Lec 2 22/01/20 (Wed)

$$V^p_c([0,T]) = V^p([0,T]) \cap C([0,T])$$

norm : $|X_0| + \|X\|_{V^p}$ (or $\|X\|_{L^\infty} + \|X\|_{V^p}$)

(iv) . For $p_1 \le p_2$, $V^{p_1} \subset V^{p_2}$

. For $\alpha_1 \ge \alpha_2$ $C^{\alpha_1}_{\text{Hölder}} \subset C^{\alpha_2}_{\text{Hölder}}$

(v) Interpolation : $1 \le p_1 < p_2 < \infty$.

$$\|X\|_{V^{p_2}} \le \|X\|_{V^{p_1}}^{p_1/p_2} \left(\sup_{t,s} |X_t - X_s| \right)^{1 - p_1/p_2}$$

($\Leftarrow$ Hölder) Prop 5.5 in [FV]

(vi) $X^n \to X$ ptwise on $[0,T]$

$$\|X\|_{V^p} \le \liminf_{n\to\infty} \|X^n\|_{V^p}$$

(vii)

$$\lim_{p' \searrow p} \|X\|_{V^{p'}} = \|X\|_{V^p}$$ Lem 5.13 in [FV].

(viii) <u>Compactness</u> : $\{X^n\} \subset C([0,T])$.

$X^n \to X$ uniformly.

- Suppose $\sup_n \|X^n\|_{V^p} < \infty$. Then, $X^n \to X$ in $V^{p'}$, $\forall p' > p$.

((vi) $\Rightarrow$ $X \in V^p$. Then apply interpolation (v))

- $V^p_\infty = \overline{C^\infty}^{\|\cdot\|_{V^p}}$ $(= C^{0,p\text{-var}})$

$C^\alpha_\infty = \overline{C^\infty}^{\|\cdot\|_{C^\alpha}}$ $(= C^{0,\alpha\text{-Hölder}})$

(VMO / CMO $= \overline{C^\infty_c}^{\|\cdot\|_{BMO}}$

- closed subspaces, Banach, separable. (Prop 5.36 in [FV])

$\to$ Polish

V^p_c, $C^\alpha_{\text{Hölder}}$ not separable

- <u>Reparametrization</u> : (Prop 1.21, Prop 5.14 in [FV])

$X \in V^p_c([0,T])$ iff $\exists$ conti. increasing function h from $[0,T]$ to $[0,1]$ and $C^{1/p}_{\text{Hölder}}$-function Y s.t.

$$X = Y \circ h.$$

1 Riemann - Stieljes integral & ODE. $(p=1)$

Prop 1: $X \in V^1_c([0,T])$, Y on $[0,T]$, piecewise continuous.

Then, the Riemann - Stieljes integral $\int_0^T Y\,dX$ exists,

linear in X and Y with the bound:

$$(*) \qquad \left| \int_0^T Y\,dX \right| \le \|Y\|_{L^\infty([0,T])} \|X\|_{V^1([0,T])} \qquad \text{Prop 2.2 in [FV]}$$

Idea: ① dX - integrable functions if Riemann sum converges

lin space, $(*)$ holds.

② Y^n, dX - integrable, $Y^n \to Y$ in $L^\infty([0,T])$

Use $(*)$ to show $\int_0^T Y\,dX = \lim_{n\to\infty} \int_0^T Y^n\,dX$

etc.

Rmk: $Z(t) = \int_0^t Y\,dX \Rightarrow |Z(t_2) - Z(t_1)| = \left|\int_{t_1}^{t_2} Y\,dX\right| \le \|Y\|_{L^\infty} \|X\|_{V^1([t_1,t_2])}$

$$\Rightarrow \|Z\|_{V^1([0,T])} \le \|Y\|_{L^\infty} \|X\|_{V^1([0,T])}$$

Lemma 2: (Gronwall's inequality)

$X \in V^1_c([0,T])$. $\Phi : [0,T] \to \mathbb{R}_+$, measurable, bdd.

Suppose

$$(\ast\ast) \quad \Phi(t) \le A + \int_0^t \Phi(s)\,|dX(s)|, \quad \forall\, 0 \le t \le T$$

for some $A, B \ge 0$. Then,

$$\Phi(t) \le A \exp\left(B\, \|X\|_{V^1([0,T])} \right), \quad \forall\, 0 \le t \le T.$$

Here, $\int_0^t \cdots |dX(s)|$ = integration w.r.t. a V^1-path $\ell(t) = \|X\|_{V^1([0,t])}$.

Pf: By iterating $(\ast\ast)$ n times, we get

$$\Phi(t) \le A + \sum_{k=1}^{n} A B^k \underbrace{\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} |dX(t_k)| \cdots |dX(t_1)|}_{= \frac{\|X\|^k_{V^1([0,t])}}{k!}}$$

$$+ \underbrace{\text{remainder}}_{\sim \frac{\|\Phi\|_{L^\infty}}{(n+1)!}}$$

$\square$

Thm 3: $X \in V^1_c([0,T])$.

V, Lipschitz

Then, $\forall\ Y_0$, $\exists!$ soln to $\Longleftrightarrow\ dY = V(Y)\,dX$

$$Y(t) = Y_0 + \int_0^t V(Y(s))\,dX(s), \qquad 0 \le t \le T.$$

Moreover, $Y \in V^1_c([0,T])$.

Pf:
$$\Gamma(Y)(t) = Y_0 + \int_0^t V(Y(s))\,dX(s)$$

① work on $[0,\tau]$, $0 < \tau \ll 1$.

$$\|\Gamma(Y)\|_{V^1([0,\tau])} \le |Y_0| + \underbrace{\|V(Y) - V(0)\|_{L^\infty([0,\tau])}}_{\le K\|Y\|_{L^\infty([0,\tau])}} \cdot \|X\|_{V^1([0,\tau])}$$
$$+ |V(0)|\,\|X\|_{V^1([0,\tau])}$$
$$\le R \quad \text{by choosing } R = 2|Y_0| \text{ and } \tau \text{ small.}$$

$\forall\ Y \in B_R \subset V^1_c([0,T])$ $\left(\Rightarrow \|Y\|_{L^\infty([0,T])} \le R\right)$

$$\| \Gamma(Y^1) - \Gamma(Y^2) \|_{V^1([0,\tau])} \leq K \| Y^1 - Y^2 \|_{L^\infty([0,\tau])} \| X \|_{V^1([0,\tau])}$$

$$\leq \frac{1}{2} \| Y^1 - Y^2 \|_{V^1([0,\tau])}$$

$\Rightarrow$ Γ is a contraction on B_R

$\Rightarrow$ $\exists!\ Y \in B_R$ s.t. $Y = \Gamma(Y)$

$\boxtimes$

Rmk: • By Gronwall's inequality,

$$Y^j(t) = Y^j(0) + \int_0^t V(Y^j(s))\, dX(s), \quad j = 1, 2. \quad X \in V^1_c([0,T])$$

$$\Rightarrow \| Y^1 - Y^2 \|_{L^\infty([0,T])} \leq | Y^1(0) - Y^2(0) | \exp\left(K \| X \|_{V^1([0,T])} \right)$$

• soln map : $Y_0 \to Y$ is in fact C^1.

$$\Phi(t, Y_0) = Y(t)$$

Let $J_t = \dfrac{\partial \Phi(t, Y_0)}{\partial Y_0}$. Then,

$$J_t = Id + \int_0^t DV(\Phi(s, Y_0))\, J_s\, dX(s)$$