Lec 1 15/01/20 (Wed)

Rough path theory & pathwise well-posedness of stochastic PDEs

(*) $dY_t = f(Y_t)\, dX_t$ $\left.\begin{matrix}\text{ODE}\\ \text{SPE}\end{matrix}\right\}$ RDE = rough diff. eqn.

$\uparrow$ given input source

ex: $\dot{X} = \xi =$ white noise

$\dot{X} = dX$ i.e. $X = B =$ Brownian motion

- By Itô integral / stochastic integration theory, we can solve (*) but the soln map $\Phi : (Y_0, B) \mapsto Y$ (Itô map) lacks continuity, in general, due to roughness of BM.

Ref:
- Friz-Hairer : Intro to rough paths
- Friz-Victoir : Multidim'l stoch. processes as rough paths (Ch 1, 5, 6)
  (Friz' website : check for errata
- Baudoin : Lecture note.

<u>FACT</u> (Prop 1.1 in [FH]) $\exists$ separable Banach space $W \subset C([0,1])$

s.t. (i) sample paths of BM lie in W a.s.

$$B(\cdot;\omega) \in W, \quad \text{a.s.}$$

(ii) map $(f,g) \longmapsto \int_0^t f(t')\,dg(t')$ (or $\int_0^t f(t')\dot{g}(t')\,dt'$),

a priori well defined on smooth functions,

extends to a continuous map from $W \times W$ into $C([0,1])$

- Given two indep BM's B^1 and B^2,

$$B = (B_1, B_2) \longmapsto \int_0^{\cdot} B^1(t)\,\underbrace{\dot{B}^2(t)\,dt}_{= dB^2(t)}$$

(may still make sense of the integral but this map is not conti)

$$Y \in \mathbb{R}^2, \quad \begin{aligned} \dot{Y}^1 &= \dot{B}^1 \\ \dot{Y}^2 &= Y^1 \dot{B}^2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} Y^1 &= B^1 \\ Y^2 &= \int_0^t B^1\,dB^2. \end{aligned}$$

$\Rightarrow$ Itô map for (*) is <u>not</u> continuous.

<u>Goal</u>: Give a meaning to the soln theory for ⊛ when X is rough.

(X as rough as BM is already non-trivial and important

- Rough path theory introduced by Terry Lyons '98 (S. Davie)
- Controlled paths by Gubinelli '04.

<u>Idea</u>: <u>Factorize</u> the "ill-posed" soln map $\Phi : B \mapsto Y$

$$\cdot \quad B(\omega) \xrightarrow{\ \Psi\ } \underbrace{(B(\omega), \mathbb{B}(\omega))}_{\text{enhanced data set}} \xrightarrow{\ \hat{\Phi}\ } Y(\omega)$$

measurable universal lift

continuous soln map (Itô-Lyons map) to a RDE (where we view $(B(\omega), \mathbb{B}(\omega))$ as deterministic data for fixed ω.)

$$B^{i,j}(s,t) = \int_s^t (B^i(r) - B^i(s))\, dB^j(r)$$

Kuo: Intro to stoch. integ. (Chap. 9) ④

iterated Itô-Wiener integral

Itô or Stratonovich ⇐ freedom to choose interpretation of stoch integral.

In general, given rough X, we do not have enough regularity/information

We need to enhance the data to $(X, \mathbb{X})$.

$$\mathbb{X}^{i,j}_{s,t} = X^{i,j}(s,t) = \int_s^t X^i_{s,r}\, dX^j_r \qquad F_{s,t} = F_t - F_s$$

Here, LHS defines RHS!!

· Gubinelli's controlled paths give a meaning to $\int_0^t Y_s\, dX_s$

⇐ can do this if Y is controlled by X.

$$Y_{s,t} = Y'_s X_{s,t} + R_{s,t} \qquad Y' = \text{Gubinelli derivative}$$

smoother

⇐ local behavior/fluctuation of Y is essentially given by that of X.

We want to solve $dY = f(Y)\,dX$.

① Riemann-Stieltjes integral & ODE $\quad V^1 = BV, \quad Lip.$

② Young integral, $\quad V^p, \quad C^{1/p}, \; 1<p<2.$

$\quad\quad$ ↳ Hölder space $\quad\quad$ · predual of $C^{1/2}$?

③ endpoint case $p=2$. $\quad V^2, \; U^2 \leftarrow$ predual of V^2.

introduced by Tataru / Koch-Tataru.

