omega_\varepsilon(s,t) = \omega(s,t) + \varepsilon\left(\|X\|_{V^1([s,t])} + \|Y\|_{V^1([s,t])}\right)$$

Also, define
$$\Psi(r) = \sup_{\substack{s,u \\ \omega_\varepsilon(s,u) \leq r}} |\Gamma_{s,u}|$$

• By conti, $\exists\ t \in [s,u]$ s.t. $\omega_\varepsilon(s,t) = \omega_\varepsilon(t,u) \leq \frac{1}{2} r$.

$$\overset{(*)}{\Longrightarrow} \quad |\Gamma_{s,u}| \leq 2\Psi\left(\frac{r}{2}\right) + r^\theta.$$

$$\Longrightarrow \quad \Psi(r) \leq 2\Psi\left(\frac{r}{2}\right) + r^\theta. \quad \left(\leq 2^2\Psi\left(\frac{r}{2^2}\right) + 2\left(\frac{r}{2}\right)^\theta + r^\theta\right)$$

iterate n times $\Longrightarrow$

(**) $$\Psi(r) \leq \underbrace{2^n \Psi\left(\frac{r}{2^n}\right)}_{\downarrow \text{claim (see next page)}\ 0} + \underbrace{\sum_{k=0}^{n-1} 2^{k(1-\theta)} r^\theta}_{= \frac{1}{1-2^{1-\theta}} \cdot r^\theta}.$$

$$
\begin{aligned}
\cdot \quad |\Gamma_{s,t}| &= \left| \int_s^t (Y_u - Y_s)\, dX_u \right| \\
&\le \|X\|_{V^1([s,t])} \, \|Y_\cdot - Y_s\|_{L^\infty([s,t])} \\
&\le \left(\|X\|_{V^1([s,t])} + \|Y\|_{V^1([s,t])} \right)^2 \\
&\le \frac{1}{\varepsilon^2} \, \omega_\varepsilon^2(s,t)
\end{aligned}
$$

$$
\Longrightarrow \lim_{n\to\infty} 2^n \Psi\left(\frac{r}{2^n}\right) \le \lim_{n\to\infty} 2^n \left(\frac{1}{\varepsilon^2} \frac{r^2}{2^{2n}} \right) = 0
$$

Hence, from (**), we obtain

$$
\Psi(r) \le \frac{1}{1-2^{1-\theta}} r^\theta
$$

$$
\Longrightarrow |\Gamma_{s,u}| \le \frac{1}{1-2^{1-\theta}} \omega_\varepsilon(s,u)^\theta .
$$

(For given s, u, X, Y, $\omega_\varepsilon(s,u) = r$

Finally, send $\varepsilon \to 0$

$\boxtimes$

<u>Prop 5</u>: $X \in V_c^p([0,T])$, $Y \in V_c^q([0,T])$, $\theta = \frac{1}{p} + \frac{1}{q} > 1$

Suppose $\exists \{X^n\}, \{Y^n\} \subset V_c^1([0,T])$

s.t. $X^n \to X$ in V_c^p

$Y^n \to Y$ in V_c^q.

Then, $\forall\ 0 \le s \le t \le T$,

Thm 6.8 in [FV]

$\int_s^t Y_u^n \, dX_u^n$ converges to a <u>limit</u>

Young's integral $\int_s^t Y_u \, dX_u$.

- The limit is indep of the choice of approx seq $\{X^n\}, \{Y^n\}$.
- For $0 \le s \le t \le T$,

$$\left| \int_s^t Y_u \, dX_u - Y_s (X_t - X_s) \right| \le \frac{1}{1 - 2^{1-\theta}} \|X\|_{V^p([s,t])} \|Y\|_{V^q([s,t])}$$

Rmk: ① $\overline{V_c^1}^{\|\cdot\|_{V^p}} = V_\infty^p$

but we have

$$V_c^p \subset V_\infty^{p+\varepsilon}$$

i.e. Given $X \in V_c^p$, $Y \in V_c^q$,

use Prop 5. with $\frac{1}{p+\varepsilon} + \frac{1}{q+\varepsilon} = \theta_\varepsilon > 1$ $\quad \theta_\varepsilon \nearrow \theta$

since $X \in V_\infty^{p+\varepsilon}$. $Y \in V_\infty^{q+\varepsilon}$.

· Recalling $\lim_{p' \searrow p} \|X\|_{V^{p'}} = \|X\|_{V^p}$,

we conclude that the Young - Loève estimate holds for $X \in V_c^p$, $Y \in V_c^q$.

② Riemann sum $\sum_{k=0}^{n-1} Y_{t_k}(X_{t_{k+1}} - X_{t_k})$ converges to $\int_s^t Y_u \, dX_u$

use $X \in V_\infty^{p+\varepsilon}$, $Y \in V_\infty^{q+\varepsilon}$ and the result from page ②

to the Young - Loève estimate on $[t_k, t_{k+1}] \subset [s,t]$

③ **Prop 6:** $X \in V^p_c$, $Y \in V^q_c$, $\frac{1}{p} + \frac{1}{q} > 1$.

$$\left\| \int_0^\cdot Y_u \, dX_u \right\|_{V^p([s,t])} \le C \|X\|_{V^p([s,t])} \left(\|Y\|_{V^q([s,t])} + \|Y\|_{L^\infty([s,t])} \right)$$

$$\le 2 \|X\|_{V^p([s,t])} \left(\|Y\|_{V^q([s,t])} + |Y_0| \right).$$

Prop 6.11 in [FV]

- $t \longmapsto \int_0^t Y_u \, dX_u$ is conti (with values in V^p_c).
- $(X, Y) \longmapsto \int_0^\cdot Y \, dX$ is bilinear & conti.

$$\begin{array}{ccc} (X, Y) & \longmapsto & \int_0^\cdot Y \, dX \\ \cap & & \cap \\ V^p_c \times V^q_c & \longrightarrow & V^p_c \end{array}$$

Lipschitz conti & Fréchet smooth