④ controlled rough paths $\quad V^p, \; C^{1/p}, \; 2 \le p < 3.$

In general, for $k \le p < k+1$, we need higher order rough paths

$$(X, \underbrace{\mathbb{X}^2, \mathbb{X}^3, \cdots, \mathbb{X}^k}_{\text{define higher order iterated integrals}})$$

· ④ with $2 \le p < 3$ is most important

since $BM \in V^p, \; C^{1/p}, \quad p = 2+, \; a.s.$

5 stochastic PDE

smoothing op in x

$$\partial_t u = \Delta u + N(u) + \sigma(u)\phi\xi \quad \text{heat eqn}$$

$$i\partial_t u = \Delta u + N(u) + u\phi\xi \quad \text{Schrödinger eqn.}$$

space-time white noise

$\Leftarrow$ solve

- $L^2(\Omega)$: using stochastic integral theory
- pathwise (much harder)

(There is also a notion of martingale soln but we are concerned with stronger notions of solns)

ex: $(\partial_t^2 - \partial_x^2)u = u^3 + u\xi$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$\frac{1}{2}-, \frac{1}{2}-$ (space, time) $\quad -\frac{1}{2}-, -\frac{1}{2}- \quad \Rightarrow$ cannot make sense of the product.

- $L^2(\Omega)$-theory: easy. see a note on my website.
- pathwise: open.

• Functions of finite p-variations and Hölder functions

$[0,T]$, $T>0$, fixed.

$$\|X\|_{V^p} = \sup_{\mathcal{P}} \left(\sum_{j=1}^{n-1} |X_{t_{j+1}} - X_{t_j}| \right)^{1/p}$$

$\uparrow$ partition of $[0,T]$

$t_0 = 0 < t_1 < \cdots < t_n = T$

Banach space with a norm $|X_0| + \|X\|_{V^p}$

• $\|\cdot\|_{V^p}$ is a semi-norm

- $V^p = V^p([0,T])$, $p>0$
- $V^p_c = V^p \cap C([0,T])$ $\quad$ $(= C^{p\text{-var}}([0,T])$ in [FV]$)$
- $V^p_{rc} = \{X \in V^p : X \text{ is right continuous}\}$

• Hölder space: Hölder continuous functions $\quad \|X\|_{C^\alpha} = \sup_{0\le s<t\le T} \dfrac{|X_t - X_s|}{|t-s|^\alpha}$

- $\alpha = 1$, Lipschitz conti func.
- $C^\alpha_{\text{Hölder}} = C^\alpha_{\text{Hölder}}([0,T]) = \begin{cases} C^{\alpha\text{-Höl}}([0,T]) & \text{in [FV]} \\ \Lambda^\alpha & \text{in Grafakos} \end{cases}$

(i) $V^p_c([0,T])$, $C^\alpha_{\text{Hölder}}([0,T])$

- Banach space
- <u>not</u> separable

(Thm 5.25
ex 5.26 in [FV].

(ii) $C^\alpha_{\text{Hölder}} \subset V^p_c$, $\alpha \geq \frac{1}{p}$

<u>Pf:</u> $$\underbrace{\left(\sum | X_{t_{j+1}} - X_{t_j} |^p \right)^{1/p}}_{\uparrow \text{ main building block of } V^p_c\text{-norm}} \leq \| X \|_{C^\alpha_{\text{Hölder}}} \left(\sum | t_{j+1} - t_j |^{\overbrace{p\alpha}^{\geq 1}} \right)^{1/p} \lesssim \| X \|_{C^\alpha_{\text{Hölder}}}$$

(iii) For $p < 1$ V^p_c
$\alpha > 1$, $C^\alpha_{\text{Hölder}}$ = const func.

(We will drop the subscript "Hölder" since we only use $0 < \alpha < 1$. and use Lip when $\alpha = 1$

<u>Pf:</u> By (ii), suffices to show $V^p_c = \{\text{const func}\}$, $p < 1$

$$|X_t - X_0| \leq \sum_{j=1}^{n-1} | X_{t_{j+1}} - X_{t_j} |$$

$$\leq \underbrace{\sup_j | X_{t_{j+1}} - X_{t_j} |^{1-p}}_{\to 0} \| X \|^p_{V^p}$$

$\to 0$ by taking $|\mathcal{P}| \to 0$, and using <u>unif conti</u> of X on $[0,T]$.